A Hierarchy of Symmetry Breaking in the Nonsymmetric Kaluza-Klein (Jordan-Thiry) Theory

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Abstract

The paper is devoted to the hierarchy of symmetry breaking in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory. The basic idea consists in a deformation of a vacuum states manifold to the cartesian product of vacuum states manifolds of every stage of a symmetry breaking. In the paper we consider a pattern of a spontaneous symmetry breaking including a hierarchy in the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory.

Introduction

In this paper we consider hierarchy of symmetry breaking in the Nonsymmetric Kaluza–Klein Theory and the Nonsymmetric Kaluza–Klein Theory with a spontaneous symmetry breaking and Higgs’ mechanism. In the second section we consider a Nonsymmetric Kaluza–Klein Theory and the Nonsymmetric Kaluza–Klein Theory with a spontaneous symmetry breaking and Higgs’ mechanism [1–6]. In the third section we develop a hierarchy of the symmetry breaking in our theory. For further development of the Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory [7–10].

Elements of the Nonsymmetric Kaluza–Klein Theory

In general non-Abelian case and with spontaneous symmetry breaking and Higgs’ mechanism

Let \( P \) be a principal fiber bundle over a space-time \( E \) with a structural group \( G \) which is a semisimple Lie group. On a space-time \( E \) we define a nonsymmetric tensor \( g_{\alpha\omega} = g_{\omega\alpha} \) such that

\[
g = \text{det}(g_{\alpha\omega}) \neq 0
\]

\( g_{\alpha\omega} \) is called as usual a skewon field (e.g., in NGT, [6,11–13] We define on \( E \) a nonsymmetric connection compatible with \( g_{\alpha\omega} \) such that

\[
\mathcal{D}_{\alpha\omega} = g_{\alpha\omega} \mathcal{D}_{\alpha\omega}^\rho (\mathcal{D}_{\rho} g)^{\omega
\}
\]

where \( \mathcal{D} \) is an exterior covariant derivative for a connection \( \mathcal{D}_{\alpha\omega} = \mathcal{D}_{\alpha\omega}^\rho (\mathcal{D}_{\rho} g)^{\omega
\} \) and \( \mathcal{D}_{\alpha\omega}^{\rho\rho} \) is its torsion. We suppose also

\[
\mathcal{D}_{\rho\omega} (\mathcal{D}_{\rho\omega}^\rho) = 0
\]

We introduce on \( E \) a second connection

\[
\mathcal{W}_{\beta} = \mathcal{W}_{\beta}^{\rho\rho} \mathcal{D}_{\rho\omega}
\]

such that

\[
\mathcal{W}_{\beta} = \mathcal{W}_{\beta}^{\rho\rho} - \frac{1}{2} \delta_{\alpha\omega} \mathcal{W}_{\rho\omega} = \frac{1}{2} (\mathcal{W}_{\rho\omega} - \mathcal{W}_{\omega\rho}) \mathcal{D}_{\rho\omega}
\]

Now we turn to nonsymmetric metrization of a bundle \( P \). We define a nonsymmetric tensor \( \gamma \) on a bundle manifold \( P \) such that

\[
\gamma = \mathcal{D}_{\alpha\omega} \otimes \mathcal{D}_{\omega\alpha} \otimes \mathcal{D}_{\beta\gamma}
\]

where \( \gamma \) is a projection from \( P \) to \( E \). On \( P \) we define a connection \( \omega \) (a 1-form with values in a Lie algebra \( g \) of \( G \)). In this way we can introduce on \( P \) (a bundle manifold) a frame \( \theta^i = (\mathcal{D}_{\alpha\omega}, \mathcal{D}_{\omega\alpha}) \) such that

\[
\theta^i = \lambda \theta^i, \quad \omega = \omega^i X_i, \quad a = 5, 6, \ldots, n + 4, \quad n = \text{dim } G, \quad \lambda = \text{const.}
\]

Thus our nonsymmetric tensor looks like

\[
\lambda = \gamma AB \theta^i \otimes \theta^i, \quad A, B = 1, 2, \ldots, n + 4
\]

\[
\lambda = \lambda \mathcal{D}_{\alpha\omega} + \mu \mathcal{K}_{\alpha\omega}
\]

where \( h_{\alpha\omega} \) is a bi-invariant t Killing-Cartan tensor on \( G \) and \( k_{\alpha\omega} \) is a right-invariant t skew- symmetric tensor on \( G, \mu = \text{const} \)

We have

\[
h_{\alpha\omega} = C_{\alpha\omega}^G C_{\alpha\omega}^G = h_{\omega\alpha}
\]

\[
K_{\alpha\omega} = - K_{\alpha\omega}
\]

Thus we can write

\[
\tau(X,Y) = g(\pi' X, \pi' Y) + \frac{1}{2} h(\omega(X), \omega(Y))
\]

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\]

\[
(\mathcal{C}_{\alpha\omega} \text{ are structural constant } s \text{ of the Lie algebra } g)
\]

\( \gamma \) is the symmetric ic part of \( \gamma \) and \( \mathcal{L} \) is the anti symmetric ic part of \( \gamma \) We have as usual

\[
[X_i, X_j] = C_{i,j}^G X_k
\]

\[
\Omega = \frac{1}{2} \mathcal{H}^{\mu\nu} \theta^i \lambda^i \theta^i
\]

\[
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\]

\[
\Omega = \frac{1}{2} \mathcal{H}^{\mu\nu} \theta^i \lambda^i \theta^i
\]

\[
d \theta^i = \frac{1}{2} \mathcal{H}^{\mu\nu} \theta^i \lambda^i \theta^i - \frac{1}{2} C_{\alpha\omega}^G \theta^i \lambda^i \theta^i \neq 0
\]

Even if the bundle \( \mathcal{L} \) is trivial, i.e. for \( \Omega = 0 \) This is different than

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in an electromagnetic case explained by Kalinowski MW [3]. Our non symmetric metrization of a principal fiber bundle gives us a right-invariant t structure on P with respect to an action of a group G on P [3]. Having P non symmetrically metrized one defines two connection s on P right- invariant t with respect to an action of a group G on P. We have

\( \gamma_{AB} = \begin{pmatrix} \xi_{\alpha\beta} & 0 \\ 0 & \lambda_{\alpha} \end{pmatrix} \) \quad (2.17)

In our left horizontal frame \( \theta^A \)

\( D_{\alpha\beta} = \gamma A D \Theta^B \) \quad (2.18)

\( Q^B D\Theta = 0 \) \quad (2.19)

where \( D \) is an exterior covariant derivative with respect to connection \( \alpha^A = \Gamma_{\alpha}^A \theta^A \) on P and \( \dot{Q}^A (\dot{\Theta}) \) its torsion. One can solve Equation (2.18)–(2.19) getting the following results

\( \omega^A B = \left( \xi^A (\xi^B) - \xi^A \xi^B \eta^C + \eta^C \eta^D \eta^A \right) \gamma^D \) \quad (2.20)

where \( \xi^A \) is an inverse tensor of \( \gamma^A \)

\( g_{\alpha\beta} \gamma^A = g_{\alpha\beta} \gamma^B = \delta^A_{\beta} \) \quad (2.21)

\( L'_{\alpha} = -L'_{\beta} \) is an Ad-type tensor on P such that

\( \xi^A g_{\alpha\beta} \mu^B = \xi^C g_{\alpha\beta} \mu^C + \xi^C g_{\alpha\beta} \mu^D g_{\mu\nu} H'_{\nu} \) \quad (2.22)

\( \Sigma^A_\alpha = \Gamma^A_{\alpha} \theta^A \) is a connection on an internal space (typical fiber) compatible with a metric \( \xi^A \) such that

\( \xi^A \Gamma^B_{\alpha} + \xi^B \Gamma^C_{\alpha} = -\xi^C g_{\alpha\beta} \) \quad (2.23)

\( \Gamma^A_{\alpha} = 0, \Gamma^A_{\beta} = \Gamma^A_{\alpha} \) \quad (2.24)

and of course \( \dot{Q}^A (\dot{\Theta}) = 0 \) where a torsion of the connection \( \Sigma^A_\alpha \) We also introduce an inverse tensor of \( g^A \)

\( g^A (gB) g^{BC} = \delta^B_{\beta} \) \quad (2.25)

We introduce a second connection on P defined as

\( W'_{\alpha} = \omega^A_{\alpha} - \frac{4}{3(a+2)} \delta^A_{\alpha} \) \quad (2.26)

\( \bar{W} \) is a horizontal one form

\( \bar{W} = \text{hor} \bar{W} \) \quad (2.27)

\( \bar{W} = -\frac{1}{2} (\bar{W}_{\alpha} - \bar{W}_{\beta}) \) \quad (2.28)

In this way we define on P all analogues of four- dimension al quantities from NGT [6,11]. It means, \( n+4 \) dimension al analogues from Moffat theory of gravitation, i.e. two connection s and a non symmetric metric \( \gamma^A_\alpha \). Those quantities are right- invariant t with respect to an action of a group G on P. One can calculate a scalar curvature of a connection \( W'_{\alpha} \) getting the following result [1-3].

\( R(W) = R(\bar{W}) = \frac{\lambda^2}{4} (2\xi^A g_{\alpha\beta} \gamma^C \gamma^D, \xi^C g_{\gamma\delta} H'_{\delta} + \bar{R}(\bar{\Theta}) \) \quad (2.29)

Where

\( R(W) = \gamma^A (R^C \gamma^D) + \frac{1}{2} R^C \gamma^D \) \quad (2.30)

is a Moffat–Ricci curvature scalar for the connection \( W'_{\alpha} \), \( R(\bar{W}) \) is a Moffat–Ricci curvature scalar for the connection \( \Sigma^A_\alpha \)

\( H' = g^{[\alpha\beta]} H'_{\alpha\beta} \) \quad (2.31)

\( L^a_{\alpha} = \gamma^A g_{\alpha\beta} \gamma^C g_{\delta\gamma} H'_{\delta\gamma} \) \quad (2.32)

Usually in ordinary (symmetric) Kaluza–Klein Theory one has

\( \lambda = \frac{2}{c^2} \) \quad where \( GN \) is a Newtonian gravitational constant and \( c \) is the speed of light. In our system of units this is the same as in Non symmetric Kaluza–Klein Theory in the electromagnetic case [3,4]. In the non- Abelian Kaluza –Klein Theory which unifies GR and Yang–Mills field theory we have a Yang–Mills lagrangian and a cosmological term.

Here we have

\( L^V_{\alpha\beta} = \frac{1}{8\pi} \gamma^A g_{\alpha\beta} (2H'_{\alpha\beta} - L^C_{\alpha\beta} H'_{\alpha\beta}) \) \quad (2.33)

and \( \bar{R}(\bar{\Theta}) \) plays a role of a cosmolog ical term.In order to incorporate a spontaneous symmetry breaking and Higgs’ mechanism in our geometrical unification of gravitation and Yang–Mills’ fields we consider a fiber bundle \( P \) over a base manifold \( E \times G/G_0 \) where E is a space-time, \( G \subset G \) are semisimple Lie groups. Thus we are going to combine a Kaluza–Klein theory with a dimension al reduction procedure.

Let \( P \) be a principal fiber bundle over \( V = E \times M \) with a structural group \( H \) and with a projection \( \pi \), where

\( M = G/G_0 \) is a homogeneous space, \( G \) is a semisimple Lie group and \( G_0 \) its semisimple Lie subgroup. Let us suppose that \( (V, Y) \) is a manifold with a non symmetric metric tensor

\( \gamma_{\alpha\beta} = \gamma_1 (\alpha) + \gamma_{\alpha\beta} \) \quad (2.34)

The signature of the tensor \( \gamma \) is \((+,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-,-)\). Let us introduce a natural Phenomenon

\( \theta^3 = \sum (\theta^{\alpha} \theta^{\beta} \theta^\gamma \) \quad (2.35)

It is convenient to introduce the following notation. Capital Latin indices with tilde \( \bar{A}, \bar{B}, \bar{C} \) run \( 1,2,3,\ldots,m+4 \), \( m = \text{dim} H \; \text{dim} M = n + \text{dim} M = n \), \( m = \text{dim} H \). Lower Greek indices \( \alpha, \beta, \gamma, \delta = 1,2,3,4 \) and lower Latin indices \( a, b, c \) run \( 1,2,3 \ldots, m+4 \). Capital Latin indices \( A, B, C \) run \( 1,2,\ldots, n \). Lower Latin indices with tilde \( \bar{a}, \bar{b}, \bar{c} \) run \( 1,2,\ldots, n \). The symbol over \( \theta^{\alpha} \) and other quantities indicates that these quantities are defined on \( V \). We have of course \( n = \text{dim} G \), \( \text{dim} G_0 = n \) \( n = n_1, \ldots, n \), \( n_1, \ldots, n \), \( m = n + n \), \( n + n \), \( m = n + n \), \( m = n + n \).

On the group \( H \) we define a bi- invariant t ( symmetric) Killing–Cartan tensor

\( h_A (B) = h_B A^B \) \quad (2.36)

We suppose \( H \) is semisimple, it means \( \text{det}(h_A) \neq 0 \). We define a skew- symmetric ic right- invariant t tensor on H

\( k_A (B) = k_{\bar{a}} A^B \bar{a} \; \bar{a} = -k_{\bar{b}} \)

Let us turn to the nonsymmetric metrization of \( P \).

\( k(X,Y) = \gamma(X,Y) + \lambda^2 \xi^A \omega(X) \omega^A(Y) \) \quad (2.37)

where

\( \xi^A = h_A + \xi K^A \) \quad (2.38)

is a nonsymmetric right-invariant t tensor on H. One gets in a matrix form (in the natural frame (2.35))
We have \( (2.53) \)
\[
\gamma_{ab} = r^2 (h_{ab} + \xi k_{ab}^* ) = r^2 g_{ab}^* .
\]
Thus we have a nonsymmetric metric on \( M = E \times G/G_0 \)
\[
(2.50)
\]
Thus we are able to write down the nonsymmetric metric on \( V = E \times M = E \times G/G_0 \)
\[
\gamma_{ab} = \left( \begin{array}{cc}
g_{ab} & 0 \\
0 & r^2 g_{ab}^* 
\end{array} \right)
\]
(2.51)
where
\[
g_{ab} = g(\alpha \beta) + g(\alpha \beta)
\]
\[
g_{ab} = h_{ab}^0 + k_{ab}^0 h_{00}
\]
\[
k_{ab}^0 = -k_{ba}^0
\]
\[
a, \beta = 1, 2, 3, 4 . \quad \tilde{a}, \tilde{b} = 5, 6, \ldots, n_1 + 4 .
\]

The signature of the tensor \( k \) is \((4, -\cdots, -\cdots, 4)\). As usual, we have commutation relations for Lie algebra of \( H, \xi \)
\[
[X_a, X_b] = C_{ab}^c X_c.
\]
(2.41)

This metrization of \( P \) is right-invariant t with respect to an action of \( H \) on \( P \). Now, we should nonsymmetrically metrize \( M = G/G_0 \), \( M \) is a homogeneous space for \( G \) (with left action of group \( G \)). Let us suppose that the Lie algebra of \( G \), \( g \) has the following reductive decomposition
\[
g = g_0 \oplus m
\]
(2.42)
where \( g_0 \) is a Lie algebra of \( G_0 \) (a subalgebra of \( g \)) and \( m \) (the complement to the subalgebra \( g_0 \)) is \( \text{Ad}_{G_0} \) invariant, \( + \) means a direct sum. Such a decomposition might be not unique, but we assume that one has been chosen. Sometimes one assumes a stronger condition for \( m \), the so called symmetry requirement
\[
[m, m] \subset g_0 .
\]
(2.43)

Let us introduce the following notation for generators of \( g \):
\[
Y_i \in g,Y_i \in g,Y_i \in g_0 .
\]
(2.44)

This is a decomposition of a basis of \( g \) according to (2.42). We define a symmetric metric on \( M \) using a Killing–Cartan form on \( G \) in a classical way. We call this tensor \( h \). Let us define a tensor field \( h(s) \) on \( G/G_0 \), \( \text{exp}(s)G \) using tensor field \( h \) on \( G \). Moreover, if we suppose that \( h \) is a bi-invariant metric on \( G \) (a Killing–Cartan tensor) we have a simpler construction. The complement \( m \) is a tangent space to the point \( [eG] \) of \( M \), \( e \) is a unit element of \( m \). We restrict \( h \) to the space \( m \) only. Thus we have \( h^1(eG) \) at one point of \( M \). Now we propagate \( h^1(eG_0) \) using a left action of the group \( G \)
\[
h^1((gG_0)) = (L_g^g)^* h^1((eG_0)).
\]
(2.50)
\( h^1((eG_0)) \) is of course \( \text{Ad}_{G_0} \) invariant t tensor defined on \( m \) and \( L_g h^0 = h^0 \)

We define on \( M \) a skew-symmetric 2-form \( \omega^0 \). Now we introduce a natural frame on \( M \). Let \( f^{\nu}_j \) be structure constant of \( s \) of the Lie algebra \( g \), i.e.
\[
[Y_j, Y_i] = f^{\nu}_j Y_i.
\]
(2.45)

\( Y_i \) are generators of the Lie algebra \( g \). Let us take a local section \( s : V^g \rightarrow G/G_0 \), of a natural bundle \( G \rightarrow G/G_0 \) where \( V \subset M = G/G_0 \). The local section \( s \) can be considered as an introduction of a coordinate system on \( M \).

Let \( \omega = \omega_{\nu}^\gamma \) be a left-invariant t Maurer–Cartan form and let
\[
\omega_{\nu}^{\gamma} = \sigma^\gamma \omega_{\nu}^{\gamma}
\]
(2.46)
Using de composition (2.42) we have
\[
\omega^\gamma Y = \omega^\nu Y + \omega^\nu Y = \theta^\nu Y + \Gamma^\gamma Y,
\]
(2.47)
It is easy to see that \( \omega \) is the natural (left-invariant t) frame on \( M \) and we have
\[
h^0 = h^0 \omega_\nu \otimes \omega_\nu,
\]
(2.48)
\[
k^0 = k_{ab}^0 \omega_\nu \wedge \omega_\nu
\]
(2.49)

According to our notation \( \tilde{a}, \tilde{b} = 5, 6, \ldots, n_1 + 4 \).

Let us introduce the following notation for generators of \( g_0 \):
\[
Y_i \in g_0,Y_i \in g_0,Y_i \in g_0 .
\]
(2.44)

\( Y_i \) are coefficients of nonholonomicity and depend on the \( s \) are structure constant s of the Lie algebra
\[
C_{ab}^c \in \mathbb{C}.
\]

Let us denote by \( \mu' \) a tangent map to \( \mu \) at a unit element. Thus \( \mu' \) is a differential of \( \mu \) acting on the Lie algebra element \( s \). Let us suppose that the connection \( \omega \) on the fiber bundle \( P \) is invariant t under group action of \( G \) on the manifold \( V = E \times G/G_0 \). According to Kobayashi [14-17] this means the following.

Let \( e \) be a local section of \( P \), \( e : V \subset U \rightarrow P \) and \( A = e^* \omega \). Then for every \( e \in G \) there exists a gauge transformation \( \rho \) such that
\[
f^* (\omega ) = Ad_{\rho_\omega} = \rho^{-1} s d \rho
\]
(2.56)
f means a pull-back of the action of the group \( G \) on the manifold \( V \). According to Hlavaty [13-25] we are able to write a general form for such an \( \omega \). Following [17] we have
\[
\omega = \omega_{\nu}^\gamma + \mu' \omega_{\nu}^\gamma + \Phi \omega_{\nu}^\gamma
\]
(2.57)
(An action of a group \( G \) on \( V = E \times G/G_0 \) means left multiplication on a homogeneous space \( M = G/G_0 \), \( \omega'_{\nu} = \omega_{\nu}^\gamma + \omega_{\nu}^\gamma - \omega_{\nu}^\gamma \), \( \omega_{\nu}^\gamma \) are components of the pull-back of the Maurer–Cartan form from the de composition (2.47). \( \omega \) is a connection defined on a fiber bundle \( Q \) over a space-time \( E \) with structural group \( C \) and a projection \( \pi' \). Moreover, \( C = G \) and \( \omega_{\nu}^\gamma \) is a 1-form with values in the Lie algebra \( g \).
This connection describes an ordinary Yang–Mills’ field gauge group \( Q = G \) on the space-time \( E \). \( \Phi \) is a function on \( E \) with values in the space \( S \) of linear maps

\[
\Phi : m \rightarrow h
\]  

(2.58)
satisfying \( \Phi \)

\[
\Phi([X_a, X_b]) = \Phi([X_a, X_b], \Phi(X))
\]  

(2.59)
Thus

\[
\omega_a = \omega_a^i Y_i, Y_i \in g,
\]

\[
\omega_b = \omega_b^i Y_i, Y_i \in g,
\]

\[
\omega_a^b = \gamma_a^b Y_i, Y_i \in m.
\]  

(2.60)
Let us write condition (2.57) in the base of left-invariant t form \( \rho, \rho^i \), which span respectively dual spaces to \( g_a \) and \( m \) \([24,25]\). It is easy to see that

\[
\Phi \Phi_a \omega_a = \Phi_a^i(x) \tilde{\gamma}_i X_a, X_a \in h
\]  

(2.61)
and

\[
\mu_a^i = \mu_a^i \rho X_a
\]  

(2.62)
From (2.59) one gets

\[
\Phi_a(x) \tilde{\gamma}_i^a = \mu_a^\gamma A \rho C_i^a
\]  

(2.63)
where \( \tilde{\gamma}_i^a \) are structure constant \( s \) of the Lie algebra \( g \) and \( C_a, \gamma \) are structure constant of the Lie algebra \( h \). Equation (2.63) is a constraint on the scalar field \( \Phi(x) \). For a curvature of \( \omega^a \) one gets

\[
H = \alpha_i H_i^j
\]  

(2.65)
\[
H^i = \gamma_i^a \rho \Phi_a^i = - H_i^j
\]  

(2.66)
\[
H = C_i^a \Phi_a^i \tilde{\gamma}_i^a - \Phi_a^i \tilde{\gamma}_i^a - \Phi_a^i \tilde{\gamma}_i^a
\]  

(2.67)
where \( \gamma_i^a \) means gauge derivative with respect to the connection \( \tilde{\gamma}_i^a \) defined on a bundle \( q \) over a space-time \( E \) with a structural group \( G \).

\[
Y_i = \alpha_i X_a
\]  

(2.68)
\( \tilde{H}_i \) is the curvature of the connection \( \tilde{m} \) in the base \( \{Y_i\} \), where \( \alpha_i \) is the matrix which connects \( \{Y_i\} \) with \( \{X_i\} \). Now we would like to remind that indices \( a, b, c \) refer to the Lie algebra \( h \), \( i, j, k \) to the space \( m \) (tangent space to \( M \)), \( l, m, n \) to the Lie algebra \( g \), and \( i, j, k \) to the Lie algebra of the group \( G \), \( g \). The matrix \( \alpha_i \) establishes a direct relation between the generators of the Lie algebra of the subgroup \( H \) iso morphic to the group \( G \).

Let us come back to a construction of the Non-symmetric Kaluza–Klein Theory on a manifold \( P \). We should define connection \( s \). First of all, we should define a connection compatible with a non-symmetric tensor \( \gamma^{\alpha \beta} \) Equation (2.51)

\[
\mathcal{V}^a B = \gamma^a BCD\tilde{\gamma}^d
\]  

(2.69)
\[
\mathcal{D} \gamma^a AB = \gamma A D \tilde{\gamma}^b B C(\gamma) \tilde{\gamma}^c
\]  

(2.70)
\[
\tilde{\gamma}^d BD(\gamma) = 0
\]

where \( \mathcal{V}^a \) is the exterior covariant derivative with respect to \( \gamma^a \) and \( \tilde{\gamma}^b B C(\gamma) \) its torsion. Using (2.51) one easily finds that the connection \( (2.69) \) has the following shape

\[
\gamma^a B = \begin{pmatrix}
\sigma_1 \omega & \sigma_2 \omega \\
0 & \sigma_2 \omega
\end{pmatrix}
\]

(2.71)
where \( \omega \) is a connection on the space-time \( E \) and on the manifold \( M = G / G_0 \) with the following properties.\[ \]  

\[
\mathcal{D}_{\alpha \beta} \gamma = g a \omega \tilde{\gamma}^d \beta (\gamma) \tilde{\gamma}^e = 0
\]

(2.72)
\[
\tilde{\gamma}^d \beta (\gamma) = 0
\]

(2.73)
\[
\tilde{\gamma}^d (\gamma) = 0
\]

(2.74)
\( \tilde{\gamma}^d \) is an exterior covariant derivative with respect to a connection \( \omega \) whereas \( \tilde{\gamma}^d \beta (\gamma) \) is a torsion of a connection \( \tilde{\gamma}^d \) is an exterior covariant derivative of a connection \( \tilde{\gamma}^d \beta (\gamma) \). On a space-time \( E \) we also define the second affine connection such that

\[
\tilde{\gamma}^d \beta = \omega - \frac{2}{3} \beta \tilde{\gamma}^d \beta\tilde{\gamma}^e = 0
\]

(2.75)
\[
\tilde{\gamma}^d = \frac{1}{2} (\tilde{\gamma}^d - \omega - \omega)
\]

(2.76)
\( \tilde{\gamma}^d \) is an exterior covariant derivative with respect to the connection \( \omega \) and \( \tilde{\gamma}^d \) its torsion. On a space-time \( E \) we define the second affine connection such that

\[
\omega = \frac{1}{2} (\tilde{\gamma}^d - \omega - \omega) + \tilde{\gamma}^d \beta
\]  

(2.77)
where \( \omega \) is an exterior covariant derivative with respect to a connection \( \omega \) and \( \tilde{\gamma}^d \) and \( \tilde{\gamma}^d \beta \) its torsion. After some calculations one finds

\[
\omega = \frac{1}{2} (\tilde{\gamma}^d - \omega - \omega) + \tilde{\gamma}^d \beta = \frac{1}{2} \tilde{\gamma}^d \beta + \tilde{\gamma}^d \beta
\]

(2.78)
We define on \( P \) a second connection

\[
\omega = \omega + \frac{A}{(m+2)} \beta \tilde{\gamma}^d \beta W
\]

(2.79)
Thus we have on \( \omega \) all \( (m+4) \) dimension al analogs of geometrical quantities from NGT, i.e. \( \omega \), \( \omega \) and \( \kappa_{ab} \).

Let us calculate a Moffat–Ricci curvature scalar for the connection

\[
W = \omega + \frac{A}{(m+2)} \beta \tilde{\gamma}^d \beta W
\]

(2.80)
Hierarchy of a Symmetry Breaking

Let us incorporate in our scheme a hierarchy of a symmetry breaking. In order to do this let us consider a case of the manifold

\[ M = M_1 \times M_2 \times \cdots \times M_{i+1} \]  
(3.1)

where

\[ \dim M_i = n_i, \quad i = 0, 1, 2, \ldots, k - 1 \]  
(3.2)

\[ \dim M = \sum_{i=0}^{k-1} n_i \]  
(3.3)

\[ M_i = \frac{G_{i+1}}{G_i} \]  
(3.4)

Every manifold \( M_i \) is a manifold of vacuum states if the symmetry is breaking from \( G_{i+1} \) to \( G_i \). Thus

\[ G_i \subset G_{i+1} \subset \cdots \subset G_k = G. \]  
(3.5)

We will consider the situation when

\[ M \simeq G / G_k. \]  
(3.6)

This is a constraint in the theory. From the chain (3.5) one gets

\[ g_0 \subset g_1 \subset \cdots \subset g_k = g. \]  
(3.7)

and

\[ g_{i+1} = g_i + m_i, \quad i = 0, 1, \ldots, k - 1. \]  
(3.8)

The relation (3.6) means that there is a diffeomorphism \( g \) onto \( G / G_k \) such that

\[ g : \prod_{i=0}^{k-1} (G_{i+1} / G_i) \rightarrow G / G_k. \]  
(3.9)

This diffeomorphism is a deformation of a product (3.1) in \( G / G_k \). The theory has been constructed for the case considered before with \( G_k \) and \( G \). The multiplet of Higgs’ fields \( \Phi \) breaks the symmetry from \( G \) to \( G_k \) (equivalently from \( G \) to \( G_k \) in the false vacuum case), \( g \) mean Lie algebras for groups \( G \) and \( G_k \) complement in a decomposition (3.8). On every manifold \( M \) we introduce a radius \( r \) (a “size” of a manifold) in such a way that \( r \ll r_i \). On the manifold \( M / G_k \) we define the radius \( r \) before. The diffeomorphism \( g \) induces a contragradient transformation for a Higgs field \( \Phi \) in such a way that

\[ g^* \Phi = (g_0^* \Phi_1, g_1^* \Phi_2, \ldots, g_k^* \Phi_k). \]  
(3.10)

The fields \( \Phi_i, i = 0, \ldots, k - 1 \).

In this way we get the following decomposition for a kinetic part of the field \( \Phi \) and for a potential of this field:

\[ L_{\text{kin}}(\Phi) = \sum_{i=0}^{k-1} L_{\text{kin}}(g_i \Phi_i) \]  
(3.11)

\[ V(\Phi) = \sum_{i=0}^{k-1} V^i(\Phi_i) \]  
(3.12)

where

\[ V^i = \int_x \sqrt{|g^i|} |\Phi_i^*| |\Phi_i| \]  
(3.13)

\[ \tilde{g} = \det(g_{\alpha\beta}) \]  
(3.14)

\[ g_{\alpha\beta} = \delta_{\alpha\beta} \]  
(3.15)

Equation (3.5) is a non symmetric tensor on a manifold \( M_i \).

\[ \varepsilon^{ij} \frac{\partial}{\partial \Phi^i} \frac{\partial}{\partial \Phi^j} (C_2 g_0^1 \Phi_1^1 \Phi_2^1 - \mu_2^0 \Phi_1^1 \Phi_2^1) \times \varepsilon^{ij} \]  
(3.16)

are structure constants of the Lie algebra \( g \). The scheme of the symmetry breaking acts as follows from the group \( G_{i+1} \) to \( G_i \) (if the symmetry has been broken up to \( G_{i+1} \)). The potential \( V^i(\Phi_i) \) has a minimum (global or local) for \( \Phi_{i+1}^{\text{min}}, k = 0 \). The value of the remaining part of the sum (3.12) for fields \( \Phi_j, j < i \), is small for the scale of energy is much lower (\( r_i > r_j, j < i \)). Thus the minimum of \( V^i(\Phi_i) \) is an approximate minimum of the remaining part of the sum (3.12)). In this way we have a descending chain of truncations of the Higgs potential. This gives in principle a pattern of a symmetry breaking. However, this is only an approximate symmetry breaking. The real symmetry breaking is from \( G \) to \( G_k \) (or to \( G_i \) in a false vacuum case). The important point here is the diffeomorphism \( g \)

\[ \Phi = \Phi_0(\Phi_{i+1}, \ldots, \Phi_k). \]  
(3.17)

\[ \Phi = \Phi_0^g \Phi_i, \quad i = 0, \ldots, k - 1. \]  
(3.18)

The shape of \( g \) is a true indicator of a reality of the symmetry breaking pattern. If

\[ \tilde{g} = \text{Id} + \delta g \]  
(3.19)

where \( \delta g \) in some sense small and \( \text{Id} \) is an identity, the sums (3.11)-(3.12) are close to the analogous formulae from the expanation of Kalinowski [5,10]. The smallness of \( g \) is a criterion of a practical application of the symmetry breaking (3.1). Moreover, a smallness of \( \delta g \) plus some natural conditions for groups \( G \) can narrow looking for grand unified models. Let us notice that the decomposition of \( M \) results in decomposition of cosmological terms

\[ \tilde{P} = \sum_{i=0}^{k-1} \tilde{P}_i \]  
(3.20)

where

\[ \tilde{P}_i = \frac{1}{g_i} \sqrt{g_i} \sqrt{|R(\tilde{G}_i)/l^4|} \]  
(3.21)

where \( \tilde{G}_i \) is a non symmetric connection on \( M_i \) compatible with the non symmetricic tensor \( \tilde{R}_{\alpha\beta\gamma} \) and \( \tilde{R}_{\gamma\alpha\beta} \) its curvature scalar. The truncation procedure can be proceeded in several ways. Finally let us notice that the energy scale of broken gauge bosons is fixed by a radius \( r_i \) at any stage of the symmetry breaking in our scheme.

Let us consider Equation (3.10) in more details. One gets

\[ A_{i+1}(y) \Phi_i(y) = \Phi_{i+1}(y), \quad y \in M_i, y \in M_i \]  
(3.22)

where

\[ g^*(y) = (A_1 | A_2 | A_3 | \ldots | A_k), \]  
(3.23)

\[ A_i = (A^i_{y\alpha\beta})_{\alpha\beta=1,2,...,n_i}, \quad i = 0,1,2,...,k \]  
(3.24)

is a matrix of Higgs’ fields transformation.

According to our assumptions one gets also:

\[ \tilde{L}_{\gamma\alpha\beta}(y) \delta g_{\gamma\alpha\beta}(y) = A_{i+1}(y) A_i^* (y) \tilde{g}_{\gamma\alpha\beta} (y) \]  
(3.25)

For \( g \) is an invertible map we have \( \det g^*(y) \neq 0 \).

We have also
We know from elementary particles physics theory that
\[ \Phi = U_{e} \otimes SU(3)_{\text{color}} \]
and that \( G_{2} \) is a group which plays the role of \( H \) in the case of a symmetry breaking from \( SU(2)_{L} \otimes U_{1}(1) \) to \( U_{e}(1) \). However, in this case because of a factor \( U(1) \), \( M = S^{2} \). Thus \( M_{i} = S^{2} \) and \( G_{2} \subset H_{i} \).

It seems that in a reality we have to do with two more stages of a symmetry breaking. Thus \( k = 3 \). We have
\[ M \approx S^{2} \times M_{1} \times M_{2} \]
\[ M_{i} = G_{i} / (SU(2) \times U(1) \times SU(3)) \]
\[ U(1) \otimes SU(3) \subset SU(2) \otimes U(1) \subset SU(3) \subset G_{j} \subset SU(3) \subset G_{k} = G_{3} \]
and
\[ G_{i} \subset H_{i} \subset H \]
\[ (U(1) \otimes SU(2) \otimes SU(3)) \otimes G_{i} \subset H_{i} \]
and
\[ G_{i} \otimes G \subset H_{i} \]
\[ M_{2} = G / G_{i} \]
\[ G \subset SU(5), SU(10), E6 \text{ or } SU(6). \]

Thus there are a lot of choices for \( G_{j}, H_{i} \), and \( H \). We can suppose for a trial that
\[ G_{2} \subset SU(3) \subset H_{i} \]
We have also some additional constraints
\[ \text{rank}(G) \geq 4 \]
Thus
\[ \text{rank}(H_{i}) \geq 4 \]
We can try with \( F4 = H_{i} \),

In the case of \( H \)
\[ \text{rank}(H) \geq \text{rank}(G) + 3 \frac{7}{2} \]
Thus we can try with \( E_{6}, E_{8} \)
\[ \text{rank}(H) \geq \text{rank}(G) + 4 \]
\[ \text{rank}(H) \geq \text{rank}(G) + \text{rank}(G) \leq \text{rank}(G) + 4 \geq \text{rank}(G) + 4 \]
In this way we have
\[ \text{rank}(H) \geq 8.5f \]
Thus we can try with
\[ H \approx E8.5g \]
But in this case
\[ \text{rank}(G) = \text{rank}(G) = 4 \]
This seems to be nonrealistic. For instance, if \( G = SO(10), E_{6} \),
\[ \text{rank}(SO(10)) = 5 \text{rank}E_{6} = 6 \]
In this case we get
\[ \text{rank}(H) = 9 \text{ rank}(H) = 10 \]
And \( H \) could be \( SO(10), SO(18), SO(20) \). In this approach we try to
consider additional dimensions connecting to the manifold M more seriously, i.e. as physical dimensions, additional space-like dimensions. We remind to the reader that gauge-dimensions connecting to the group H have different meaning. They are dimensions connected to local gauge symmetries (or global) and they cannot be directly observed. Simply saying we cannot travel along them. In the case of a manifold M this possibility still exists. However, the manifold M is diffeomorphically equivalent to the product of some manifolds M_i, i = 0, 1, 2, ..., k−1, with some characteristic sizes r. The radii r_i represent energy scales of symmetry breaking. The lowest energy scale is a scale of weak interactions (Weinberg–Glashow–Salam model) r_0 ≈ 10^{−16} cm. In this case this is a radius of a sphere S. The possibility of this "travel" will be considered in the concept explained by Kalinowski [26]. In this case a metric on a manifold M can be dependent on a point x ∈ E (parametrically). It is interesting to ask on a stability of a symmetry breaking pattern with respect to quantum fluctuations. This difficult problem strongly depends on the details of the model. Especially on the Higgs sector of the practical model. In order to preserve this stability on every stage of the symmetry breaking we should consider remaining Higgs' fields (after symmetry breaking) with zero mass. According to S. Weinberg, they can stabilize the symmetry breaking in the range of energy

\[ \frac{1}{r_i^2} < E < \frac{1}{r_i^1}, \quad i = 0, 1, 2, ..., k−1, 6 \]  

(3.58)
i.e. for a symmetry breaking from G_i to G_i

It seems that in order to create a realistic grand unified model based on non symmetric Kaluza–Klein (Jordan–Thiry) theory it is necessary to use a more general condition (3.61) which goes to the mentioned terms \( f(\Phi, \Phi') \). The conditions (3.63) plus a consistency (3.66) avoid those terms in the Higgs' potential. This problem demands more investigation. φ (g) = [gG_i]

It seems that the condition (3.9) could be too strong. In order to find a more general condition we consider a simple example of (3.5). Let G_{0} = e and K = 2 in this case we have

\[ \{\phi \} \subset G \subset G_i = G \]  

(3.67)

\[ M_i = G_i, \quad M_i = G / G_i \]  

(3.68)

\[ g : G \times G / G_i \to G \]  

(3.69)

In this way G \times G / G_i is diffeomorphically equivalent to G. Moreover, we can consider a fibre bundle with base space G / G_i and a structural group G_i with a bundle manifold G. This construction is known in the theory of induced group representation done by Trautman [27]. The projection φ : G \to G_i is defined by φ (g) = [gG_i]. The natural extension of (3.69) is to consider a fibre bundle (G_i, G_0, G, φ_i). In this way we have in a place of (3.60) a local condition

\[ g : G_i \times U \to G \]  

(3.70)

where U ⊂ G / G_i is an open set. Thus in a place of (3.9) we consider a local diffeomorphism

\[ g : M_i \times M_i \times \ldots \times M_i , \quad \to G / G_i \]  

(3.71)

where

\[ U = U_i \times U_i \times U_i , \quad i = 0, 1, 2, ..., k−1 \]  

(3.72)

i.e.

\[ \phi_i : G / G_i \to G_i / G_i \]  

(3.73)
in a unique way. This could give us a fibration of $G/G_0$ in $\prod_{\mathcal{I}} (G_{\mathcal{I}}/G_{\mathcal{I}}).$

For $g \in G_{\mathcal{I}}$, we simply define

$$\phi((gG_{\mathcal{I}})) = [gG_{\mathcal{I}}] \quad (3.74)$$

If $g \not\in G_{\mathcal{I}}$, we define

$$\phi((gG_{\mathcal{I}})) = [G_{\mathcal{I}}] \quad (3.75)$$

Thus in general

$$\phi((gG_{\mathcal{I}})) = [\rho(g)G_{\mathcal{I}}] \quad (3.76)$$

where

$$\rho(g) = g \cdot \begin{cases} 1 & g \in G_{\mathcal{I}} \\ 0 & g \not\in G_{\mathcal{I}} \end{cases} \quad (3.77)$$

Thus in a place of (3.9) we have to do with a structure

$$\{G/G_0 \prod_{\mathcal{I}} (G_{\mathcal{I}}/G_{\mathcal{I}}), \phi_1, \phi_2, \ldots, \phi_{\mathcal{I}}\} \quad (3.78)$$

such that

$$g_1 \cdot \phi_2 = \id \quad (3.79)$$

where

$$\phi_2 = \prod_{\mathcal{I}} \phi_{\mathcal{I}} \quad (3.80)$$

This generalizes (3.9) to the local conditions (3.71). Now we can repeat all the considerations concerning a decomposition of Higgs’ fields using local diffeomorphisms $g_1 (\phi_2)$ in the place of $g (\phi_2)$. Let us also notice that in the chain of groups it would be interesting to consider as

$$G = SU(2) \otimes SU(2) \otimes SU(4) \quad (3.81)$$

suggested by Salam and Pati, where $SU(4)$ unifies $SU(2)_{\text{ext}} \otimes U(1)_{\text{em}}$. This will be helpful in our future consideration concerning extension to super symmetric $U(2,2)$ which unifies $SU(2) \otimes SU(2)_k$ to the super Lie group $U(2,2)$ considered by Mohapatra. Such models on the phenomenological level incorporate fermions with a possible extension to the super symmetric $SO(10)$ model. They give a natural framework for lepton flavour mixing going to the neutrino oscillations incorporating see-saw mechanism for mass generations of neutrinos. In such approaches the see-saw mechanism is coming from the grand unified models. Our approach after incorporating manifolds with anticommuting parameters, super Lie groups, super Lie algebras and in general supermanifolds (superfibrebundles) can be able to obtain this. However, it is necessary to develop a formalism (in the language of supermanifolds, superfibrebundles, super Lie groups, super Lie algebras) for non symmetric connections, non symmetric Kaluza–Klein (Jordan–Thiry) theory. In particular we should construct an analogue of Einstein–Kaufmann connection for supermanifold, a non symmetric Kaluza–Klein (Jordan–Thiry) theory for superfibrebundle with super Lie group. In this way we should define first of all a non symmetric $c$-tensor on a super Lie group and afterwards a non symmetric $c$-metricization of a superfibrebundle. Let us notice that on every stage of symmetry breaking, i.e. from $G_0$ to $G_1$, we have to do with group $G_1$ (similar to the group $G_0$). Thus we can have to do with a true and a false vacuum cases which may complicate a pattern of a symmetry breaking.

References


