Abstract

It is proved that the Hilbert class field of a real quadratic field $(\mathcal{Q}(f))$ modulo a power $m$ of the conductor $f$ is generated by the Fourier coefficients of the Hecke eigenform for a congruence subgroup of level $f$.

Keywords: Class field; Real multiplication

Introduction

The Kronecker’s *Jugendtraum* is a conjecture that the maximal unramified abelian extension (The Hilbert class field) of any algebraic number field is generated by the special values of modular functions attached to an abelian variety. The conjecture is true for the rational field and imaginary quadratic fields with the modular functions being an exponent and the $j$-invariant, respectively. In the case of an arbitrary number field, a description of the abelian extensions is given by class field theory, but an explicit formula for the generators of these abelian extensions, in the sense sought by Kronecker, is unknown even for the real quadratic fields.

The problem was first studied by Hecke [1]. A description of abelian extensions of real quadratic number fields in terms of coordinates of points of finite order on abelian varieties associated with certain modular curves was obtained in studies of Shimura [2]. Stark formulated a number of conjectures on abelian extension of arbitrary number fields, which in the real quadratic case amount to specifying generators of these extensions using special values of Artin $L$-functions [3]. Based on an analogy with complex multiplication, Manin suggested to use the so-called “pseudo-lattices” $\mathbb{Z} + \mathbb{Z} \theta$ in $\mathbb{R}$ having non-trivial real multiplications to produce abelian extensions of real quadratic fields [4]. Similar to the case of complex multiplication, the endomorphism ring $\mathcal{O}_k = \mathbb{Z} + \mathbb{Z} \theta$ of pseudo-lattice $\mathbb{Z} + \mathbb{Z} \theta$ is an order in the real quadratic field $\mathbb{Q}(\theta)$, where $\mathcal{O}_k$ is the ring of integers of $\mathbb{Q}$ and $\theta$ is the conductor of $\mathcal{O}_k$. Manin calls these pseudo-lattices with real multiplication.

The aim of our note is a formula for generators of the Hilbert class field of real quadratic fields based on a modularity and a symmetry of complex and real multiplication. To give an idea, let

$$\Gamma_0(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{SL}_2(\mathbb{Z}) | a \equiv d \equiv 1 \mod N, c \equiv 0 \mod N \right\}$$

be a congruence subgroup of level $N \geq 1$ and $\mathbb{H}$ be the Lobachevsky half-plane; let $\text{SL}_2(\mathbb{Z}) = \mathbb{H} / \Gamma_0(N)$ be the corresponding modular curve and $S_2(\Gamma_0(N))$ the space of all cusp forms on $\Gamma_0(N)$ of weight 2. Let $\zeta_{n,\theta}(\mathbb{Z})$ be elliptic curve with complex multiplication by an order $\mathcal{O}_k = \mathbb{Z} + \mathbb{Z} \theta$ in the field $\mathbb{Q}(\sqrt{-D})$ [5]. Denote by $K_n(k) = k(j(\zeta_{n,\theta}(\mathbb{Z})))$ the Hilbert class field of $k$ modulo conductor $f \geq 1$ and let $N = fD$; then $\text{Jac}(X(f))$ is the Jacobian of modular curve $X(f)$. There exists an abelian subvariety $A_j \subset \text{Jac}(X(f))$, such that its points of finite order generate $K_n(k)$, [6,7], Section 8. The $K_n(k)$ is a CM-field, i.e., a totally imaginary quadratic extension of the totally real field $\mathbb{K}_n$ generated by the Fourier coefficients of the Hecke eigenform $\phi(z) \in S_2(\Gamma_0(f))$ [2]. In particular, there exists a holomorphic map $X_n(f) \to \mathbb{K}_n$, where $X_n(f)$ is a Riemann surface such that $\text{Jac}(X_n(f)) \cong A_j$; we refer to the above as a modularity of complex multiplication.

Recall that (twisted homogeneous) coordinate ring of an elliptic curve $\mathbb{C}$ is isomorphic to a Sklyanin algebra, [8]; the norm-closure of a self-adjoint representation of the Sklyanin algebra by the linear operators on a Hilbert space $\mathcal{H}$ is isomorphic to a noncommutative torus $\mathcal{A}_\phi$ [9] for the definition.

Whenever elliptic curve $\mathcal{E}(\mathbb{C}) \cong \mathbb{C}^2$ has complex multiplication, the noncommutative torus $\mathcal{A}_\phi$ has real multiplication by an order $\mathcal{O}_k = \mathbb{Z} + \mathbb{Z} \theta$ in the field $\mathbb{R} \supset \mathbb{Q}(\sqrt{-D})$; moreover, it is known that $f = m^2$ for the minimal power $m$ satisfying an isomorphism:

$$\text{Cl}(\mathcal{O}_k) \cong \mathcal{A}_\phi(\mathbb{R})$$

where $\text{Cl}(\mathcal{O})$ and $\text{Cl}(\mathcal{O}_k)$ are the ideal class groups of orders $\mathcal{O}$ and $\mathcal{O}_k$, respectively. We shall refer to (2) as a symmetry of complex and real multiplication. The noncommutative torus with real multiplication by $\mathcal{O}_k$ will be denoted by $\mathcal{A}_\phi(\mathbb{R})$.

Remark 1: The isomorphism (2) can be calculated using the well-known formula for the class number of a non-maximal order $\mathcal{O} = \mathbb{Z} + \mathbb{Z} \theta$ of a quadratic field $\mathbb{Q}(\sqrt{-D})$.

$$h_{\mathcal{O}_k} = \frac{h_{\mathcal{O}_k}}{\epsilon} \prod_{p} \left( 1 - \frac{1}{p} \right)$$

where $h_{\mathcal{O}_k}$ is the class number of the maximal order $\mathcal{O}_k$, $\epsilon$ is the index of the group of units of $\mathbb{Z} + \mathbb{Z} \theta$ in the group of units of $\mathcal{O}_k$, $p$ is a prime number and $\frac{h_{\mathcal{O}_k}}{\epsilon} \prod_{p}$ is the Legendre symbol [10,11].

The (twisted homogeneous) coordinate ring of the Riemann surface $X(\mathbb{Z})$ is an AF-algebra $\mathbb{C}(\mathbb{Z})$ linked to a holomorphic differential $\phi(z) dz$ on $X(\mathbb{Z})$, see Section 2.2, Definition 1 and Remark 5 for the details; the Grothendieck semigroup $K_n(\mathbb{Z})$ is a pseudo-lattice $\mathbb{Z} + \mathbb{Z} \theta_1 + \ldots + \mathbb{Z} \theta_n$ in the number field $\mathbb{K}_n$, where $n$ equals the genus of $X_n$. Moreover, a holomorphic map $X(\mathbb{Z}) \to \mathbb{K}_n$ induces the $\mathbb{C}$-algebra homomorphism $\mathcal{A}_\phi(\mathbb{R}) \to \mathbb{K}_n$ between the corresponding coordinate rings, so that the following diagram commutes:

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The Sklyanin algebra $S_{\theta}(C)$ is a free C-algebra on four generators and six relations:

$$
\begin{align*}
\theta x_1 - x_1 & = \alpha (x_1 x_3 + x_4 x_1), \\
\theta x_2 + x_1 & = x_1 x_3 - x_1, \\
\theta x_2 - x_1 & = \beta (x_1 x_3 + x_4 x_1), \\
x_1 x_3 + x_4 x_1 & = x_2 x_3 - x_1, \\
x_1 x_3 - x_1 & = \gamma (x_1 x_3 + x_4 x_1), \\
x_1 x_3 + x_4 x_1 & = x_1 x_3 - x_1,
\end{align*}
$$

where $\alpha + \beta + \gamma = 0$; such an algebra corresponds to a twisted homogeneous coordinate ring of an elliptic curve in the complex projective space $\mathbb{C}P^2$ given by the intersection of two quadratic surfaces of the form $E_{\theta}(C) = \{ (u,v,w,z) \in \mathbb{C}P^2 | u^2 + v^2 + w^2 + z^2 - 1 = 0 \}$.

Being such a ring means that the algebra $S_{\theta,\nu}$ satisfies an isomorphism

$$
\text{Mod}(S_{\theta,\nu}(C)) \cong \text{Tors} \text{ Coh}(S_{\theta,\nu}(C)),
$$

where $\text{Coh}$ is the category of quasi-coherent sheaves on $E_{\theta}(C)$, $\text{Mod}$ the category of graded left modules over the graded ring $S_{\theta,\nu}(C)$ and $\text{Tors}$ the full sub-category of $\text{Mod}$ consisting of the torsion modules, [8].

If one sets $x_i = u, x_i = u, x_i = v, x_i = v$, there exists a self-adjoint representation of the Sklyanin $*$-algebra $S_{\theta,\nu}(C)$ by linear operators on a Hilbert space $\mathcal{H}$, such that its norm-closure is isomorphic to $A_\nu$, namely, $A_\nu \cong S_{\theta,\nu}(C) / I_\nu$, where $A_\nu$ is a dense sub-algebra of $A_\nu$ and $I_\nu$ is an ideal generated by the "scaled unit" relations $x_i x_i = x_i x_i = 1$, where $\nu > 0$ is a constant. Thus the algebra $A_\nu$ is a coordinate ring of elliptic curve $E(\mathcal{C})$, such that isomorphic elliptic curves correspond to the stably isomorphic (Morita equivalent) noncommutative tori; this fact explains the modular transformation law in (4). In particular, if $\{ C \}$ is complex multiplication by an order $\Omega = \mathbb{Z} + i \mathbb{Q}$, in a quadratic field $t = \sqrt{-d}$, then $A_\nu$ has real multiplication by an order $\mathcal{O} = \mathbb{Z} + i \mathbb{Q}$ in the quadratic field $t = \sqrt{-d}$, where $\mathcal{O}$ is the smallest integer satisfying an isomorphism $\mathbb{Z}[r] \cong \mathbb{Z}[r]$, [16]; the isomorphism is a necessary and sufficient condition for $A_\nu$ to have the same endomorphism ring $\mathcal{O}$. For the constraint $|f| = n$, see remark 6.

**AF-algebra of the Hecke eigenform**

An AF-algebra (Approximately Finite C*-algebras) is defined to be the norm closure of an ascending sequence of finite dimensional C*-algebras $M_0$, where $M_0$ is the C*-algebra of the $n \times n$ matrices with entries in $\mathbb{C}$. Here the index $n = (n_1, \ldots, n_k)$ represents the semi-simple matrix algebra $M_n = M_{n_1} \otimes \cdots \otimes M_{n_k}$. The ascending sequence mentioned above can be written as $M_0 = \mathbb{C} \rightarrow M_1 \rightarrow \cdots$, where $M_0$ are the finite dimensional C*-algebras and $\phi : M_0 \rightarrow M_0$ the homomorphisms. One has two sets of vertices $V_1, \ldots, V_1$ and $V_2, \ldots, V_2$, joined by $b_1$ edges whenever the summand $M_i$ contains $b_1$ copies of the summand $M_i$ under the embedding $\phi$. As $i$ varies, one obtains an infinite graph called the Bratteli diagram of the AF-algebra. The matrix $B = (b_1)$ is known as a partial multiplicity matrix; an infinite sequence of $B$ defines a unique AF-algebra. An AF-algebra is called stationary if $B = \text{const} = B$, [14], when two non-similar matrices $B$ and $B'$ have the same characteristic polynomial, the corresponding stationary AF-algebras will be called companion AF-algebras.
Let $N \geq 1$ be a natural number and consider a (finite index) subgroup of the modular group given by the formula:

$$\Gamma(N) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z}) \mid a = d = 1 \mod N, c = 0 \mod N \right\}. \quad (7)$$

Let $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ be the upper half-plane and let $\Gamma(N)$ act on $\mathbb{H}$ by the linear fractional transformations; consider an orbifold $\mathbb{H}/\Gamma(N)$. To compactify the orbifold at the cusps, one adds a boundary to $\mathbb{H}$, so that $\overline{\mathbb{H}} = \mathbb{H} \cup \{(x,0)\mid x \in \mathbb{R}\}$ and the compact Riemann surface $X(N) = \overline{\mathbb{H}}/\Gamma(N)$ is called a modular curve. The meromorphic functions $\{z\}$ on $\mathbb{H}$ that vanish at the cusps and such that

$$\phi \left( \frac{az + b}{cz + d} \right) = (cz + d)^k \phi(z), \forall \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(N), \quad (8)$$

are called cusp forms of weight two; the (complex linear) space of such forms will be denoted by $S_2(\Gamma(N))$. The formula $\phi(z) \mapsto \phi(z)dz$ defines an isomorphism $S_2(\Gamma(N)) \cong \Omega_{\text{even}}(X(N))$, where $\Omega_{\text{even}}(X(N))$ is the space of all holomorphic differentials on the Riemann surface $X(N)$. Note that $\dim(S_2(\Gamma(N))) = \dim(\Omega_{\text{even}}(X(N))) = g$, where $g = \text{genus}(N)$ is the genus of the surface $X(N)$. A Hecke operator, $T_\nu$, acts on $S_2(\Gamma(N))$ by the formula $T_\nu \phi = \sum_{\gamma \in \Gamma(N)} \lambda(\gamma) \phi(\gamma z)$, where $\lambda(\gamma) = \sum_{\gamma(z) = z} \gamma \in \Gamma(N)$ and $\phi(z) \mapsto \phi(z + q)$. The $T_\nu$ are the Fourier series of the cusp form $\phi$ at $q = e^{2\pi i}$. Further, $T_\nu$ is a self-adjoint linear operator on the vector space $S_2(\Gamma(N))$ endowed with the Petersson inner product; the algebra $\mathcal{A}_1 = \mathbb{Z}[T_0, T_1, \ldots]$ is a commutative algebra. Any cusp form $\phi \in S_2(\Gamma(N))$ that is an eigenvector for one (and hence all) of $T_\nu$ is referred to as a Hecke eigenform. The Fourier coefficients $c(n)$ of $\phi$ are algebraic integers, and we denote by $K_\phi = \mathbb{Q}(c(n))$ an extension of $\mathbb{Q}$ by the coefficients of $\phi$. Then $K_\phi$ is a real algebraic number field of degree $1 \leq \text{deg}(K_\phi) \leq g$, where $g$ is the genus of the surface $X(N)$.

It is known that $\text{deg}(K_\phi) = 2g - 2$, where $g$ is the genus of the Riemann surface $X(N)$. The Jacobian $J(X(N))$ is the Jacobian that coincides with $A_0$ and is defined over $\mathbb{Q}$.

**Proof of Theorem 1**

**Definition 1.** Let $A \subseteq \text{Jac}(X(N))$ be an abelian variety associated to the Hecke eigenform $\phi(z) \in S_2(\Gamma(N))$. [Definition 6.6.3. By $X^1(N)$ we shall understand the Riemann surface of genus $g$, such that $\text{Jac}(X^1(N)) \cong A$.]

By $\phi(z)dz \in \Omega_{\text{even}}(X^1(N))$ we denote the image of the Hecke eigenform $\phi(z)dz \in \Omega_{\text{even}}(X^1(N))$ under the holomorphic map $X(N) \to X^1(N)$.

**Remark 3.** The surface $X^1(N)$ is correctly defined. Indeed, since the abelian variety $A_0$ is the product of $g$ copies of an elliptic curve with the complex multiplication, there exists a holomorphic map from $A_0$ to the elliptic curve. For a Riemann surface $X$ of genus $g$ covering the elliptic curve $E_{CM}$ by a holomorphic map (such a surface and a map always exist), one gets a period map $X \to E_{CM}$ by closing the arrows of a commutative diagram $A_0 \to E_{CM} \to X$. It is easy to see that the Jacobian of $X$ coincides with $A_0$ and we set $X^1(N) = X_0$.

**Lemma 1.** $g(X^1(N)) = \text{deg}(K_\phi)$. [Proof. By definition, abelian variety $A_0$ is the quotient of $C^g$ by a lattice of periods of the Hecke eigenform $\phi(z) \in S_2(\Gamma(N))$ and all its conjugates $\phi'(z)$ on the Riemann surface $X^1(N)$. These periods are complex algebraic numbers generating the Hilbert class field $K^0$ over imaginary quadratic field $k = \sqrt{-D}$ modulo conductor $f = 2, 6, 7$, Section 8. The number of linearly independent periods equals the total number of the conjugate eigenforms $\phi'(z)$, i.e., $|\sigma| = n = \dim(A_0)$. Since real dimension $\dim(A_0) = 2n$, we conclude that $\text{deg}(K^0) = 2n$ and, therefore, $\text{deg}(K_\phi) = n$. But $\text{dim}(A_0) = 2g - 2$ and one gets $g(X^1(N)) = (\text{deg}(K_\phi) + 1)$. Lemma 1 follows.]

**Corollary 1.** $g(X^1(N)) = |\text{Cl}(R)|$. [Proof. Because $K_\phi$ is the Hilbert class field over $k$ modulo conductor $f$, we must have $\text{Gal}(K^0/k) \cong \text{Cl}(R)$, (11) where $\text{Gal}(K^0/k)$ is the Galois group of the extension $K^0/k$ and $\text{Cl}(R)$ is the class group of ring $R_f$. But $|\text{Gal}(K^0/k)| = \text{deg}(K^0/k)$ and by lemma 1 we have $\text{deg}(K_\phi/k) = g(X^1(N))$. In view of this and isomorphism (11), one gets $|\text{Cl}(R)| = |\text{Gal}(K_\phi/k)| = g(X^1(N))$. Corollary 1 follows.]

**Lemma 2.** $g(f_\phi(N)) = \text{deg}(K_\phi).$ [Proof. It is known that $\dim(A_0) = \dim(K_\phi)$. [15], Proposition 6.6.4. But abelian variety $A_0 \subseteq \text{Jac}(X^1(N))$ and, therefore, $\dim(A_0) = \dim(\text{Jac}(X^1(N))) = g(X^1(N))$], hence the lemma.]

**Corollary 2.** $\text{deg}(K_\phi) = |\text{Cl}(R)|$. [Proof. From lemma 2 and corollary 1 one gets $\text{deg}(K_\phi) = |\text{Cl}(R)|$. In view of this and equality (2), one gets the conclusion of corollary 2.]

**Lemma 3.** [Basic lemma] $\text{Gal}(K_\phi/k) \cong \text{Cl}(R_f)$. [Proof. Let us outline the proof. In view of lemma 2 and corollaries 1-2, we denote by $h$ the single integer $g(f_\phi(N)) = |\text{Cl}(R_f)| = |\text{Cl}(R)| = \text{deg}(K_\phi).$ Since $\text{deg}(K_\phi) = h$, there exist $(\phi_0, \ldots, \phi_h)$ conjugate Hecke eigenforms $\phi(z) \in S_2(\Gamma(N))$. [15], Theorem 6.5.4; thus one gets $h$ holomorphic forms $\phi_0, \ldots, \phi_h$ on the Riemann surface $X^1(N)$. Let $A_0$, $A_1$, $\ldots$, $A_h$ be the corresponding stationary AF-algebras; the $A_0$ are companions AF-algebras, see Section 1.2. Recall that the characteristic polynomial for the partial multiplicity matrices $B_0$ is $B_h$ of companions AF-algebras $A_0$. The $A_0$ is the simplest; it is a minimal polynomial of degree $h$ and let $\lambda_i, \ldots, \lambda_h$ be the roots of such a polynomial, compare with studies of Effros [14], Corollary 6.3. Since $|\text{det}(R_f) - 1|$, the numbers $\lambda_i$ are algebraic units of the field $K_\phi$. Moreover, $\lambda_i$ are algebraically conjugate and can be taken for generators of the extension $K_\phi/k$; since $\text{deg}(K_\phi/k) = h = |\text{Cl}(R_f)|$ there exists a natural action of group $\text{Cl}(R_f)$ on these generators. The...
In the case of elliptic curves with complex multiplication, the number of points on the curve over a finite field $F_q$ is equal to $q + 1 - a_q$, where $a_q$ is the $q$-th Fourier coefficient of the modular form associated to the curve.

Remark 4. Any isomorphism $\phi$ between two number fields $K_1$ and $K_2$ is induced by an automorphism of the ring of integers $\mathcal{O}_K$. In the particular case of $p$-adic fields, there are only finitely many possibilities up to $\mathbb{C}$-isomorphism. This is a consequence of the theory of Lubin-Tate formal groups.

Remark 5. A Hecke algebra is a ring of operators on the space of modular forms. The operators are defined as integrals against a fixed test function. A Hecke operator $T_n$ is defined as $T_n(f) = \sum_{m|n} \mu(n/m) f(mn)$. The operators satisfy the Hecke commutation relations $T_m T_n = T_{mn}$ for $(m,n) = 1$.

Examples

Example 1. Let $K = \mathbb{Q}(\sqrt{15})$. The class field theory says that $K$ has a subfield $L = \mathbb{Q}(\sqrt{-5})$. The Galois group of the extension $K/L$ is isomorphic to the group of units of the ring of integers of $K$. The unit group of $K$ is generated by a single element, the norm of which is $2$.

Example 2. Let $K = \mathbb{Q}(\sqrt{-2})$. The class field theory says that $K$ has a subfield $L = K(\sqrt{5})$. The Galois group of the extension $K/L$ is isomorphic to the group of units of the ring of integers of $K$. The unit group of $K$ is generated by a single element, the norm of which is $2$. The extension $K/L$ is an example of a non-abelian extension.
and one finds a generator of \( K(\sqrt{D}) \) modulo conductor \( f \) is the Hilbert class over \( \mathbb{Q}(\sqrt{D}) \).

Thus the field \( \mathbb{Q}(\sqrt{19}) \) is the Hilbert class field of \( \mathbb{Q}(\sqrt{432}) \).


table 1:

<table>
<thead>
<tr>
<th>D</th>
<th>f</th>
<th>( CI(\mathbb{Q}(\sqrt{D})) )</th>
<th>( \mathbb{Q}(\sqrt{D}) ) modulo conductor</th>
<th>f</th>
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<tr>
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<td>( \mathbb{Q}(\sqrt{3}) )</td>
<td>1</td>
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<td>( \mathbb{Q}(\sqrt{7}) )</td>
<td>1</td>
</tr>
<tr>
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<td>( \mathbb{Q}(\sqrt{11}) )</td>
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<td>( \mathbb{Q}(\sqrt{-2+8\sqrt{14}}) )</td>
<td>8</td>
</tr>
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<td>( \mathbb{Q}(\sqrt{-1+\sqrt{15}}) )</td>
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<tr>
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<td>( \mathbb{Q}(\sqrt{7+\sqrt{35}}) )</td>
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</table>

Table 1: Square-free discriminants \( 2 \leq D \leq 101 \).

\[ 3x^2 + 9y^2 - 4 \times 14 = 0. \] The bi-quadratic equation \( x^2 + 3x^2 + 9 - 2\sqrt{14} = 0 \) has discriminant \( -27 + 8\sqrt{14} \) and one finds a generator of \( K_2 \) to be \( \sqrt{-27 + 8\sqrt{14}} \). Thus the field \( \mathbb{Q}(\sqrt{-27 + 8\sqrt{14}}) \) is the Hilbert class over \( \mathbb{Q}(\sqrt{14}) \).

Remark 7. Table 1 above lists quadratic fields for some square-free discriminants \( 2 \leq D \leq 101 \). The conductors \( f \) and \( f \) satisfying equation (2) were calculated using tables for the class number of non-maximal orders in quadratic fields posted at www.numbertheory.org; the site is maintained by Keith Matthews. We focused on small conductors; the interested reader can compute the higher conductors using a pocket calculator. In contrast, computation of generator \( \alpha \) of the Hilbert class field require the online program SAGE created by William A. Stein. We write an explicit formula for \( x \) or its minimal polynomial \( p(x) \) over \( f \).

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References


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