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Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

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Abstract

In this paper, we prove that equation $E \equiv u_t - u_{x^2t} + u_x f(u) - au_x u_{x^2} - buu_{x^3} = 0$ is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where *f* is smooth function on *u* and *a*, *b* are arbitrary constans. We find Three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitham equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

$$E \equiv u_t - u_{x^2t} + u_x f - au_x u_{x^2} - buu_{x^3} = 0, \tag{1}$$

and we show that this equation is self-adjoint and quasi self-adjont. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) $b \neq 0$, a = arbitrary constant, 2) $b = 0$, $a \neq 0$, and 3) $b = 0$, $a = 0$. Clarkson, Mansfield and Priestly [3] are concerned with symmetry reductions of the non-linear third order partial differential equation given by $u_t - \in u_{x^2t} + (k-u)u_x - uu_{x^3} - \beta u_x u_{x^2} = 0$, where $\in$, k, and β are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of (1) are:

Camassa-Holm (CH) equation $u_t - u_{x^2t} + (k+3u)u_x = uu_{x^3} + 2u_x u_{x^2}$, k-arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let $f = k + 3u$, $a = 2$, $b = 1$ in (1).

Degas peris-Procesi (DP) equation $u_t - u_{x^2t} + (k+4u)u_x = uu_{x^3} + 3u_x u_{x^2}$, k-arbitrary (real), is another equation of this class (let $f = k + 4u$, $a = 3$, $b = 1$ in (1).

Fornberg Whitham (FW) equation $u_t - u_{x^2t} + (1+u)u_x = uu_{x^3} + 3u_x u_{x^2}$, is another equation of this class (let $f = 1 + u$, $a = 3$, $b = 1$ in (1)).

BBM equation $u_t - u_{x^2t} + u_x + (uu_x) = 0$, is another equation of this class (let $f = 1 + u$, $a = 0$, $b = 0$ in (1)).

Preliminaries

In this section, we recall the procedure in literature of Ibragimov [1]. Let us introduce the formal Lagrangian

$$L \equiv vE, \tag{2}$$

where $v = v(t, x)$ is a new dependent variable.

We define the adjoint equation by $E^* \equiv \frac{\delta L}{\delta u} = 0$. Here

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - D_i D_j D_k \frac{\partial}{\partial u_{ijk}} + \cdots \qquad i,j,k = 1,2,$$

is the variational derivative and D_i is the operator of total diferentiation.

An equation $E = 0$ is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution $v = u$ is identical with the original equation.

An equation $E = 0$ is said to be quasi- self-adjoint [5] if there exists a function $v = \varphi(u)$, $\varphi'(u) \neq 0$ such that $E^*|_{v=\varphi(u)} = \lambda E$ with an undetermined coefficient λ. Eq.(1) is said to have a nonlocal conservation law if there exits a vector $C = (C^1, C^2)$ satisfying the equation

$$D_t(C^1) + D_x(C^2) = 0, \tag{3}$$

on any solution of the system of differential equations comprising (E) and the adjoint equation (E^*). We say that orginal equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry $v = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \phi(x,t,u)\partial_u$ is obtained by the following formula :

$$C^i = \xi^i L + W[\frac{\partial L}{\partial u_i} - D_j(\frac{\partial L}{\partial u_{ij}}) + D_j D_k(\frac{\partial L}{\partial u_{ijk}})]$$
$$+ D_j(W)[\frac{\partial L}{\partial u_{ij}} - D_k(\frac{\partial L}{\partial u_{ijk}})] + D_j D_k(W)\frac{\partial L}{\partial u_{ijk}}, \tag{4}$$

where $i, j, k = 1,2$ and $W = \phi - \xi^i u_i$. (Here ∂_x means $\frac{\partial}{\partial x}$).

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order n, containing p independent and q dependent variables is given as follows

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1,\cdots,l, \tag{5}$$

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involving $x=(x^1,\cdots,x^p)$, $u=(u^1,\cdots,u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of transformations acting on the variables of system (5): $\overline{x}_i = x^i + \in \xi^i(x,u)+O(\in^2)$, $\overline{u}_j = u^j + \in \phi^j(x,u)+O(\in^2)$, where i=1,⋯, p, j = 1,⋯, q. ξ^i, φ^j are the infinitesimal of the transformations for the independent and dependent variables, respectively, and $\in$ is the transformation parameter. We consider the general vector field v as the infinitesimal generator associated with the above group $v=\sum_{i=1}^p \xi^i(x,u)\partial_{x^i}+\sum_{j=1}^q \phi^j(x,u)\partial_{u^j}$. A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (5) under the infinitesimal transformation leads to the invariance conditions. (Theorem 2.36 of studies of Olver [6], Theorem 6.5 of literature of Olver [7]).

$$v^n[\Delta_v(x,u^n)]=0,\quad \Delta_v(x,u^n)=0,\quad v=1,\cdots,r, \tag{6}$$

where v^n is called the n^{th} order prolongation of the infinitesimal generator given by $v^n = v+\sum_{j=1}^q\sum_k \phi_k^j(x,u^{(n)})\partial_{u_k^j}$, where $k=(i_1,\cdots i_\alpha)$, $1\le i_\alpha \le p$, $1\le \alpha\le n$, and the sum is over all k's of order $0<\#k\le n$. If $\#k=\alpha$, the coeficent ϕ_k^j of $\partial_{u_k^j}$, will depend only on α'th and lower order derivatives of u and $\phi_j^k(x,u^n)=D_k(\phi_j-\sum_{i=1}^p(\xi^i u_i^j))+\sum_{i=1}^p \xi^i u_{k,i}^j$, where $u_i^j := \partial u^j/\partial x^i$ and $u_{k,i}^j := \partial u_k^j/\partial x^i$.

Adjoint Equation and Classical Symmetry Method

Formal Lagrangian for Eq. (1) is

$$L = vE = v[u_t - u_{x^2t}+u_x f - a u_x u_{x^2} - b u u_{x^3}]. \tag{7}$$

Therefore, the adjoint equation E^* to Eq. (1) is

$$f v_x + v_t + a u_{x^2} v_x + a u_x v_{xx} = 3b u_{x^2} v_x + 3b u_x v_{xx} + b u v_{xxx} + v_{xxt}. \tag{8}$$

Upon setting $v = u$ it becomes

$$u_t = u_{x^2t} - u_x f - 2a u_x u_{x^2} + 6b u_x u_{x^2} + b u u_{x^3}.$$

Hence, Eq. (1) is self-adjoint if and only if it has the form

$$a = 2b. \tag{9}$$

Consider again Eq. (1), and substitute

$$v=\varphi(u),\qquad v_t=\varphi' u_t,$$

$$v_x=\varphi' u_x,\qquad v_{xx}=\varphi' u_{x^2}+\varphi'' u_x^2,$$

$$v_{xxx}=\varphi' u_{x^3}+3\varphi'' u_x u_{x^2}+\varphi''' u_x^3,$$

$$v_{xxt}=\varphi' u_{x^2t}+\varphi'' u_t u_{x^2}+\varphi''' u_x^2 u_t+2\varphi'' u_x u_{xt},$$

in the adjoint equation (8), then

$$-f\varphi' u_x-\varphi' u_t-2a\varphi' u_x u_{x^2}-a\varphi'' u_x^3+6b\varphi' u_x u_{x^2}$$

$$+3b\varphi'' u_x^3+b\varphi' u u_{x^3}+3b\varphi'' u u_x u_{x^2}+b\varphi''' u u_x^3$$

$$+\varphi' u_{x^2t}+\varphi'' u_{x^2} u_t+\varphi''' u_x^2 u_t+2\varphi'' u_x u_{xt}$$

$$=\lambda(u_t-u_{x^2t}+u_x f-a u_x u_{x^2}-b u u_{x^3}).$$

Hence, Eq. (1) is quasi self-adjoint if and only if it has the form

$$a=2b,\ v=-\lambda u+\varepsilon \tag{10}$$

In this section, we will perfom Lie group method for Eq. (1) on $(x^1 = x,\ x^2 = t,\ u^1 = u)$, $(\tilde{x},\tilde{t},\tilde{u})=(x,t,u)+\in(\xi(x,t,u),\tau(x,t,u),\phi(x,t,u))+O(\in^2)$, where $\varepsilon \le 1$ the group parameter and $\xi^1=\xi$, $\xi^2=\tau$ and $\phi^1=\phi$ are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form $v=\xi(x,t,u)\partial_x+\tau(x,t,u)\partial_t+\phi(x,t,u)\partial_u$ and the third porolongation of v is the vector field

$$v^{(3)}=v+\phi^x\partial_{u_x}+\phi^t\partial_{u_t}+\phi^{x^2}\partial_{u_{x^2}}+\phi^{xt}\partial u_{xt}+\cdots+\phi^{ttt}\partial u_{ttt},$$

with coefficent

$$\phi^k = D_k(\phi-\sum_{i=1}^2 \xi^i u_i^j)+\sum_{i=1}^2 \xi^i u_{k,i}, \tag{11}$$

where D_k is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

$$v^{(3)}[u_t-u_{x^2t}+u_x f-a u_x u_{x^2}-a u u_{x^3}]=0, \tag{12}$$

whenever $E = 0$. The condition (12) is equvalent to

$$\phi^t - b u_{x^3}\phi+(f-a u_{x^2})\phi^x - a u_x \phi^{x^2} - b u \phi^{x^3} - \phi^{x^2t}=0, \tag{13}$$

whenever $E = 0$. Substituting (11) into (13), yields the determining equations. There are three cases to consider:

a and *b* ≠ 0 are arbitrary constants

In this case, complete set of determining equation is:

$$\xi_u=0, \tag{14}$$

$$\tau_u=0, \tag{15}$$

$$\tau_x=0, \tag{16}$$

$$\phi_{u^2}=0, \tag{17}$$

$$a\phi_{ux^2}+a\phi_u+a\tau_t-3a\xi_x=0, \tag{18}$$

$$\xi_{x^2}-2\phi_{ux}=0, \tag{19}$$

$$3bu\phi_{ux}+a\phi_x-3b\xi_{x^2}u+2\xi_{xt}+\phi_{ut}=0, \tag{20}$$

$$2\xi_x-\phi_{ux^2}=0, \tag{21}$$

$$\xi_t-b\phi_x-\phi_{ux^2}-bu\tau_t+3bu\xi_x=0, \tag{22}$$

$$a\xi_{x^2}u-2a\phi_{ux}=0, \tag{23}$$

$$\xi_{x^2t}+bu\xi_{x^3}+\phi f_u+\phi_{ux^2}f+f\xi_x+\tau_t f=a\phi_{x^2}+\xi_t+2\phi_{xtu}+3bu\phi_{ux^2}, \tag{24}$$

$$-bu\phi_{x^3}+\phi_t+\phi_x f-\phi_{x^2t}=0. \tag{25}$$

With substituting (14) – (17) into (18) – (23) we have

$$\phi=c_1+\frac{1}{b}\alpha'(t),\quad \tau=-c_1t+c_2,\quad \xi=\alpha(t). \tag{26}$$

With substituting (26) into (24) – (25) we have

$$f=-1+K(bu+1), \tag{27}$$

where c_1, c_2 and K are arbitrary constants. With substituting (27) into determining system, we have

$$\phi=\frac{-c_1(bu+1)}{b},\quad \tau=c_1t+c_2,\quad \xi=-c_1t+c_3,$$

where c_i, i = 1,2,3 are arbitrary constants.

Theorem 3.1.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1=-t\partial_x+t\partial_t-\frac{(bu+1)}{b}\partial_u,\quad v_2=\partial_t,\quad v_3=\partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1=-t\partial_x+t\partial_t-\frac{(bu+1)}{b}\partial_u.$$

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We have

$$W = -u - \frac{1}{b} - tu_t + tu_x. \tag{28}$$

The right-hand side of (4) is written

$$\begin{aligned}
C^1 &= W(v - D_x^2(v)) + (D_x W)(D_x v) - D_x^2(W)v,\\
C^2 &= W[vf - avu_{x^2} + D_x(avu_x) - D_x^2(buv) - 2D_t D_x(v)]\\
&+ D_x(W)[-avu_x + D_x(buv) + D_t(v)] + D_t(W)[D_x(v)]\\
&-2D_x D_t(W)[v] - D_x^2(W)[buv].
\end{aligned} \tag{29}$$

We eliminate the term $\xi^i L$ since the Lagrangian L is equal to zero on solution of Eq.(1). Substituting in (29), the expression (7) for L and (28) for W, we obtain

$$\begin{aligned}
C^1 &= -uv - \frac{1}{b}v - tu_t v + tu_x v + uv_{xx} + \frac{1}{b}v_{xx} + tu_t v_{xx} - tu_x v_{xx}\\
&-u_x v_x - tu_{xt}v_x + tv_x u_{x^2} + u_{x^2}v + tu_{x^2 t}v - tu_{x^3}v,
\end{aligned} \tag{30}$$

and

$$\begin{aligned}
C^2 &= -u(vf - bvu_{x^2} - buv_{xx} - 2v_{xt})\\
&-(b^{-1}(vf - bvu_{x^2} - buv_{xx} - 2v_{xt}))\\
&-tu_t(vf - bvu_{x^2} - buv_{xx} - 2v_{xt})\\
&+tu_x(vf - bvu_{x^2} - buv_{xx} - 2v_{xt}) - u_x(buv_x - bvu_x + v_t)\\
&-(tu_{xt})(buv_x - bvu_x + v_t) + (tu_{x^2})(buv_x - bvu_x + v_t) - 2u_t v_x\\
&-tu_{t^2}v_x + u_x v_x + tu_{xt}v_x + buvu_{x^2} + tbuvu_{x^2 t} - tbuvu_{x^3}\\
&+4u_{xt}v + 2tu_{xt^2}v - 2u_{x^2}v - 2u_{x^2 t}v.
\end{aligned} \tag{31}$$

We can eliminate u_t by using Eq.(1) and then substitute in (30) and (31) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$\begin{aligned}
C^1 &= \frac{-1}{b}(t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2 t})ub\\
&-t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2 t})u_{x^2}b - tu_x ub\\
&+tu_{xt}u_x b + tu_{x^3}ub - tu_{x^2 t}ub + u^2 b - 2uu_{x^2}b + u_x{}^2 b + u - u_{x^2}),
\end{aligned} \tag{32}$$

$$\begin{aligned}
C^2 &= -u(uf - 2buu_{xx} - 2u_{xt}) - (b^{-1}(uf - 2buu_{xx} - 2u_{xt}))\\
&-t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2 t})(uf - 2buu_{xx} - 2u_{xt})\\
&+tu_x(uf - 2buu_{xx} - 2u_{xt}) - u_x(u_t) - tu_{xt}(u_t)\\
&+(tu_{x^2})(u_t) - 2(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2 t})u_x\\
&-tu_{t^2}u_x + u_{xt} + tu_{xt}u_x + bu^2 u_{x^2} + tbu^2 u_{x^2 t} - tbu^2 u_{x^3} + 4u_{xt}u\\
&+2tu_{xt^2}u - 2uu_{x^2} - 2u_{x^2 t}u.
\end{aligned}$$

Where $f = -1 + K(bu + 1)$.

a is an arbitrary nonzero constant and *b* = 0.

In this case Eq.(1) is not self adjoint because $a \neq 2b$. Complete set of determining equation is:

$$\phi_{uu} = 0, \tag{33}$$

$$\xi_u = 0, \tag{34}$$

$$\xi_t = 0, \tag{35}$$

$$\tau_u = 0, \tag{36}$$

$$\tau_x = 0, \tag{37}$$

$$3a\xi_x = a\phi_{ux^2} + a\tau_t - a\phi_u, \tag{38}$$

$$\phi_{ut} + a\phi_x = 0, \tag{39}$$

$$-2\phi_{ux} + \xi_{x^2} = 0, \tag{40}$$

$$2\xi_x - \phi_{ux^2} = 0, \tag{41}$$

$$a\xi_{x^2} - 2a\phi_{ux} = 0, \tag{42}$$

$$\tau_t f + \phi f_u + \phi_{ux^2} f = 2\phi_{uxt} + f\xi_x + a\phi_{x^2}, \tag{43}$$

$$\phi_t + \phi_x f = \phi_{x^2 t}. \tag{44}$$

Now, by considering Eq. (33) – (42) it is not to hard to find that the components ξ, τ and φ of infinitesimal generators become

$$\phi = u\frac{dF_1(t)}{dt} - \frac{x}{a}\frac{d^2F_1(t)}{dt^2} + F_2(t), \quad \tau = -F_1(t) + c_2, \quad \xi = c_1. \tag{45}$$

To find complete solution of the above system, we consider Eq. (43) and $l = \dim \mathrm{Spam}_{\mathbb{R}}\{f_u, f, 1\}$. Three general cases are possible:

3.2.i) $l = 1$, then f = constant;

3.2.ii) $l = 2$, then $f_u = \alpha f + \beta$;

3.3.iii) $l = 3$, then $\alpha f_u + \beta f + \gamma \neq 0$, $\alpha \neq 0$.

Case 3.2.i). With substituting f = constant in determining system (33)-(44), we have $\varphi = c_1$, $\tau = c_2$, $\xi = c_3$, where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.2.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = \partial_u.$$

Case 3.2.ii). With integrating from $f_u = \alpha f + \beta$ with respect to u, we obtain

$$f = \frac{-\beta}{\alpha} + Ce^{\alpha u}, \tag{46}$$

where C is an integrating constant. With substituting (46) into Eq. (43)-(44) and Eq. (45), we have

$$\xi = c_1, \qquad \tau = -c_1 t, \qquad \phi = \frac{c_1(C\alpha - e^{-\alpha u}\beta)}{C\alpha^2}. \tag{47}$$

Theorem 3.2.2. *Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:*

$$v = \partial_x - t\partial_t + \frac{C\alpha - e^{-\alpha u}\beta}{C\alpha^2}\partial_u. \tag{48}$$

Case 3.2.iii). The Eq. (43) leads to $\varphi = 0$, $\tau = c_1$, $\xi = c_2$.

Theorem 3.2.3. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = \partial_t, \quad v_2 = \partial_x.$$

b = 0, *a* = 0.

Complete set of determining equation is

$$\xi_u = 0, \tag{49}$$

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$$\xi_t = 0, \tag{50}$$

$$\tau_u = 0, \tag{51}$$

$$\tau_x = 0, \tag{52}$$

$$\phi_{ut} = 0, \tag{53}$$

$$\phi_{uu} = 0, \tag{54}$$

$$\phi_{ux^2} = 2\xi_x, \tag{55}$$

$$2\phi_{ux} = \xi_{x^2}, \tag{56}$$

$$\phi_t + f\phi_x = \phi_{x^2t}, \tag{57}$$

$$f\tau_t + f\xi_x + \phi f_u = 0. \tag{58}$$

To find a complete solution of the above system we consider Eq. (58) and with assumption $f/f_u \neq 0$ we rewrite:

$$\phi = \frac{-f}{f_u}(\tau_t + \xi_x). \tag{59}$$

Two general cases are possible:

$3.3.i)\ \dfrac{f}{f_u} = c, \qquad 3.3.ii)\ \dfrac{f}{f_u} = h(u),$

where c is constant.

Case 3.3.i).

With integrating from $f/f_u \neq c$ with respect to u, we have

$$f = Le^{u/c}, \tag{60}$$

where L is an integrating constant. Then the Eq. (58) reduce to

$$\phi = -c(\tau_t + \xi_x). \tag{61}$$

With substituting Eq. (61) into determining equation, we have

$$\xi = c_1, \quad \tau = c_2 t + c_3, \quad \phi = -cc_2, \tag{62}$$

where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.3.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = t\partial_t - c\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = t\partial_t - c\partial_u.$$

We have

$$W = -c - tu_t. \tag{63}$$

The right-hand side of (4) is written

$$C^1 = W(v - v_{xx}) + (D_x(W))[v_x] - D_x^2(W)[v], \tag{64}$$

$$C^2 = W[vf - 2v_{xt}] + D_x(W)[v_t] + D_t(W)[v_x] - 2D_xD_t(W)[v]. \tag{65}$$

Substituting in (64) and (65), the expression (7) for L and (63) for W, we obtain

$$C^1 = -cv + cv_{xx} - tvu_t + tv_{xx}u_t - tv_xu_{tx} + tvu_{txx}, \tag{66}$$

$$C^2 = -cvf - u_tv_x + 2cv_{xt} - tu_{xt}v_t$$

$$-tv_xu_{t^2} - tvfu_t + 2vu_{tx} + 2tu_{xt^2}v + 2tu_tv_{xt}. \tag{67}$$

We can eliminate u_t by using Eq. (1) and obtain

$$C^1 = -cv + cv_{xx} + tvfu_x + tv_{xx}u_{x^2t} - tfv_{xx}u_x - tv_xu_{tx}, \tag{68}$$

$$C^2 = -u_{x^2t}v_x + fu_xv_x + 2cv_{xt} - tu_{xt}v_t - tv_xu_{t^2} - tvfu_{x^2t} + tvf^2u_x$$

$$+2vu_{tx} + 2tu_{xt^2}v - cvf + 2tv_{xt}u_{xxt} - 2tfu_xv_{xt}. \tag{69}$$

Now, we substitute in (68) and (??) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$C^1 = -cu + cu_{x^2} + tufu_x + tu_{x^2}u_{x^2t} - tfu_{x^2}u_x - tu_xu_{tx}, \tag{70}$$

$$C^2 = -cuf - u_{x^2t}u_x + fu_x^2 + 2cu_{xt} - tu_{xt}u_t - tu_xu_{t^2} + 2tu_{xt}u_{xxt}$$

$$-2ftu_xu_{xt} - tufu_{x^2t} + tuf^2u_x + 2uu_{tx} + 2tu_{xt^2}u, \tag{71}$$

where $f = Le^{u/c}$.

Case 3.3.ii). By considering Eq. (49) – (54), we find that the components ξ, τ and φ are $\xi = \xi(x)$, $\tau = \tau(t)$ and $\phi = A(x)u + B(x,t)$. By considering Eq. (55) and (56) we have

$$\xi = c_1 \exp 2x + c_2 \exp -2x + c_3,$$

$$A(x) = c_1 \exp 2x - c_2 \exp -2x + c_4.$$

By considering Eq. (57) we have

$$\tau = ft^2(2c_1 \exp 2x + 2c_2 \exp -2x) + c_5t + c_6,$$

where c_i, $i = 1..6$ are arbitrary constants.

From the following identity:

$$A(x)u + B(x,t) = \frac{-f}{f_u}(\tau_t + \xi_x),$$

we find that $c_1 = c_2 = 0$ and $\phi = -(f/f_u)c_5$. Hence we have two particular cases:

$$\frac{f}{f_u} = Ku, \qquad \frac{f}{f_u} \neq Ku = g(u),$$

where K is an arbitrary nonzero constant. For the first case, we have

$$\xi = c_3, \quad \tau = c_5t + c_6, \quad \phi = -Kuc_5,$$

and for the second case, we have

$$\xi = c_3, \quad \tau = c_6, \quad \phi = 0.$$

Theorem 3.2. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when* $f/f_u = Ku$ *are*

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_t - u\partial_u,$$

and when $f/f_u \neq Ku = g(u)$ are

$$v_1 = \partial_x, \quad v_2 = \partial_t,$$

where K is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry $v = t\partial_t - u\partial_u$, we find that $W = -u - tu_t$. Therefore, we have the conserved vector with the following components:

$$C^1 = -u^2 + uu_{xx} - tuu_{xxt} + tfuu_x + tu_{xx}u_{xxt}$$

$$-tfu_xu_{xx} - u_{x^2} - tu_xu_{xt} + uu_{xx} + tuu_{xxt},$$

$$C^2 = -u^2f - tufu_{xxt} + tuf^2u_x + 2uu_{xt} + 2tu_{xxt}u_{xt} - 2ftu_{xt}u_x$$

$$-u_xu_t - tu_{xt}u_t + 4u_{xt}u + 2tu_{ttx}u - 2u_{xxt}u_x + 2fu_x^2 - 2u_{tt}u_x,$$

where $f/f_u \neq Ku = g(u)$.

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