Helgason-Schifman Formula for Semisimple Lie Groups of Arbitrary Rank

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Abstract

This paper extends the Helgason-Schifman formula for the H-function on a semisimple Lie group of real rank one to cover a semisimple Lie group G of arbitrary real rank. A set of analytic R-valued cocycles are deduced for certain real rank one subgroups of G. This allows a formula for the c-function on G to be worked out as an integral of a product of their resolutions on the summands in a direct-sum decomposition of the maximal abelian subspace of the Lie algebra g of G. Results about the principal series of representations of the real rank one subgroups are also obtained, among other things.

Keywords: Helgason-Schifman formula; Spherical functions; H-function; Semi simple Lie group

Introduction

Let G be a semisimple Lie group with finite center and Lie algebra, g. Define a Cartan involution on G as an involutive automorphism θ of G whose set of fixed points, Gθ = {x ∈ G : θ(x) = x}, is a maximal compact subgroup of G: We say K and θ are associated whenever K = Gθ. In this case, set t = {X ∈ g : θX = X} and P = {X ∈ g:θX = −X}. Then t is the Lie algebra of K and we have the decompositions g = t ⊕ p and G = K exp P commonly called the Cartan decompositions of g and G, respectively, associated to θ. Now choose a maximal abelian subspace, a, of p and let a* be its dual vector space. For any λ ∈ a*, consider the subspace gλ of g defined as gλ = {X ∈ g : [X, a] = λa}. The c-function in this case is then given as cλ(θ(X)) = (1 + |X|^2)^{1/2} d|X|^2.

An analogous expression has been sought for other examples of G; starting in 1960 with the work of Bhanu-Murthy, whose study entails a group-by-group consideration, while the case of an arbitrary G is not known. A common feature of the computation of the H-function for higher-than-one rank groups, which is used to compute the H-function on a group-by-group basis, is its relationship with the finite-dimensional representations of G. The above mentioned relationship is as follows: the H-function of G relative to a minimal parabolic subgroup satisfies the relation cλ(θ(X)) = Φλ(x)μ, where Φλ is a finite dimensional irreducible holomorphic representation of Gc, simply connected group such that Gc ≤ Gθ, with highest weight λ and μ is any unit vector in the sum of the weight spaces for weights that restricts to λ on a [2].

We give the computation in the case of G = SL(3, R). Let us write the subgroup N of G as N = {nx: x ∈ N}. Then, from the above relation, it may be shown that cλ(θ(x)) = (1 + x^2 + z^2)(1 + y^2 + (x+y)^2) for every n ∈ N. The c-function in this case is then given as c(λ) = ∫N dν(dx) = (1 + x^2 + z^2)(1 + y^2 + (x+y)^2) for every n ∈ N. The c-function in this case is then given as c(λ) = ∫N dν(dx) = (1 + x^2 + z^2)(1 + y^2 + (x+y)^2) for every n ∈ N.

For any G; with R -rank one and Lie algebra g, there is an explicit expression for the H-function which was independently established by Helgason and Schifman [1]. Indeed the expression is completely defined on θ (N) and we have it as

\[ \lambda'(H(n)) = \frac{1}{2} \log[(1 + |X|^2)^{1/2} d|X|^2] \]

where \( \lambda' \) is half of the only positive real root of \( (g, a) \).

\[ n = \exp X \exp Y \in \theta(N), \quad |X| = -B(X, \theta X) \quad \text{and} \quad B \quad \text{is the Killing form} \]

on g. This may also be written as cλ(θ(X)) = (1 + |X|^2)^{1/2} d|X|^2.

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formula for $SL(3, \mathbb{R})$, which may be expressed in terms of gamma function. However, our interest here is to find the generalization of the expression for $e^{\omega(x,k)}$ that would work for every semisimple group $G$ [3]. In order to generalize the methods in the last paragraph to every semisimple Lie group $G$ we seek the earlier mentioned relationship of $H$ in terms of $m := R$-rank (G). In this paper, we give an expression, in 2 for $H$ which makes the harmonic analysis on $G \mathbb{R}$ -rank dependent. Indeed this expression leads to a generalization of the $R$-rank one Helgason-Schiffman formula [1] to arbitrary rank as contained in 3. This general formula reduces to the H-function for $SL(3, \mathbb{R})$, without using the method of the highest weight theorem for finite dimensional representations of $G$.

The Decomposition of the H-function

We start with Theorem 2.1 below which plays a fundamental role in what follows.

**Theorem** Let $G$ be of $\mathbb{R}$ -rank $m$. Then we have

$$H(x) = \sum_{\alpha} t_{x,\alpha}(x) X_{\alpha}, \quad x \in G,$$

Where $a = \text{span}_{\mathbb{R}} \{X_{\alpha}, \ldots, X_{\alpha}\}$. In particular, each $x \mapsto t_{x,\alpha}(x)$ a logarithm function and is analytic on $G$.

**Proof:**

The proof is essentially the same as in ([3], Theorem 2.1) and so is omitted.

Before going on, we give the following notations which are required for what follows below. We know that the $\mathbb{R}$-rank $(G) = m = \dim (a)$. For each $\alpha \in \{1, \ldots, m\}$ choose a semisimple subalgebra $g_{\alpha}$ of $g$ with a Cartan decomposition $g_{\alpha} = t_{\alpha} \oplus p_{\alpha}$ such that $\{0\} \neq t_{\alpha} \subset t$ and $p_{\alpha} \subset p$. Fix a maximal abelian proper subspace $a_{\alpha}$ of $p_{\alpha}$ (assume throughout that $a_{\alpha}$ is one-dimensional). Fix also a compatible order on non-zero restricted roots; here there are at most two roots which are positive with respect to this order, which we denote by $\Delta_{\alpha}$ and $\pm a_{\alpha}$. Thus, denoting by $\Delta = \Delta(\alpha, a_{\alpha})$ the set of restricted roots of the pair $(g_{\alpha}, a_{\alpha})$, then $\Delta_{\alpha} = \{-a_{\alpha}, -\alpha, \alpha, a_{\alpha}\}$ with a corresponding positive system $\Delta_{\alpha}^{+} = \{\alpha, 2\alpha\}$. We denote by $H_{\alpha}$ the linear functional on $a_{\alpha}$ which equals one half the largest positive restricted root of $\Delta$. We decompose a into a direct sum of one-dimensional $a_{\alpha}$ subspaces $a_{\alpha}, 1 \leq j \leq m$, that is, $a = \oplus_{\alpha} a_{\alpha}$, with $\dim (a_{\alpha}) = 1$.

We employ the groups $SL(3, \mathbb{R})$ and $Sp(2, \mathbb{R})$ to illustrate examples of the decomposition in the Theorem 2.1 above.

For the real rank 2 group $SL(3, \mathbb{R})$ a maximal abelian subspace, $a$, of $p$ is

$$a = \left\{ \begin{array}{c} a_{1} \ 0 \ 0 \\ 0 \ a_{2} \ 0 \\ 0 \ 0 \ -(a_{1} + a_{2}) \end{array} \right\} : a_{1}, a_{2} \in \mathbb{R}.$$ \[1\]

We may then choose

$$\left\{ \begin{array}{c} a_{1} \ 0 \ 0 \\ 0 \ 0 \ 0 \\ 0 \ -a_{1} \end{array} \right\} : a_{1} \in \mathbb{R}$$

$$\left\{ \begin{array}{c} 0 \ 0 \ 0 \\ 0 \ a_{2} \ 0 \\ 0 \ 0 \ -a_{2} \end{array} \right\} : a_{2} \in \mathbb{R}$$

as $a_{1}$ and $a_{2}$, respectively, each of which is one-dimensional. In the case of $G =$ $Sp(2, \mathbb{R}), a$ maximal abelian subspace is

$$\left\{ \begin{array}{c} s \ 0 \ 0 \\ 0 \ t \ 0 \\ 0 \ 0 \ -s \end{array} \right\} : s, t \in \mathbb{R}.$$ \[2\]

Thus

$$\left\{ \begin{array}{c} s \ 0 \ 0 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ -s \end{array} \right\} : s \in \mathbb{R}$$

$$\left\{ \begin{array}{c} 0 \ 0 \ 0 \\ 0 \ 0 \ 0 \\ 0 \ 0 \ -t \end{array} \right\} : t \in \mathbb{R}.$$

be chosen as $a_{1}$ and $a_{2}$, respectively.

It is clear that the case $m=1$ reduces to the situation of Helgason-Schiffmann. Next we discuss some of the properties of each of the maps $x \mapsto t_{x,\alpha}(x)$. To this end let $d_{\alpha}(x) = \exp(t_{x,\alpha}(x), X_{\alpha})$, $x \in G, 1 \leq j \leq m$.

**Corollary**

We have $a(x) = \prod_{i=1}^{m} d_{\alpha}(x), x \in G$.

This corollary generalizes an equivalent expression for $SL(m+1, \mathbb{R})$, established in [4] to any semisimple Lie group with finite center and of any real rank. One of the major applications of the $H$-function, and now of Theorem 2.1, is its contribution to the compact picture of the induced representations on semisimple Lie groups. This contribution relies on the cocycle nature of $H$. In anticipation of a similar use to be made of the maps $x \mapsto t_{x,\alpha}(x)$ we establish the following proposition.

**Proposition**

Let there be given $\alpha \in \{1, \ldots, m\}$, the map $x \mapsto t_{x,\alpha}(x)$ induces an analytic $\mathbb{R}$-valued cocycle on $G$.

**Proof**

Since $G / AN = K$ the subgroup $K$ may be regarded as a transitive homogeneous space for $G$ acting from the left. We denote this action as $G \times K \rightarrow K : (x,k) \mapsto x(k) := k(xk)$. In this context the function $x \mapsto a(x)$ induces an $A$-valued map $G \times K \rightarrow A : (x,k) \mapsto a(xk)$ given simply as $a(xk) := a(xk)$ and which satisfies

(i) $a(1) = 1$,

(ii) $a(x_{1}x_{2} : k) = a(x_{1}, x_{2}[k])a(x_{2} : k)$, and

(iii) $a(x : x^{-1}[k]) = a(x^{-1} : k)^{-1}$ (cf. [7], p.84).

Now going over, from the map $(x,k) \mapsto a(x,k)$, to a (via the $H$-function) and then to $\mathbb{R}$ (via each of $t_{x,\alpha}(x)$), we may define the map $x \mapsto \mu_{\alpha}(x, k)$, and denote it by $t_{x,\alpha}(x,k)$.

Using Theorem 2.1 above, properties (i), (ii) and (iii) of $a(x : k)$ become

(i) $t_{x,\alpha}(1) = 0$,

(ii) $t_{x,\alpha}(x_{1}, x_{2}[k] : k) = t_{x,\alpha}(x_{1} : x_{2}[k]) + t_{x,\alpha}(x_{2} : k)$, and

(iii) $t_{x,\alpha}(x : x^{-1}[k]) = -t_{x,\alpha}(x^{-1} : k)$.

The real rank 1 case of the last proposition is contained in Proposition 3.1 of [5]. It is known that the $H$-function vanishes on the maximal compact subgroup $K$. The implication is that each of the
The H-function is known to be completely defined on \( \tilde{N} = \tilde{\theta}(N) \), where \( N = \exp(n) \), \( n = \bigoplus_{\alpha \in \Delta_+} G_\alpha \) and \( \tilde{\theta} \) is the Cartan involution of G associated to K. The decomposition of a in Theorem 2.1 means we consider the complete understanding of each of \( t_m(j) \), on the direct sum of \( G \). The result is raised to a power depending on \( l \), with 1's on the diagonal. For each \( l \) with \( l \neq 0 \), we give here an approach for the computation of the c-function on \( SL(3, \mathbb{R}) \) as the integral of complex indices of two polynomials. The above situation may be generalised to the c-function on \( SL(m+1, \mathbb{R}) \). To this end we take \( n \) to be a lower triangular matrix, \( (X_{ij})_{i=1}^{m+1} \), with 1's on the diagonal. For each \( l \) with \( 1 \leq l \leq m \), a generalisation of the above computations is obtained by forming the sum of the squares of \( C \), minors of size 1-by-1 obtained from the first l columns of \( (X_{ij})_{i=1}^{m+1} \). The result is raised to a power depending on \( l \), and the analogue of the c-function above is the integral over \( \mathbb{R} \) of the product of m expressions raised to their respective powers.

It is however known that the above construction techniques given for the c-function of \( G = SL(m+1, \mathbb{R}) \) do not extend to other real semisimple Lie groups with finite center. For this reason the earlier expression given as \( e^{2i\lambda(H(x))} = \Phi_\lambda(x)u \int \Phi_\lambda(x)u \) is always resorted to when ever the c-function of specific groups are needed, with the attendant restriction that there exists a simply connected group \( G^C \), such that \( G \subseteq G^C \) and with a finite-dimensional irreducible holomorphic representation, \( \Phi_\lambda \). We give here an approach for the computation of the above j-function (hence the c-function) for any real rank \( m \) connected semisimple Lie group with finite center, which will establish the exact contribution of \( m \) as earlier seen in the case of \( SL(m+1, \mathbb{R}) \).

**Theorem**

Let \( \alpha_j = \alpha_j \) and \( \tilde{N}_j = \tilde{N}_{a_j} \) where every \( n_j \) is of the form \( n_j = \exp Y_j, \exp Z_j \), \( Y_j, Z_j \in g_{-a_j} \). Introduce parameters that describe members of each \( N_j, 1 \leq j \leq m \), such that \( N = N_1 \ldots N_m \). Then, for every \( \alpha \in \Delta ', (g, a) \),

\[
j(\alpha) = \left\{ \begin{array}{l}
\int_{x-a_j}^{x} \prod_{i=1}^{m} \left( 1 + (a_i Y_i) \right)^{2j} dY_i, \text{ if each } 2a_i \neq a_j, \\
\int_{x-a_j}^{x} \prod_{i=1}^{m} \left( 1 + (a_i Y_i) \right)^{2j} + 2Q_{ij}(Z_i) |dY_i|, \text{ if each } 2a_i = a_j,
\end{array} \right.
\]

Where \( a_j \) is chosen appropriately and \( Q_{ij} \) is a quadratic form.

**Proof**

If \( \alpha \in \Delta ', (g, a) \) then a choice may be made to have \( a_j = a_j \), \( \alpha_j = a_j \). Hence if \( 2a_j \in \Delta ' \), then \( \alpha_j = a_j \), while if \( 2a_j \notin \Delta ' \), then \( \alpha_j = 2a_j \), where is as defined under Theorem 2.1. Therefore
\[ e^{2\alpha(H(\tilde{n}))} = e^{2\alpha[\sum_{j=1}^{m} e^{2\alpha_j(t_m, j(\tilde{n}), X_j)]} \]

\[ = \prod_{j=1}^{m} \left[ e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} \right] \]

\[ = \prod_{j=1}^{m} \left[ e^{2\mu_j(t_m, j(\tilde{n}), X_j)} \right], \quad \text{if each } 2\alpha_j \not\in \Delta, \]

\[ = \prod_{j=1}^{m} \left[ e^{2\mu_j(t_m, j(\tilde{n}), X_j)} \right] \mu_j, \quad \text{if each } 2\alpha_j \in \Delta, \]

Hence we restrict our computations to \( e^{2\mu_j(t_m, j(\tilde{n}), X_j)} \).

If we recall the definition of \( \mu_j \) above, then

\[ \frac{1}{2} \mu_j = \begin{cases} \frac{1}{2} (\alpha_j), & \text{if each } 2\alpha_j \not\in \Delta, \\ \frac{1}{2} (\alpha_j), & \text{if each } 2\alpha_j \in \Delta, \end{cases} \]

each of which is not a root of the pair \((g, a)\). Hence \( \mu_j \) is a short root of \((g, a)\), and we may take the root-space decomposition

\[ g_j = (m_j @ a_j) \oplus \bigoplus_{\beta \not\in \Delta} g_{\beta}, \]

where \( m_j @ a_j \) is the centraliser of \( a_j \) in \( g(\mu_j) = g_{a_j} \). By construction \( g(\mu_j) = g_{a_j} \), each \( g(\mu_j) \) is stable under the restriction of the Cartan involution of \( g \) and is therefore simple.

Denote by \( g(\mu_j) \) the analytic subgroup of \( g(\mu_j) \) corresponding to \( g(\mu_j) \), while \( K \) and \( G(\mu_j) \) may be taken to be the connected groups \( K(\mu_j) = K \cap g(\mu_j) \) and \( M(\mu_j) = M \cap g(\mu_j) \) with \( M(\mu_j) = M \cap K(\mu_j) \) as the corresponding \( M \)-group. Thus the symmetric group \( G(\mu_j) / K(\mu_j) \) has rank one, where each \( G(\mu_j) \) is a real rank one semisimple Lie group with finite center. Hence we may define a quadratic form, \( Q(\mu_j) \), as

\[ Q_{\mu_j}(X) = \frac{4}{H_{\mu_j}(H_{\mu_j})} \left[ a \in a, X = g(\mu_j) \right], \]

where \( H_{\mu_j} \) is such that \( \mu_j(\tilde{H}_{\mu_j}) = 2 \) and \( \{.,.\} \) is the restriction of the Killing form to \( a, X = a \).

It therefore follows that \( e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} \) is the \( e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} \) for the real rank one semisimple Lie group \( G(\mu_j) \) (with \( \mu_j \) given in terms of \( \alpha_j \) as above). Hence

\[ e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} = \begin{cases} (1 + \frac{1}{2} \alpha_j(Y)), & \text{if each } 2\alpha_j \not\in \Delta, \\ (1 + \frac{1}{2} \alpha_j(Y)), & \text{if each } 2\alpha_j \in \Delta, \end{cases} \]

as required.

Corollary

Let \( \alpha \in \Delta^+ \). Then the function \( n \mapsto \tilde{e}^{2\alpha_j(t_m, j(\tilde{n}), X_j)} \) on \( N \) are polynomials in the Lie algebra coordinates of \( N \).

Computation of \( e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} \) : the case of SL(3, \( \mathbb{R} \)).

We start by restricting the members of \( \Delta^+ = \{ (e_i - e_j), (e_j - e_i), (e_j - e_j) \} \) to \( a_i \) and \( a_j \) to have

\[ (e_i - e_j)(\text{diag}(a_i, 0, -a_i)) = (e_i - e_j)(\text{diag}(a_j, 0, -a_j)) = 2a_i, \quad \text{for } a_i, \quad \text{and } (e_j - e_i)(\text{diag}(0, -a_i, a_i)) = (e_j - e_i)(\text{diag}(0, -a_j, a_j)) = 2a_j, \quad \text{for } a_j. \]

If we now require, in addition to the earlier requirements of Example 3.1, that \( a_i > 0 \) and \( a_j > 0 \), we may define \( \alpha_j : a_i \to \mathbb{R} \) and \( \alpha_i : a_j \to \mathbb{R} \) as \( \alpha_j(H_i) = a_i, H_i \in a_i \) and \( \alpha_i(H_j) = a_j, H_j \in a_j \), respectively. These are respectively the restrictions \((e_i - e_j)|_a \) and \((e_j - e_i)|_a \), with \( 2a_i = (e_i - e_i)|_a \) and \( 2a_j = (e_j - e_j)|_a \).

If we then define \( g(\alpha_j) = a_i \oplus g_{a_i} \oplus g_{a_j} \oplus g_{-a_i} \oplus g_{-a_j} \) and \( g(\alpha_i) = a_j \oplus g_{a_j} \oplus g_{a_i} \oplus g_{-a_j} \oplus g_{-a_i} \) (since \( m = 0 \)), then \( \tau(j) = g(\alpha_j) \cap \tau \) and \( \tau(j) = g(\alpha_i) \cap \tau \) with \( N = \exp(g_{\alpha_i} \oplus g_{\alpha_j}) \). The restriction of members of \( \Delta^+ \) to \( a_i \) shows that \( 2a_i \in \Delta \) and we may conclude that each \( g(\alpha_j) \) is isomorphic with a real rank one (semi-)simple Lie algebra with \( \Delta_j = \{ \pm \alpha_j \} \), so that

\[ e^{2\alpha_j(t_m, j(\tilde{n}), X_j)} = (1 + \frac{1}{2} Q_{\alpha_j}(Y))^2 + 2Q_{\alpha_j}(Z_j) \]

For \( n = \exp(Y + Z_{\alpha_j}) \), \( 1 \leq j \leq 2 \). This is as computed earlier in Example 3.1.

Another approach to the construction of \( g(\mu_j) \) is as follows. Let \( m_j \) be the centraliser of \( a_j \) in \( g \). It may be shown that \( m_j \) is stable under the restriction of the Cartan involution and that the analytic subgroup, \( M_j \), of \( G \) corresponding to \( m_j \), is the centraliser of \( a_j \). We set \( m_j = m_j \cap K \) and \( M(\mu_j) = M_j \cap K \).

Let us now choose \( \alpha \) to be a short root of the pair \((g, a)\), i.e., \( \alpha \in \Delta \) such that \( \frac{1}{2} \alpha \not\in \Delta \). We may choose \( \alpha_j \) by restrictions as in Computation 3.4 and compute the algebra \( g_{m_j} = \{ X \in g : ad(HX) = \alpha(HX), \forall H \in a_j \} \), from which we now define \( g(\alpha_j) = m_j @ a_j @ g_{a_j} \oplus g_{a_j} \oplus g_{-a_j} \oplus g_{-a_j} \).

We are now in a position to employ Proposition 2.3 to construct the compact picture of the induced representation on \( G(\mu_j) \) fix \( j \in \{1, \ldots, m \} \). Let \( A_j = \exp(\alpha_j) \), \( \lambda_j \in \{ \alpha_j \} = \alpha_j + i\alpha_j \) and define \( \xi_j : A_j \to \mathbb{C}^* \cap \mathbb{C} \}, \) the requirement \( \xi_j(\alpha) = e^{i\lambda_j(\alpha)} \). \( \xi_j \) is a quasi-character of \( A_j \) and is unitary if \( \lambda_j \in i\mathbb{N} \). We therefore have the following.

Proposition

The map \((x, k) \mapsto \xi_j(a_k(x, k))\), for \( x \in G(\mu_j), k \in K(\mu_j) \), is an analytic \( C^* \)-valued cocycle.

Proof

By Proposition 2.3.

Setting \( \rho_j = \frac{1}{2} \sum_{\alpha \in \Delta} \dim(g_{\alpha_j}) \beta \), we define \( \mathcal{R}_{\sigma_j, \lambda_j} \) as

\[ (\mathcal{R}_{\sigma_j, \lambda_j}(x)f)(k) = e^{i\lambda_j(x, k)} \int f(x^{-1}K), \]

\[ x \in G(\mu_j), k \in K(\mu_j), \quad \text{with } f \in h(\sigma_j), \]

where \( h(\sigma_j) = [g \in L^2(K(\mu_j)) : g(\mathbb{R}^m) = \sigma_j(\mathbb{R}^m)g(x), m \in M(\mu_j) \cap K(\mu_j), \tau \in G(\mu_j)] \), \( \sigma_j \) a finite-dimensional unitary representation on \( K(\mu_j) \), Details of the construction of \( \mathcal{R}_{\sigma_j, \lambda_j} \) may be found in [5].

Proposition

\( \mathcal{R}_{\sigma_j, \lambda_j} \) is an irreducible unitary representation of \( G(\mu_j) \) on...
for and is the resolution of is of class-1 if, and it reduces to the left-regular representation on Y_j := \{x \in G(\mu) : t_{\omega}(x : k) = 0, \forall k \in K\}.

Proof

The cocycle relations proved in Proposition 2.3 for t_{w_j} give \pi_{\sigma_{\omega},j}(1) = 1 and \pi_{\sigma_{\omega},j}(y) = \pi_{\sigma_{\omega},j}(x)\pi_{\sigma_{\omega},j}(y), \forall x, y \in G(\mu) while the continuity of the map (x, f) \mapsto \pi_{\sigma_{\omega},j}(x)f of G(\mu) \times h(\sigma_j) into h(\sigma_j) the irreducibility and unitarity of \pi_{\sigma_{\omega},j}(x) are established exactly as in the case of the principal series on G.

If x \in Y_j, then from the same cocycle properties of t_{w_j}, we have that x^{-1} \in Y_j. Thus t_{\omega,j}(x^{-1} \cdot k) = t_{\omega,j}(\cdot k) = 0.

It is known that each of the real rank one semisimple Lie groups, G(\mu) admits the induced representations, Ind_{\mu}^{G(\mu)} which may be restricted to K(\mu) to get all the principal series of representations of G(\mu). In this light a consequence of the above Proposition is the following.

Corollary

Let \sigma_j be a finite-dimensional irreducible unitary representation of M(\mu) and \lambda, \in i\mathfrak{g}'. The representations \pi_{\sigma_j,\lambda} exhausts the unitary principal series of G(\mu).

We are now encouraged to define the spherical functions x \mapsto \phi_j(x), x \in G(\mu) corresponding to the class 1 members of \pi_{\sigma_j,\lambda}. With respect to the spherical function, \phi_j(x) = \int_{G} e^{i\lambda(x)X(x)}\,dK(x) of G, we refer to \phi_j as the resolution of the spherical function \pi_j.

The Plancherel measure \mu is supported on the set of real-valued \lambda and is of the form

\[d\mu(\lambda) = \text{const.} \frac{d\lambda}{|\lambda|^2}\]

where \lambda is the Lebesgue measure on the dual of the real vector space and the function c is given explicitly as a product of beta-functions by the following formula,

\[c(\lambda) = \prod_{a \in \Phi} \left(\frac{1}{2} m_a \lambda^2 + \frac{1}{4} \lambda^2 a(a')\right)\]

where the product is over the positive roots relative to some ordering, \(m_a\) is the multiplicity of the root \(a\), and \(a' \in a\) is the dual root corresponding to \(a\), that is,

\[\lambda(a') = \left(\frac{2\lambda(a)}{a, a}\right)\]

The explicit calculation (3.1) of \(c(\lambda)\) is due to Bhanu - Murthy [7] for the split groups and to Gindikin and Karpelevic in the general case [1].

We define a representation \(\pi\) on a (locally convex) space \(V\) to be of class-1 whenever the subspace \(V^{(\omega)} := \{v \in V : \pi(\omega)v = v, \forall k \in K\}\) of all K-invariant vectors in \(V\), is of dimension 1. It is known [8] that class-1 representations are associated with spherical functions on \(G\) (which are the matrix coefficients of these representations), and that, for irreducible \(\sigma\), the (unitary) principal series, \(\pi_{\sigma,\lambda}\) is of class-1 if, and only if, \(\sigma\) is the trivial representation on \(M\). Let us therefore denote \(\pi := \pi_{\sigma,\lambda}\) and set the matrix coefficient of \(\pi_{\sigma,\lambda}\) defined by the function \(1\), as \(\varphi_{\sigma,\lambda}\) given as

\[\varphi_{\sigma,\lambda}(x) = (\pi_{\sigma,\lambda}(x)\cdot 1, 1)\]

where \(x \in G, \lambda, \sigma, 1 \in L^1(K)\) and \(\pi\) is an inner product on \(L^1(K)\). The function \(\varphi_{\sigma,\lambda}\) is spherical and, has the integral representation

\[\varphi_{\sigma,\lambda}(x) = \int_{G} e^{i\lambda(x)X(x)}\,dK(x)\]

as given above.

The result of Theorem 3.2 leads to the following product formula for the spherical functions, \(\Phi_j\), in a direction different from the Gindikin-Karpelevic product formula for spherical functions.

Theorem

Every spherical function, \(\varphi_{\sigma,\lambda} \in G\), on \(G\) is of the form

\[\varphi_{\sigma,\lambda}(x) = \prod_{j=1}^{m} \phi_{j}(x)\]

where each \(\phi_j(x)\) is the resolution of \(\varphi_{\sigma,\lambda}(x)\) on each summand in the direct sum \(\oplus_{j=1}^{m} \sigma_j\).

Proof

We first note that

\[(\pi_{\sigma,\lambda}(x)f)(k) = (\xi_{\lambda,\omega}(x^{-1}k)^{-1}f(x^{-1}k))\]

\[= e^{-\omega \log(a(x^{-1}k)^{-1})} e^{-\rho \log(a^{-1}k)} f(x^{-1}k)\]

\[= e^{-\lambda(x)X(x)} f(x^{-1}k)\]

\[= e^{-\lambda(x)X(x)} f(x^{-1}k)\]

\[= e^{-\lambda(x)X(x)} f(x^{-1}k)\]

which is substituted into \(\varphi_{\sigma,\lambda}(x) = (\pi_{\sigma,\lambda}(x)\cdot 1, 1)\) gives

\[\varphi_{\sigma,\lambda}(x) = \prod_{j=1}^{m} (\int_{G} e^{i\lambda(x)X(x)}\,dK(x)\,d\lambda_{(j)}(x))\]

The expression \(\int_{G} e^{i\lambda(x)X(x)}\,dK(x)\,d\lambda_{(j)}(x)\) is the resolution of \(\varphi_{\sigma,\lambda}(x)\) on each \(j\) and is denoted as \(\varphi_{\sigma,\lambda}(x)\).

The product formula above explains that spherical functions, \(\varphi_{\sigma,\lambda}(x)\) on any real rank \(m\) group \(G\) is the product of its resolutions, \(\varphi_{\sigma,\lambda}(x)\) on each of the \(1\)-dimensional subspaces, \(\omega\) of a. It implies that spherical functions on real rank \(m\) groups can be studied through its resolutions, on some \(1\)-dimensional subspace.

References
