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Principal Curvatures of Homogeneous Hypersurfaces in a Grassmann Manifold $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ by the G_2-action

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Abstract. We compute the principal curvatures of homogeneous hypersurfaces in a Grassmann manifold $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ by the G_2-action. As applications, we show that there is a unique orbit which is an austere submanifold, and that there are just two orbits which are proper biharmonic homogeneous hypersurfaces. We also show that the austere orbit is a weakly reflective submanifold.

1. Introduction

It is an interesting and important subject to compute the principal curvatures of homogeneous hypersurfaces in symmetric spaces. Such calculations are given by many researchers. In 1970's, regarding homogeneous hypersurfaces in the simply connected, compact Riemannian symmetric spaces of rank one, except for the Cayley projective plane, their principal curvatures were obtained in [14], [15] and [4]. Later Verhóczki got the result in the case of the Cayley projective plane in [16] and [3]. As for the higher rank cases, the computations have been progressed recently. Homogeneous hypersurfaces are given as orbits by the cohomogeneity one actions. Kollross classified cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type in [12]. By his classification, in the higher rank cases, it is known that most of the cohomogeneity one actions are the Hermann actions. Then the principal curvatures of homogeneous hypersurfaces by the Hermann actions of cohomogeneity one are obtained ([1], [2], [3], [5], [17], [19]). Among homogeneous hypersurfaces by the "exceptional" cohomogeneity one actions, that is, non-Hermann type, the principal curvatures of homogeneous hypersurfaces in $G_2/SO(4)$ by the $SU(3)$-action and those in G_2 by the $SU(3) \times SU(3)$-action are computed, respectively ([18]). In this paper, we compute the principal curvatures of homogeneous hypersurfaces in a Grassmann manifold $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ by the G_2-action, which is an "exceptional" cohomogeneity one action (Theorem 1). Here the Lie group G_2 is defined as the automorphism group of the algebra $\mathbb{O}$ of octonions which

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is 14-dimensional and simple. We denote by $\mathrm{Im}\mathbb{O}$ the seven-dimensional Euclidean space of all imaginary part of octonions and by $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ the Grassmann manifold of all three-dimensional oriented subspaces in $\mathrm{Im}\mathbb{O}$.

Next we summarize our computation method. Let $\tilde{N}$ be an irreducible Riemannian symmetric space of compact type and N a singular orbit of $\tilde{N}$ by an isometric action of cohomogeneity one. Principal curvatures have been calculated by making use of the fact that homogeneous hypersurfaces by this action are tubular hypersurfaces around N. In particular, they can be obtained by applying the formula in [17] under the following assumption:

1. The singular orbit N is totally geodesic.
2. The tangent space T_pN at a point $p \in N$ is invariant by the Jacobi operator R_u for a unit normal vector u of N at p. Here R_u is defined by $R_u(v) = R(v,u)u$ for $v \in T_p\tilde{N}$ with respect to the curvature tensor R of $\tilde{N}$.

This method is used in the cases of the Hermann actions on $\tilde{N}$ of rank bigger than one. On the other hand, we remark that we cannot apply it in our case. Actually, the singular orbit $G_2/SO(4)$ of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ by the G_2-action is totally geodesic, but the tangent space of $G_2/SO(4)$ at the base point o is not invariant by the Jacobi operator (Remark 1). So we focus on the decomposition of the Lie algebra $\mathfrak{g}_2$ of G_2. Let H be the subgroup of $SO(4)$ whose adjoint representation fixes an element of the normal space of $G_2/SO(4)$ at o. We decompose $\mathfrak{g}_2$ into the invariant subspaces by the adjoint representation of H, which is described in the equation (2) in Section 4. Depending on the decomposition, we compute the eigenvalues of the shape operator.

The study of homogeneous hypersurfaces contributes to the progress in the research of various geometric properties through construction of interesting examples. In this paper, we show examples of austerity, proper biharmonicity and weakly reflectiveness. Austerity is a condition related to the symmetrical arrangement of eigenvalues of the shape operator. Since austere submanifolds form a special class of minimal submanifolds, it is fascinating to study them ([7], [8], [10]). Proper biharmonicity is the concept introduced as critical maps of the geometric variational problem regarding the tension fields of maps between Riemannian manifolds. The classification of biharmonic homogeneous hypersurfaces in Riemannian symmetric spaces has been made progress in recent years ([9], [13]). The computation results of the principal curvatures and the multiplicities are used especially in [9]. The notion of weakly reflective submanifolds is a generalization of that of reflective submanifolds ([8]). Weakly reflective submanifolds have interesting properties such as austerity. We provide the new examples concerning those three properties in Section 5 and 6.

This paper is organized as follows. In Section 2, we recall some basic facts about the algebra $\mathbb{O}$ of octonions and the exceptional Lie group G_2, and give our main theorem (Theorem 1). Then, in Section 3, we describe the Riemannian symmetric pairs of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ and $G_2/SO(4)$ explicitly to investigate the structure of $\mathfrak{g}_2$. In Section 4, we obtain the decomposition of $\mathfrak{g}_2$ and compute the principal curvatures of homogeneous hypersurfaces in $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$

by the G_2-action. As applications, we prove that the orbit of $\Phi = 0$ is an austere submanifold (Corollary 1), and the orbits of $\Phi = \pm\frac{1}{\sqrt{10}}$ are the only proper biharmonic homogeneous hypersurfaces of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ by the G_2-action (Corollary 2) in Section 5. Further, in Section 6, we show that the orbit of $\Phi = 0$ is a weakly reflective submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ (Proposition 4).

2. Preliminaries and the main theorem

First of all, we review some necessary basic facts to state the main theorem. Let $\mathbb{H} = \{x + yi + zj + wk \mid x, y, z, w \in \mathbb{R}\} \cong \mathbb{R}^4$ $(i^2 = j^2 = k^2 = -1, ij = -ji = k)$ be the algebra of quaternions and $\mathrm{Im}\mathbb{H}$ be the subspace of imaginary quaternions. We denote the group of unit quaternions by $Sp(1) \subset \mathbb{H}$. The algebra $\mathbb{O}$ of octonions is a normed algebra given by $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$, where the multiplication on $\mathbb{O}$ is given by $(a + b\varepsilon)(c + d\varepsilon) = (ac - \bar{d}b) + (da + b\bar{c})\varepsilon$ ([7]). Here $\bar{a}$ is the quaternionic conjugation for $a \in \mathbb{H}$. Let $\mathrm{Im}\mathbb{O} = \mathrm{Im}\mathbb{H} \oplus \mathbb{H}\varepsilon$ be the subspace of all imaginary part of octonions. We define the alternating trilinear form φ on $\mathrm{Im}\mathbb{O}$ by

$$\varphi(x, y, z) = \langle x, yz \rangle ,$$

where $\langle , \rangle$ is the natural inner product on $\mathbb{O} \cong \mathbb{R}^8$. The 3-form φ is called the *associative calibration* on $\mathrm{Im}\mathbb{O}$ ([7] p.113 Definition 1.5). The Lie group G_2 is defined by

$$G_2 = \mathrm{Aut}(\mathbb{O}) = \{g \in GL_8(\mathbb{R}) \mid g(xy) = g(x)g(y), \text{ for any } x, y \in \mathbb{O}\} .$$

It is well known that the Lie group G_2 is 14-dimensional and simple ([7]). Every automorphism of $\mathbb{O}$ fixes the subspace $\mathbb{R} \cdot 1 \subset \mathbb{O}$ and leaves the subspace $\mathrm{Im}\mathbb{O}$ invariant. We also have the facts that G_2 is a subgroup of $SO(\mathrm{Im}\mathbb{O}) \cong SO(7)$, and that the following holds:

$$G_2 = \{g \in O(7) \mid g^*\varphi = \varphi\} .$$

Let $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ be the Grassmann manifold of all three-dimensional oriented subspaces in $\mathrm{Im}\mathbb{O}$. $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is isomorphic to the Riemannian symmetric space $SO(7)/SO(3) \times SO(4)$. We equip $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ with the Riemannian metric defined by the Killing form on the Lie algebra $\mathfrak{so}(7)$. Since the value of φ does not depend on the choice of orthonormal basis of a three-dimensional oriented subspace, φ may be considered as a function on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. Moreover, we have $|\varphi(\zeta)| \leq 1$ for all $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ ([7] p.113 Theorem 1.4). Then we define level sets $M(\Phi)$ of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ for $-1 \leq \Phi \leq 1$ by

$$M(\Phi) = \{\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O}) \mid \varphi(\zeta) = \Phi\} .$$

If $\zeta \in \widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is the canonically oriented imaginary part of some quaternion subalgebra of $\mathbb{O}$, then the oriented three-plane ζ is said to be an *associative subspace*. The set of all associative subspaces is called the *associative Grassmann manifold* denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. The Lie group G_2 acts transitively on each $M(\Phi)$ $(-1 \leq \Phi \leq 1)$ and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, respectively. The level set $M(1)$ coincides with $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Reversing the orientation of subspaces in $M(-1)$, we see that $M(-1)$ is isometric to $M(1) = \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. $M(1)$ and $M(-1)$ are totally geodesic singular orbits. For $-1 < \Phi < 1$, $M(\Phi)$ are principal orbits of codimension one and diffeomorphic to $G_2/SO(3)$. In particular, φ is an isoparametric function. If we take the normal vector field $-\frac{\mathrm{grad}\varphi}{||\mathrm{grad}\varphi||}$ for the hypersurfaces $M(\Phi)$, we obtain the following theorem.

THEOREM 1. *The principal curvatures of homogeneous hypersurfaces $M(\Phi)$ $(-1 < \Phi < 1)$ are*

$$\begin{aligned} \mu_1(\Phi) &= 0\,, \\ \mu_2(\Phi) &= \frac{1}{2\sqrt{30(1-\Phi^2)}}\left(-3\Phi + \sqrt{8+\Phi^2}\right), \\ \mu_3(\Phi) &= \frac{1}{2\sqrt{30(1-\Phi^2)}}\left(-3\Phi - \sqrt{8+\Phi^2}\right) \end{aligned}$$

with multiplicities $\nu_1 = 5$, $\nu_2 = 3$, $\nu_3 = 3$, respectively.

3. The Riemannian symmetric pairs of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$

In Section 3 and 4, we denote $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ by $\tilde{M}$ and M, respectively. Concerning the Killing form B of the Lie algebra $\mathfrak{so}(7)$ of $SO(7)$, we have $B(X,Y) = 5\mathrm{tr}(XY)$ for $X, Y \in \mathfrak{so}(7)$, where $\mathrm{tr}(XY)$ denotes the trace of the 7×7 matrix XY. Then B is negative definite on $\mathfrak{so}(7)$. Thus $-B$ is an $\mathrm{Ad}(SO(7))$-invariant inner product on $\mathfrak{so}(7)$. The Lie group $SO(7)$ acts transitively on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. We take $o = \mathrm{Im}\mathbb{H}$ as the base point. Then the isotropy subgroup at o is $SO(3)\times SO(4)$. The tangent space $T_o\tilde{M}$ of $\tilde{M}$ at o is identified with the orthogonal complement $\tilde{\mathfrak{m}}$ of the Lie algebra $\tilde{\mathfrak{k}}$ of $SO(3)\times SO(4)$ in $\mathfrak{so}(7)$ with respect to the Killing form B, where

$$\tilde{\mathfrak{k}} = \left\{ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \,\middle|\, X_1 \in \mathfrak{so}(3),\, X_2 \in \mathfrak{so}(4) \right\},$$

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix} \,\middle|\, X \in M_{4,3}(\mathbb{R}) \right\}.$$

The orthogonal decomposition $\mathfrak{so}(7) = \tilde{\mathfrak{k}} + \tilde{\mathfrak{m}}$ is a reductive decomposition of $\mathfrak{so}(7)$, that is, $\tilde{\mathfrak{m}}$ satisfies $\mathrm{ad}(\tilde{\mathfrak{k}})\tilde{\mathfrak{m}} \subset \tilde{\mathfrak{m}}$. The restriction $-B|_{\tilde{\mathfrak{m}}}$ of $-B$ to $\tilde{\mathfrak{m}}$ induces an $SO(7)$-invariant Riemannian metric $\tilde{g}$ on $\tilde{M}$. We define an inner automorphism $\tilde{\sigma}$ of $SO(7)$ by $\tilde{\sigma}(g) = sgs^{-1}$ for $g \in SO(7)$, where

$$s = \begin{pmatrix} E_3 & 0 \\ 0 & -E_4 \end{pmatrix}$$

and E_i is the $i \times i$ unit matrix. Then $\tilde{\sigma}$ is involutive, and the standard decomposition with respect to $\tilde{\sigma}$ coincides with $\mathfrak{so}(7) = \tilde{\mathfrak{k}} + \tilde{\mathfrak{m}}$. Hence $(SO(7), SO(3) \times SO(4), \tilde{\sigma}, -B|_{\tilde{\mathfrak{m}}})$ is a Riemannian symmetric pair of $\tilde{M}$. We denote the Riemannian connection of $\tilde{M}$ by $\tilde{\nabla}$.

Now we consider the associative Grassmann manifold $M = \widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. First, we define an action of $Sp(1) \times Sp(1)$ on $\mathbb{O}$ as follows: For the pair of unit quaternions $(q_1, q_2) \in Sp(1) \times Sp(1)$ and $a + b\varepsilon \in \mathbb{O} = \mathbb{H} \oplus \mathbb{H}\varepsilon$, we set

$$\rho(q_1, q_2)(a + b\varepsilon) = q_1 a q_1^{-1} + (q_2 b q_1^{-1})\varepsilon\,.$$

Then we can see that $\rho(q_1, q_2)$ belongs to G_2 and ρ is an action of $Sp(1) \times Sp(1)/\mathbb{Z}_2 \cong SO(4)$ on $\mathbb{O}$. The action of G_2 on M is transitive, and the isotropy subgroup of G_2 at o is $\rho(Sp(1) \times Sp(1))$. Thus M is diffeomorphic to $G_2/SO(4)$ ([7] p.114 Theorem 1.8). The involutive linear transformation s in the above is given by $s = \rho(1, -1) \in G_2$. Therefore $\tilde{\sigma}(G_2) = G_2$ and the restriction $\sigma = \tilde{\sigma}|_{G_2}$ is an involutive automorphism of G_2. The Lie algebra $\mathfrak{g}_2$ of G_2 is the Lie subalgebra of $\mathfrak{so}(7)$ and $\sigma(\mathfrak{g}_2) = \mathfrak{g}_2$. We have the standard decomposition $\mathfrak{g}_2 = \mathfrak{k} + \mathfrak{m}$. Then $\mathfrak{k}$ and $\mathfrak{m}$ satisfy $\mathfrak{k} \subset \tilde{\mathfrak{k}}$ and $\mathfrak{m} \subset \tilde{\mathfrak{m}}$. The restriction $\tilde{g}|_{\mathfrak{m}}$ induces a G_2-invariant Riemannian metric on M. Thus $(G_2, SO(4), \sigma, \tilde{g}|_{\mathfrak{m}})$ is a Riemannian symmetric pair of M, and M is a totally geodesic submanifold of $\tilde{M}$.

We describe $\mathfrak{k}$ and $\mathfrak{m}$ explicitly. Let $\{i, j, k, \varepsilon, i\varepsilon, j\varepsilon, k\varepsilon\}$ be the standard basis of $\mathrm{Im}\mathbb{O}$, and $\mathfrak{sp}(1) = \mathrm{Im}\mathbb{H}$ be the Lie algebra of $Sp(1)$. We give an action of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ on $\mathrm{Im}\mathbb{O}$ as follows: For $(q_1, q_2) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ and $a + b\varepsilon \in \mathrm{Im}\mathbb{H} \oplus \mathbb{H}\varepsilon$,

$$d\rho(q_1, q_2)(a + b\varepsilon) = q_1 a - a q_1 + (q_2 b - b q_1)\varepsilon\,.$$

Let $E_{ij}(i, j = 1, \dots, 7)$ be the 7×7 matrix, where the entry in i-th row and j-th column equals 1 and all the other entries vanish. Furthermore, let us define the matrices $A_{ij} = E_{ij} - E_{ji} (i \neq j)$, which are elements of $\mathfrak{so}(7)$. The matrices

$$k_1 = A_{54} + A_{76}\,, \qquad k_2 = A_{57} + A_{64}\,, \qquad k_3 = A_{65} + A_{74}$$

are derived from the action of $d\rho(0, i)$, $d\rho(0, j)$ and $d\rho(0, k)$, respectively. Similarly, the matrices

$$k_4 = 2A_{32} + A_{45} + A_{76}\,, \qquad k_5 = 2A_{13} + A_{46} + A_{57}\,, \qquad k_6 = 2A_{21} + A_{47} + A_{65}$$

are derived from the action of $d\rho(i, 0)$, $d\rho(j, 0)$ and $d\rho(k, 0)$, respectively. It can be seen that $\{k_1, k_2, k_3\}$ and $\{k_4, k_5, k_6\}$ generate ideals $d\rho(\{0\} \oplus \mathfrak{sp}(1))$ and $d\rho(\mathfrak{sp}(1) \oplus \{0\})$ of $d\rho(\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)) = \mathfrak{k}$, respectively.

Since

$$g(xy) = g(x)g(y)$$

for $g \in G_2$ and $x, y \in \mathrm{Im}\mathbb{O}$, it holds that

$$\tilde{X}(xy) = \tilde{X}(x)y + x\tilde{X}(y)$$

for $\tilde{X} \in \mathfrak{g}_2$. Putting $x = i$ and $y = j$, we have

$$\tilde{X}(ij) = \tilde{X}(k) = \tilde{X}(i)j + i\tilde{X}(j)\,. \tag{1}$$

Let B_{ij} $(i = 1, 2, 3, 4,\ j = 1, 2, 3)$ be the 4×3 matrix, where the entry in i-th row and j-th column equals 1 and all the other entries vanish. Let $\tilde{X}$ be an element of $\mathfrak{m}$, where

$$\tilde{X} = \begin{pmatrix} 0 & -{}^tX \\ X & 0 \end{pmatrix}, \qquad X = \sum_{\substack{i=1,2,3,4,\\ j=1,2,3}} x_{ij} B_{ij}\ (x_{ij} \in \mathbb{R})\,.$$

Then the both sides of the equation (1) can be expressed as follows:

$$\tilde{X}(k) = x_{13}\varepsilon + x_{23} i\varepsilon + x_{33} j\varepsilon + x_{43} k\varepsilon\,,$$

$$\tilde{X}(i)j + i\tilde{X}(j) = (x_{31} - x_{22})\varepsilon + (x_{41} + x_{12})i\varepsilon + (-x_{11} + x_{42})j\varepsilon + (-x_{21} - x_{32})k\varepsilon\,.$$

Thus we have

$$\begin{cases} x_{13} = x_{31} - x_{22}\,, \\ x_{23} = x_{41} + x_{12}\,, \\ x_{33} = -x_{11} + x_{42}\,, \\ x_{43} = -x_{21} - x_{32}\,. \end{cases}$$

Therefore we obtain eight elements generating $\mathfrak{m}$ as follows:

$$m_1 = A_{37} + A_{51}\,, \qquad m_2 = A_{53} + A_{71}\,,$$

$$m_3 = A_{52} + A_{61}\,, \qquad m_4 = A_{37} + A_{62}\,,$$

$$m_5 = A_{63} + A_{72}\,, \qquad m_6 = A_{41} + \frac{1}{2}A_{72} - \frac{1}{2}A_{63}\,,$$

$$m_7 = A_{42} + \frac{1}{2}A_{53} - \frac{1}{2}A_{71}\,, \qquad m_8 = A_{43} + \frac{1}{2}A_{61} - \frac{1}{2}A_{52}\,.$$

4. Computation of the principal curvatures

In this section, we compute the principal curvatures of homogeneous hypersurfaces $M(\Phi)$. First, we describe the normal space $T_o^{\perp}M$ at $o = \mathrm{Im}\mathbb{H}$. We denote by $\mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$ the space of linear homomorphisms of $\mathrm{Im}\mathbb{H}$ to $\mathbb{H}$. We can naturally identify the tangent space $T_o\tilde{M} = \tilde{\mathfrak{m}}$ with $\mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$. We define an inner product $\langle , \rangle$ on $\mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$ as follows: For $\phi, \psi \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$, $\langle \phi, \psi \rangle = \langle \phi(i), \psi(i) \rangle + \langle \phi(j), \psi(j) \rangle + \langle \phi(k), \psi(k) \rangle$. Then it is related with the Killing metric $\tilde{g}$ by the equation $\langle , \rangle = \frac{1}{10}\tilde{g}$. It is known that for $\phi \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$, ϕ is a tangent vector of M if and only if $\phi(i)i + \phi(j)j + \phi(k)k = 0$. The following statement holds for the normal space $T_o^{\perp}M$:

PROPOSITION 1. *For $\psi \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$, ψ is an element of $T_o^{\perp}M$ if and only if there exists $\lambda \in \mathbb{H}$ such that $\psi(x) = \lambda x$ for $x \in \mathrm{Im}\mathbb{H}$.*

PROOF. Suppose that for $\psi \in \mathrm{Hom}_{\mathbb{R}}(\mathrm{Im}\mathbb{H}, \mathbb{H})$, there exists $\lambda \in \mathbb{H}$ such that $\psi(x) = \lambda x$ for $x \in \mathrm{Im}\mathbb{H}$, which is denoted by ψ_λ. Then we have for any $\phi \in T_oM$,

$$\begin{aligned}\langle \phi, \psi_\lambda \rangle &= \langle \phi(i), \psi_\lambda(i) \rangle + \langle \phi(j), \psi_\lambda(j) \rangle + \langle \phi(k), \psi_\lambda(k) \rangle \\ &= \langle \phi(i), \lambda i \rangle + \langle \phi(j), \lambda j \rangle + \langle \phi(k), \lambda k \rangle \\ &= -\langle \phi(i)i + \phi(j)j + \phi(k)k, \lambda \rangle = 0\,.\end{aligned}$$

Therefore ψ_λ is an element of $T_o^{\perp}M$. The map $\lambda \mapsto \psi_\lambda$ of $\mathbb{H}$ into $T_o^{\perp}M$ is an injective real linear homomorphism. Since $\dim_{\mathbb{R}} \mathbb{H} = \dim T_o^{\perp}M$, it is surjective. □

Let ψ_λ be an element of $T_o^{\perp}M$ for $\lambda \in \mathbb{H}$. For $x \in \mathrm{Im}\mathbb{H}$ and $(q_1, q_2) \in Sp(1) \times Sp(1)$, we have

$$\begin{aligned}(\mathrm{Ad}(\rho(q_1, q_2))\psi_\lambda)(x) &= \rho(q_1, q_2)\psi_\lambda(\rho(q_1, q_2)^{-1}(x)) \\ &= \rho(q_1, q_2)(\lambda(q_1^{-1} x q_1)) \\ &= q_2(\lambda q_1^{-1} x q_1)q_1^{-1} \\ &= (q_2 \lambda q_1^{-1})x\,.\end{aligned}$$

Therefore the map $\lambda \mapsto \psi_\lambda$ is equivariant with respect to $Sp(1) \times Sp(1)/\mathbb{Z}_2 = SO(4)$. In particular, the isotropy subgroup $\rho(Sp(1) \times Sp(1))$ acts transitively on the unit sphere of $T_o^{\perp}M$. Hence we see that tubular hypersurfaces around M are homogeneous by the G_2-action.

Take a normal vector $\tilde{Z} \in \tilde{\mathfrak{m}}$ which corresponds to ψ_1. Then we have $\tilde{Z} = A_{51} + A_{62} + A_{73}$. Let us consider the geodesic $\gamma : \mathbb{R} \to \tilde{M}$ defined by $\gamma(t) = (\exp t\tilde{Z}) \cdot o$. We denote by M^t the G_2-orbits through $\gamma(t)$. Then the geodesic γ intersects orthogonally all the tubular hypersurfaces M^t. Therefore the tangent vector $\dot{\gamma}(t)$ is a normal vector to M^t. We denote by $A_{\dot{\gamma}(t)}$ the shape operator of M^t with respect to $\dot{\gamma}(t)$. Putting $g(t) = \exp t\tilde{Z}$, we can express $g(t)$ by the following matrix:

$$g(t) = \begin{pmatrix} \cos t & 0 & 0 & 0 & -\sin t & 0 & 0 \\ 0 & \cos t & 0 & 0 & 0 & -\sin t & 0 \\ 0 & 0 & \cos t & 0 & 0 & 0 & -\sin t \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ \sin t & 0 & 0 & 0 & \cos t & 0 & 0 \\ 0 & \sin t & 0 & 0 & 0 & \cos t & 0 \\ 0 & 0 & \sin t & 0 & 0 & 0 & \cos t \end{pmatrix}.$$

Let $g_1(t)$, $g_2(t)$ and $g_3(t)$ be the first, second and third columns of $g(t)$, respectively. Then $\gamma(t)$ is spanned by $g_1(t)$, $g_2(t)$ and $g_3(t)$. The value of φ for $\gamma(t)$ is

$$\varphi(\gamma(t)) = \langle (\cos t)i + (\sin t)i\varepsilon, \{(\cos t)j + (\sin t)j\varepsilon\}\{(\cos t)k + (\sin t)k\varepsilon\} \rangle = \cos 3t\,.$$

Hence we have $M^t = M(\cos 3t)$.

Now we compute the principal curvatures of M^t $(0 < t < \frac{\pi}{3})$. We denote by H the subgroup of $\rho(Sp(1) \times Sp(1))$ whose adjoint representation fixes $\tilde{Z}$. Then we have $H = \{\rho(q,q) \mid q \in Sp(1)\}$ which is isomorphic to $SO(3)$, and it coincides with the isotropy subgroup of G_2-action at $\gamma(t)$ $(0 < t < \frac{\pi}{3})$. The Lie algebra $\mathfrak{h}$ of H is given by $\{d\rho(q,q) \mid q \in \mathrm{Im}\mathbb{H}\}$. We define the subspace $\mathfrak{k}''$ of $\mathfrak{k}$ by $\mathfrak{k}'' = \{d\rho(-q,q) \mid q \in \mathrm{Im}\mathbb{H}\}$. Clearly, $\mathfrak{k}''$ is an invariant subspace in $\mathfrak{k}$ by the adjoint representation of H. We put $k_4' = k_1 - k_4 = d\rho(-i,i)$, $k_5' = k_2 - k_5 = d\rho(-j,j)$, and $k_6' = k_3 - k_6 = d\rho(-k,k)$, where k_i are defined in Section 2. Then the subspace $\mathfrak{k}''$ is spanned by k_4', k_5' and k_6'. We define the subspace $\mathfrak{m}'$ of $\mathfrak{m}$ by $\mathfrak{m}' = \{\tilde{X} \in \mathfrak{m} \mid \mathrm{ad}\tilde{Z}(\tilde{X}) = [\tilde{Z}, \tilde{X}] = 0\}$ and denote by $\mathfrak{m}''$ the orthogonal complement of $\mathfrak{m}'$ in $\mathfrak{m}$. We remark that the subspace $\mathfrak{m}'$ is invariant by the Jacobi operator $R_{\tilde{Z}} = -(\mathrm{ad}\tilde{Z})^2$, however the subspace $\mathfrak{m}''$ is not. It is easily seen that $\mathfrak{m}'$ and $\mathfrak{m}''$ are invariant subspaces of $\mathfrak{m}$ by the adjoint representation of H. Thus we obtain the direct sum decomposition:

$$\mathfrak{g}_2 = \mathfrak{h} + \mathfrak{k}'' + \mathfrak{m}' + \mathfrak{m}''. \tag{2}$$

By the straightforward computation, we see that $\mathfrak{m}'$ is spanned by m_1, m_2, m_3, m_4 and m_5, and $\mathfrak{m}''$ is spanned by m_6, m_7 and m_8, where m_i are defined in Section 2.

Instead of the homogeneous hypersurfaces M^t, we consider $N^t = g(t)^{-1}M^t$ which is the orbit through o by the action of $g(t)^{-1}G_2 g(t)$. Since $g(t)^{-1}{}_*\dot{\gamma}(t) = \dot{\gamma}(0)$, $\dot{\gamma}(0)$ is a normal vector to N^t at o, and the principal curvatures of the shape operator $A_{\dot{\gamma}(t)}$ coincide with those of $A_{\dot{\gamma}(0)}$. Under the identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, we have $\dot{\gamma}(0) = \tilde{Z}$. We compute the principal curvatures of the shape operator $A_{\tilde{Z}}$ using the result of Takagi and Takahashi ([14]). Let G be a Lie transformation group of a manifold $\tilde{N}$ with the Lie algebra $\mathfrak{g}$. For $X \in \mathfrak{g}$, we denote by X^* the vector field on $\tilde{N}$ induced by the action of the 1-parameter subgroup $\exp tX$.

PROPOSITION 2 ([14] Proposition 1). *Let $\tilde{N}$ be a Riemannian manifold with the Riemannian connection ∇ and N an orbit in $\tilde{N}$ under a Lie transformation group G of isometries of $\tilde{N}$. The shape operator A_z of N for a normal vector $z \in T_p^\perp N$ at $p \in N$ is expressed as*

$$A_z(x) = -(\nabla_z X^*)_N \quad \text{for} \quad x \in T_pN\,,$$

where X is an element of the Lie algebra $\mathfrak{g}$ of G such that $X_p^* = x$ and $(\nabla_z X^*)_N$ is the tangential component of $\nabla_z X^*$ to the submanifold N at p.

We also use the following fact:

PROPOSITION 3 (cf. [11] p.185 Theorem 2.4). *Let $(\tilde{G}, \tilde{K}, \tilde{\sigma}, \tilde{g})$ be a Riemannian symmetric pair with the Riemannian symmetric Lie algebra $(\tilde{\mathfrak{g}}, \tilde{\sigma}, \tilde{g})$ and $(\tilde{M}, \tilde{g})$ the corresponding Riemannian symmetric space. For the standard decompostion $\tilde{\mathfrak{g}} = \tilde{\mathfrak{k}} + \tilde{\mathfrak{m}}$ and the*

Riemannian connection $\tilde{\nabla}$, *the following holds*: *For* $Y \in \tilde{\mathfrak{m}}$,

$$(\tilde{\nabla}_Y X^*)_o = \begin{cases} [X, Y] & (X \in \tilde{\mathfrak{k}}), \\ 0 & (X \in \tilde{\mathfrak{m}}). \end{cases}$$

We consider a Lie transformation group $G(t) = g(t)^{-1} G_2 g(t)$. Since $g(t)$ commutes with any $k \in H$, H is a subgroup of $G(t)$ and is the isotropy subgroup of $G(t)$ at o. By the equation (2), we have the direct sum decomposition of the Lie algebra $\mathfrak{g}(t)$ of $G(t)$:

$$\mathfrak{g}(t) = \mathrm{Ad}(g(t)^{-1})\mathfrak{g}_2 = \mathfrak{h} + \mathrm{Ad}(g(t)^{-1})(\mathfrak{k}'' + \mathfrak{m}' + \mathfrak{m}'') . \tag{3}$$

We abbreviate $\mathrm{Ad}(g(t)^{-1})(\mathfrak{k}'' + \mathfrak{m}' + \mathfrak{m}'')$ to $\mathfrak{n}(t)$. The tangent space $T_o N^t$ at o is generated by $\tilde{X}^*_o$ for $\tilde{X} \in \mathfrak{n}(t)$. Let $\tilde{X}_{\tilde{\mathfrak{k}}}$ and $\tilde{X}_{\tilde{\mathfrak{m}}}$ be the components of $\tilde{\mathfrak{k}}$ and $\tilde{\mathfrak{m}}$ for $\tilde{X} \in \mathfrak{n}(t)$, respectively. Then we note that $\tilde{X}^* = (\tilde{X}_{\tilde{\mathfrak{k}}})^* + (\tilde{X}_{\tilde{\mathfrak{m}}})^*$ and $(\tilde{X}^*)_o = (\tilde{X}_{\tilde{\mathfrak{m}}})^*_o$. Therefore under the identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, we have $\tilde{X}^*_o = \tilde{X}_{\tilde{\mathfrak{m}}}$. We define the subspace $\tilde{\mathfrak{n}}(t)$ of $\tilde{\mathfrak{m}}$ by $\tilde{\mathfrak{n}}(t) = \{\tilde{X}_{\tilde{\mathfrak{m}}} \mid \tilde{X} \in \mathfrak{n}(t)\}$. In fact, $\tilde{\mathfrak{n}}(t)$ does not depend on t as it is the orthogonal complement of $\tilde{Z}$ in $\tilde{\mathfrak{m}}$. So we simply denote it by $\tilde{\mathfrak{n}}$. We describe the shape operator $A_{\tilde{Z}}$ as a linear transformation of $\tilde{\mathfrak{n}}$. Since $\mathrm{ad}\tilde{Z}(\tilde{X}) = 0$ for $\tilde{X} \in \mathfrak{m}'$, we have $\mathrm{Ad}(g(t)^{-1})\tilde{X} = \tilde{X}$. Hence $\mathrm{Ad}(g(t)^{-1})\mathfrak{m}' = \mathfrak{m}' \subset \mathfrak{m} \subset \tilde{\mathfrak{m}}$. Applying Propositions 2 and 3 for $\tilde{X} \in \mathfrak{m}'$, we have

$$A_{\tilde{Z}}(\tilde{X}_{\tilde{\mathfrak{m}}}) = A_{\tilde{Z}}(\tilde{X}^*_o) = -((\tilde{\nabla}_{\tilde{Z}}(\tilde{X}^*))_o)_{N^t} = 0 .$$

Similarly, we compute $\mathrm{Ad}(g(t)^{-1})$ on $\mathfrak{k}'' + \mathfrak{m}''$ and get the followings:

$$\begin{aligned}
\tilde{X}_1 &= g^{-1}(t) m_6 g(t) \\
&= (\sin t)A_{54} + \frac{1}{2}\sin 2t(A_{32} - A_{76}) + (\cos t)A_{41} + \frac{1}{2}\cos 2t(A_{72} - A_{63}) , \\
\tilde{X}_2 &= g^{-1}(t) m_7 g(t) \\
&= (\sin t)A_{64} + \frac{1}{2}\sin 2t(A_{13} - A_{57}) + (\cos t)A_{42} + \frac{1}{2}\cos 2t(A_{53} - A_{71}) , \\
\tilde{X}_3 &= g^{-1}(t) m_8 g(t) \\
&= (\sin t)A_{74} + \frac{1}{2}\sin 2t(A_{21} - A_{65}) + (\cos t)A_{43} + \frac{1}{2}\cos 2t(A_{61} - A_{52}) , \\
\tilde{X}_4 &= g^{-1}(t) k'_4 g(t) \\
&= (2\cos^2 t)A_{23} + (2\sin^2 t)A_{67} + (2\cos t)A_{54} + (2\sin t)A_{14} + \sin 2t(A_{72} - A_{63}) , \\
\tilde{X}_5 &= g^{-1}(t) k'_5 g(t) \\
&= (2\cos^2 t)A_{31} + (2\sin^2 t)A_{75} + (2\cos t)A_{64} + (2\sin t)A_{24} + \sin 2t(A_{53} - A_{71}) , \\
\tilde{X}_6 &= g^{-1}(t) k'_6 g(t) \\
&= (2\cos^2 t)A_{12} + (2\sin^2 t)A_{56} + (2\cos t)A_{74} + (2\sin t)A_{34} + \sin 2t(A_{61} - A_{52}) .
\end{aligned}$$

By using Propositions 2 and 3, we can express the shape operators $A_{\tilde{Z}}((\tilde{X}_i)_{\tilde{\mathfrak{m}}})$ for $\tilde{X}_i$ $(i = 1, \dots, 6)$ as follows:

$$\begin{aligned} A_{\tilde{Z}}((\tilde{X}_i)_{\tilde{\mathfrak{m}}}) &= A_{\tilde{Z}}((\tilde{X}_i^*)_o) \\ &= -((\tilde{\nabla}_{\tilde{Z}}(\tilde{X}_i^*))_o)_{N^t} \\ &= -(\{\tilde{\nabla}_{\tilde{Z}}(\tilde{X}_i)^*_{\tilde{\mathfrak{k}}} + \tilde{\nabla}_{\tilde{Z}}(\tilde{X}_i)^*_{\tilde{\mathfrak{m}}}\}_o)_{N^t} \\ &= -[(\tilde{X}_i)_{\tilde{\mathfrak{k}}}, \tilde{Z}]_{\tilde{\mathfrak{n}}} \\ &= -[(\tilde{X}_i)_{\tilde{\mathfrak{k}}}, \tilde{Z}]\,. \end{aligned}$$

Here $([(\tilde{X}_i)_{\tilde{\mathfrak{k}}}, \tilde{Z}])_{\tilde{\mathfrak{n}}} = [(\tilde{X}_i)_{\tilde{\mathfrak{k}}}, \tilde{Z}]$ holds, since $\tilde{g}([(\tilde{X}_i)_{\tilde{\mathfrak{k}}}, \tilde{Z}], \tilde{Z}) = 0$. Then we obtain

$$\begin{aligned} A_{\tilde{Z}}(\tilde{X}_1)_{\tilde{\mathfrak{m}}} &= (\sin t)A_{41} + \sin 2t(A_{72} - A_{63})\,, \\ A_{\tilde{Z}}(\tilde{X}_2)_{\tilde{\mathfrak{m}}} &= (\sin t)A_{42} + \sin 2t(A_{53} - A_{71})\,, \\ A_{\tilde{Z}}(\tilde{X}_3)_{\tilde{\mathfrak{m}}} &= (\sin t)A_{43} + \sin 2t(A_{61} - A_{52})\,, \\ A_{\tilde{Z}}(\tilde{X}_4)_{\tilde{\mathfrak{m}}} &= 2\{(\cos t)A_{41} + \cos 2t(A_{63} - A_{72})\}\,, \\ A_{\tilde{Z}}(\tilde{X}_5)_{\tilde{\mathfrak{m}}} &= 2\{(\cos t)A_{42} + \cos 2t(A_{71} - A_{53})\}\,, \\ A_{\tilde{Z}}(\tilde{X}_6)_{\tilde{\mathfrak{m}}} &= 2\{(\cos t)A_{43} + \cos 2t(A_{52} - A_{61})\}\,. \end{aligned}$$

Putting $T_1 = \begin{pmatrix} \cos t & -2\sin t \\ \frac{1}{2}\cos 2t & \sin 2t \end{pmatrix}$ and $T_2 = \begin{pmatrix} \sin t & 2\cos t \\ \sin 2t & -2\cos 2t \end{pmatrix}$, we have

$$\begin{aligned} ((\tilde{X}_1)_{\tilde{\mathfrak{m}}}, (\tilde{X}_4)_{\tilde{\mathfrak{m}}}) &= (A_{41}, (A_{72} - A_{63}))T_1\,, \\ A_{\tilde{Z}}((\tilde{X}_1)_{\tilde{\mathfrak{m}}}, (\tilde{X}_4)_{\tilde{\mathfrak{m}}}) &= (A_{41}, (A_{72} - A_{63}))T_2\,. \end{aligned}$$

It follows that $A_{\tilde{Z}}(A_{41}, (A_{72} - A_{63})) = (A_{41}, (A_{72} - A_{63}))T_2T_1^{-1}$. By changing the choice of bases for the pairs $((\tilde{X}_2)_{\tilde{\mathfrak{m}}}, (\tilde{X}_5)_{\tilde{\mathfrak{m}}})$ and $((\tilde{X}_3)_{\tilde{\mathfrak{m}}}, (\tilde{X}_6)_{\tilde{\mathfrak{m}}})$, the same equations hold. The eigenvalues of the matrix

$$T_2T_1^{-1} = \frac{1}{\sin 3t}\begin{pmatrix} -\cos 3t & 2 \\ 1 & -2\cos 3t \end{pmatrix}$$

are $\frac{1}{2\sqrt{(1-\Phi^2)}}(-3\Phi \pm \sqrt{8 + \Phi^2})$, where $\Phi = \cos 3t$ $(0 < t < \frac{\pi}{3})$. We have $||\dot{\gamma}(t)|| = ||\tilde{Z}|| = \sqrt{30}$ and $(\mathrm{grad}\varphi)_{\gamma(t)} = -c\dot{\gamma}(t)$ for some $c > 0$ with respect to the Riemannian metric $\tilde{g}$ on $\tilde{M}$. Then the eigenvalues of the shape operator with respect to the unit normal vector field $\frac{1}{\sqrt{30}}\dot{\gamma}(t) = -\frac{\mathrm{grad}\varphi}{||\mathrm{grad}\varphi||}$ are given in Theorem 1.

REMARK 1. The tangent space $T_oM = \mathfrak{m}$ is not invariant by the Jacobi operator $R_{\tilde{Z}} = -(\mathrm{ad}\tilde{Z})^2$. It means that the second condition we mention in Introduction is not satisfied.

5. Applications

Austere submanifolds in Euclidean space were introduced by Harvey and Lawson ([7]). First, we give the precise definition of austerity which is found in the paper [10]. Let M^m be an m-dimensional immersed submanifold of an n-dimensional Riemannian manifold $(N^n, \tilde{g})$ with the second fundamental form II. Then M is *austere* if for any normal vector field υ, the eigenvalues of the quadratic form $II_\upsilon(X, Y) = \tilde{g}(II(X, Y), \upsilon)$ are symmetrically arranged around zero on the real line at each point. By using the result of Theorem 1, we have the following corollary.

COROLLARY 1. *The orbit of $\Phi = 0$ is an austere submanifold of* $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$.

PROOF. The principal curvatures of $M(0)$ are

$$\mu_2(0) = \frac{1}{\sqrt{15}}, \qquad \mu_3(0) = -\frac{1}{\sqrt{15}}$$

with the multiplicities $\nu_2 = 3$, $\nu_3 = 3$, respectively. Since they are in oppositely signed pairs, $M(0)$ is an austere submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. □

In 1983, J. Eells and L. Lemaire extended the notion of harmonic maps to biharmonic maps ([6]). It is an interesting subject to classify the biharmonic hypersurfaces in Riemannian symmetric spaces. Those hypersurfaces have been studied by many researchers so far. For example, see the papers [9] and [13]. In Corollary 2, we give the new example of proper biharmonic hypersurfaces in compact Riemannian symmetric spaces. We recall some basic facts concerning the biharmonic hypersurfaces in Riemannian symmetric spaces. For details and proofs, see the paper [9]. Let (M^m, g) and $(N^n, \tilde{g})$ be Riemannian manifolds and $\phi\colon M \to N$ be a smooth map with the tension field $\tau(\phi)$. A smooth map ϕ is said to be *harmonic* if it is a critical point of the *energy functional*

$$E(\phi) = \int \frac{1}{2}|d\phi|^2 dv$$

under compactly supported variations. More generally, a smooth map ϕ is said to be *biharmonic* if it is a critical point of the *bienergy functional*

$$E_2(\phi) = \int \frac{1}{2}|\tau(\phi)|^2 dv$$

under compactly supported variations. Non-harmonic biharmonic maps are called *proper biharmonic maps*.

THEOREM 2 ([9] Theorem 3). *Let $N = G/K$ be an irreducible compact semi-simple Riemannian symmetric space equipped with the Killing metric. Then a hypersurface $\phi\colon M \to G/K$ with constant mean curvature is proper biharmonic if and only if its shape operator A*

has constant square norm

$$|A|^2 = \frac{1}{2}\,.$$

Applying Theorem 2 and the result of Theorem 1, we obtain the following corollary.

COROLLARY 2. *The only proper biharmonic homogeneous hypersurfaces of* $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ *by the* G_2*-action are the orbits of*

$$\Phi = \pm\frac{1}{\sqrt{10}}\,.$$

PROOF. The square norm $|A|^2$ is computed as

$$|A|^2 = 3\{\mu_2(\Phi)\}^2 + 3\{\mu_3(\Phi)\}^2 = \frac{5\Phi^2+4}{10(1-\Phi^2)}\,.$$

Thus the biharmonic equation $|A|^2 = \frac{1}{2}$ is rewritten as $10\Phi^2 = 1$. Hence $\Phi = \pm\frac{1}{\sqrt{10}}$. □

REMARK 2. The Lie group $SO(6)$ also acts on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ as the cohomogeneity one action. It is known that there exists a unique proper biharmonic homogeneous hypersurface by the $SO(6)$-action ([9]).

6. Weakly reflective submanifolds

In this section, we show that the orbit of $\Phi = 0$ is a weakly reflective submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. The notion of weakly reflective submanifolds was introduced by Ikawa, Sakai and Tasaki in 2009 ([8]). It is an austere submanifold with a reflection for each normal direction. We give the precise definition of weakly reflective submanifolds. Let M^m be an m-dimensional submanifold of an n-dimensional Riemannian manifold N^n. For each normal vector $\xi \in T_p^{\perp}M$ at each point $p \in M$, if there exists an isometry σ_ξ of N which satisfies

$$\sigma_\xi(p) = p\,, \qquad (d\sigma_\xi)_p\xi = -\xi\,, \qquad \sigma_\xi(M) = M\,,$$

then we call M a *weakly reflective submanifold* and σ_ξ a *reflection* of M with respect to ξ. In the case where M is a hypersurface, σ_ξ is independent of the choice of ξ at each point p. We also note that if M is an extrinsic homogeneous submanifold in N, that is, an orbit of an isometric action of a Lie group on N, then it suffices to check that the condition to be a weakly reflective submanifold only at one point of M is satisfied.

PROPOSITION 4. *The orbit of* $\Phi = 0$ *is a weakly reflective submanifold of* $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$.

PROOF. Since the orbit $M(0)$ is a homogeneous submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$, it suffices to check the condition to be a weakly reflective submanifold only at one point $p \in M(0)$, whose positively oriented orthonormal basis is $\{i, j, \varepsilon\}$. We take a normal vector $\xi = \mathrm{grad}\varphi \in T_p^{\perp}M(0)$ at $p \in M(0)$. Let σ_ξ be an isometry of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ defined by

$$\sigma_\xi = -s = \begin{pmatrix} -E_3 & 0 \\ 0 & E_4 \end{pmatrix},$$

where s is defined in Section 2, and E_i is the $i \times i$ unit matrix. Then we have

$$\sigma_\xi(p) = \sigma_\xi\{i, j, \varepsilon\} = \{-i, -j, \varepsilon\} = \{i, j, \varepsilon\} = p\,.$$

Since $s = \rho(1, -1)$ is an element of G_2 and the degree of φ is odd, it holds that $\sigma_\xi^*\varphi = (-s)^*\varphi = ((-E_7)s)^*\varphi = s^*(-E_7)^*\varphi = s^*(-\varphi) = -s^*\varphi = -\varphi$. As it can be seen that the isometry σ_ξ satisfies $(d\sigma_\xi)_p(\mathrm{grad}(\sigma_\xi^*\varphi)_p) = (\mathrm{grad}\varphi)_p$, the second equation $(d\sigma_\xi)_p\xi = -\xi$ holds. We also see that $\sigma_\xi(M(\Phi)) = M(-\Phi)$. In particular, we have $\sigma_\xi(M(0)) = M(0)$. Therefore the orbit $M(0)$ is a weakly reflective submanifold of $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. $\square$

REMARK 3. It is known that a weakly reflective submanifold is an austere submanifold (cf. [8] p.440 Proposition 2.5). Thus the austerity of $M(0)$ follows from Proposition 4.

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