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Psyquandles, Singular Knots and Pseudoknots

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Abstract. We generalize the notion of biquandles to *psyquandles* and use these to define invariants of oriented singular links and pseudolinks. In addition to psyquandle counting invariants, we introduce Alexander psyquandles and corresponding invariants such as Alexander psyquandle polynomials and Alexander-Gröbner psyquandle invariants of oriented singular knots and links. We consider the relationship between Alexander psyquandle colorings of pseudolinks and p-colorings of pseudolinks. As a special case we define a generalization of the Alexander polynomial for oriented singular links and pseudolinks we call the *Jablan polynomial* and compute the invariant for all pseudoknots with up to five crossings and all 2-bouquet graphs with up to 6 classical crossings.

1. Introduction

Biquandles are algebraic structures whose axioms are motivated by the oriented Reidemeister moves in knot theory. First suggested in the mid 1990s [6] and later developed in the 2000s [12, 5, 4], biquandles have been used since their introduction to define invariants of classical and virtual oriented knots and links, [2, 4, 5, 14, 15, 17, 16].

Singular knots and links are 4-valent spatial graphs considered up to *rigid vertex isotopy*, where we may regard a vertex as the result of two strands of a knot or link getting stuck together in a fixed position. Singular knots and links are important in the study of *Vassiliev invariants*; see [7, 20, 21]. In particular, a singular knot or link with exactly one singular crossing is a 2-*bouquet graph*.

Pseudoknots are knots whose diagrams consist of usual crossings and *precrossings* – classical crossings where we cannot tell which strand goes on top. This definition, statistical in nature, is motivated by applications in molecular biology, such as modeling knotted DNA, where data often comes inconclusive with respect to which crossing it represents, [9, 10, 11].

Motivated by effectiveness of biquandles in distinguishing oriented knots and links, we introduce *psyquandles* and use them to define invariants of oriented singular knots and links and oriented pseudoknots and pseudolinks. A psyquandle is a biquandle with additional structure in the form of operations at singular crossings or precrossings.

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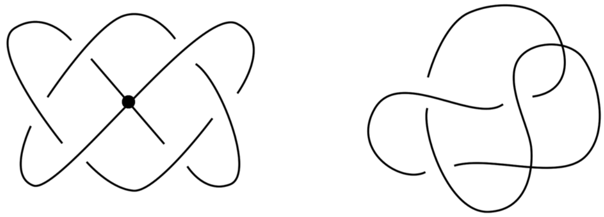
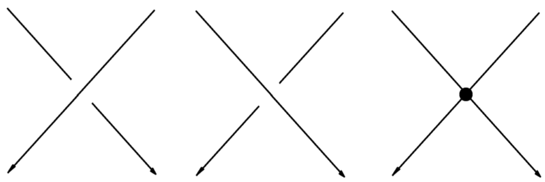


2-bouquet graph 6_7^k psuedoknot $5_2.7$

The paper is organized as follows. In Section 2 we review the basic combinatorics of oriented singular knots and links and pseudoknots and pseudolinks. In Section 3 we introduce psyquandles and prove that psyquandle colorings of singular knots and links and of pseudoknots and pseudolinks define invariants. In Section 4 we introduce a particular type of psyquandle we call *Alexander psyquandles* and use these to define analogs of the Alexander polynomials and Alexander-Gröbner invariants for oriented singular knots and links and for oriented pseudoknots and pseudolinks. We consider the relationship between Alexander psyquandle colorings and p-colorings of pseudolinks as defined in [9]. We introduce the *Jablan polynomial* which generalizes the Alexander polynomial to the case of pseudolinks and singular links. We end in Section 5 with some questions for future work.

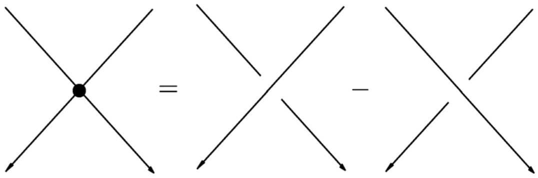
2. Singular Knots and Pseudoknots

Singular knots and links are rigid vertex isotopy classes of 4-valent spatial graphs. That is, a singular link diagram has classical crossings and 4-valent vertices which are required to maintain a fixed cyclic ordering around the vertices. Geometrically, we can think of singular links as links with transverse self-intersections, each of which is fixed inside a small neighborhood. An *oriented singular knot or link* has oriented strands which pass through at each crossing and vertex; that is, the orientations are as pictured below.



Singular knot theory finds applications in the study of *Vassiliev invariants*, integer-valued invariants of singular knots and links which satisfy the *Vassiliev skein relation*:
See [7, 20, 21] for more about singular knots.

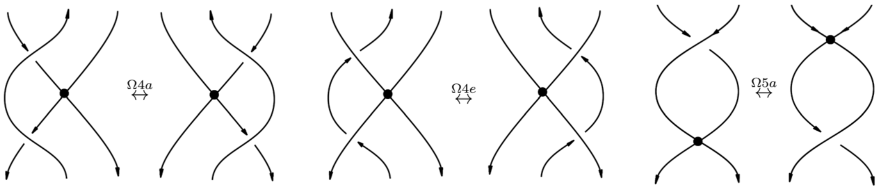
EXAMPLE 1. A 2-*bouquet graph* is a singular knot with exactly one singular crossing. 2-Bouquet graphs come in two types: K-type 2-bouquet graphs form knots if the sin-



gular crossing is replaced with a classical crossing, while L-type 2-bouquet graphs form 2-component links when the singular crossing is replaced with a classical crossing. The second listed author classified 2-bouquet graphs with up to six classical crossings in [18].

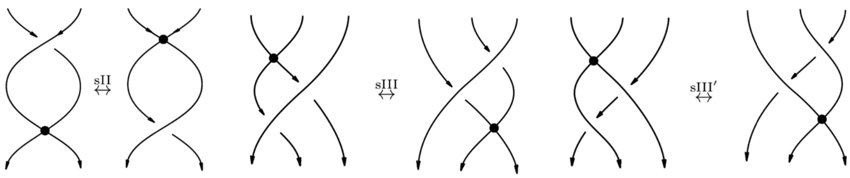
In [1] a generating set of three oriented singular moves is identified and shown to generate the remaining oriented singular moves:

THEOREM 1 (BEHY). *In the presence of the oriented classical Reidemeister moves, the three moves below generate the complete set of oriented singular moves.*

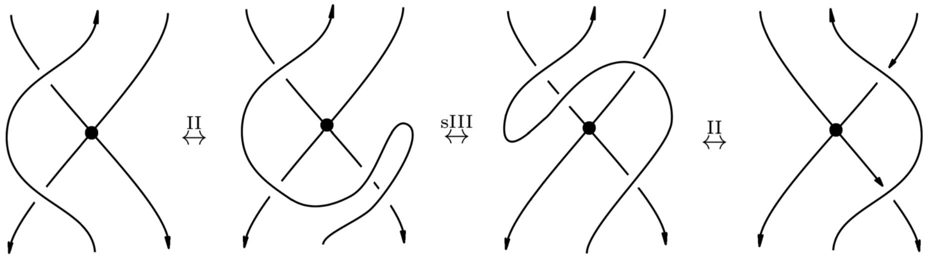


For our purposes it will be easier to use an alternative generating set of singular moves.

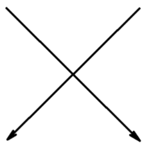
PROPOSITION 1. *In the presence of classical Reidemeister moves, the three moves below generate the complete set of oriented singular moves.*



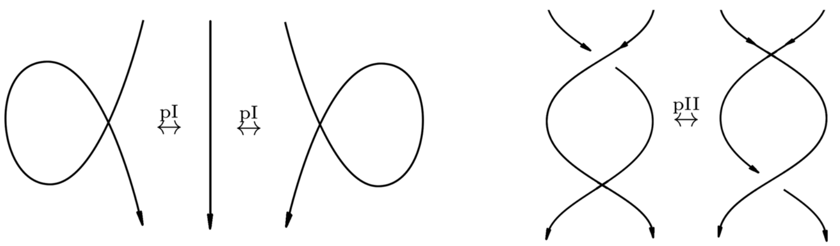
PROOF. It suffices to show that the moves in Theorem 1 can be obtained using our preferred moves and the oriented classical Reidemeister moves. Since move sII is the same as move $\Omega 5a$, we need only to show that moves $\Omega 4a$ and $\Omega 4e$ can be obtained using the classical Reidemeister moves and moves sII, sIII and sIII$'$. Then consider the case of $\Omega 4a$; we will obtain it using sIII and two classical Reidemeister II moves.
The case of sIII$'$ $\Rightarrow$ $\Omega 4e$ is similar. $\square$

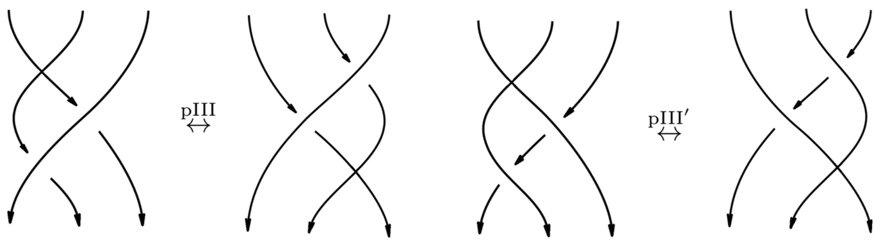


Pseudoknots are knots and links whose diagrams in addition to classical crossings include some *precrossings*, classical crossings in which it is unknown which strand goes over and which strand goes under. While the concept originated in biology where limited resolution in pictures of knotted molecules makes it difficult to tell which strand in on top, the current mathematical study of pseudoknots was initiated in [8] and continued in papers such as [9, 10, 11]. A precrossing is drawn as an undecorated self-intersection:



The Reidemeister moves for pseudoknots (see [11] etc.) are, conveniently, very similar to our preferred set of Reidemeister moves for oriented singular knots:

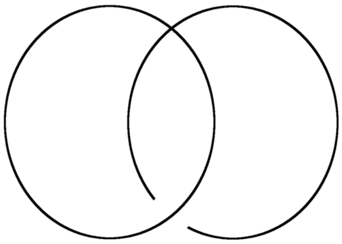




Indeed, after replacing singular crossings with precrossings, the only difference is the addition of a Reidemeister I-style move with a precrossing, no analog of which exists for singular knots.

A *resolution* of a pseudolink diagram is an assignment of classical crossing type to each of the precrossings in the diagram. A powerful invariant of pseudolinks is the *weighted resolution set* or *WeRe set*, the discrete probability distribution consisting of the set of resolution link types and their associated probabilities with the assumption that both crossing resolutions are equally probable.

EXAMPLE 2. The pseudolink below has the listed WeRe set where 0_2 is the unlink of two components and L2a1 is the Hopf link.



$$\left\{\left(0_2, \frac{1}{2}\right), \left(L2a1, \frac{1}{2}\right)\right\}$$

3. Psyquandles

The similarity of the singular Reidemeister moves with the pseudoknot Reidemeister moves suggests introducing a single algebraic structure for coloring these objects with new operations at the singular crossings or precrossings.

Recall (see [4] for example) that a *biquandle* is a set X with operations $\,\underline{\triangleright}\,, \,\overline{\triangleright}\, : X \times X \to X$ satisfying

(i) For all $x \in X$, $x \,\underline{\triangleright}\, x = x \,\overline{\triangleright}\, x$,

(ii) For all $x, y \in X$, the maps $\alpha_y, \beta, y : X \to X$ and $S : X \times X \to X \times X$ defined by

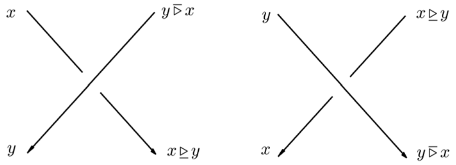
$$\alpha_y(x) = x \,\overline{\triangleright}\, y,\ \beta_y(x) = x \,\underline{\triangleright}\, y \text{ and } S(x, y) = (S_1(x, y), S_2(x, y)) = (y \,\overline{\triangleright}\, x, x \,\underline{\triangleright}\, y)$$

are invertible, and

(iii) For all $x, y, z \in X$ the *exchange laws* are satisfied:

$$\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\underline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\underline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\triangleright}\, y) &= (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z)\,.
\end{aligned}$$

The biquandle axioms are motivated by the classical Reidemeister moves where we label the *semiarcs* in a knot diagram (the edges in the graph obtained from the diagram by making each crossing a 4-valent vertex) as shown:



Axiom (ii) is equivalent to the *adjacent labels rule*, which says that the colors of any two adjacent semiarcs determine the colors of the other two.

DEFINITION 1. A *psyquandle* is a biquandle X with two additional binary operations $\underline{\bullet}, \overline{\bullet} : X \times X \to X$ satisfying the conditions

(p/si) For all $x, y \in X$, the maps $\alpha'_y, \beta'_y : X \to X$ and $S' : X \times X \to X \times X$ defined by

$$\alpha'_y(x) = x \,\overline{\bullet}\, y, \;\; \beta'_y(x) = x \,\underline{\bullet}\, y \text{ and } S'(x, y) = (S'_1(x, y), S'_2(x, y)) = (y \,\overline{\bullet}\, x, x \,\underline{\bullet}\, y)$$

are invertible,

(p/sii) For all $x, y \in X$ there exist unique $w, z \in X$ such that

$$\begin{aligned} x \,\underline{\triangleright}\, y &= z \,\overline{\bullet}\, y \\ y \,\overline{\triangleright}\, x &= w \,\overline{\bullet}\, x \\ w \,\underline{\triangleright}\, z &= y \,\underline{\bullet}\, z \\ z \,\overline{\triangleright}\, w &= x \,\underline{\bullet}\, w \end{aligned}$$

and

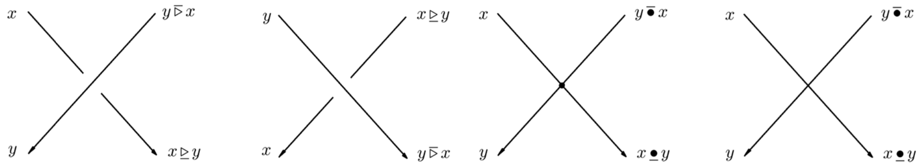
(p/siii) For all $x, y, z \in X$ we have the *mixed exchange laws*:

$$\begin{aligned} (x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\bullet}\, y) &= (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\bullet}\, z) \\ (x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\bullet}\, y) &= (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\underline{\bullet}\, z) \\ (x \,\overline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\overline{\triangleright}\, y) &= (x \,\overline{\bullet}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z) \\ (x \,\underline{\triangleright}\, y) \,\underline{\bullet}\, (z \,\underline{\triangleright}\, y) &= (x \,\underline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\ (x \,\overline{\triangleright}\, y) \,\underline{\bullet}\, (z \,\overline{\triangleright}\, y) &= (x \,\underline{\bullet}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z) \\ (x \,\underline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\underline{\triangleright}\, y) &= (x \,\overline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \end{aligned}$$

A psyquandle is *pI-adequate* if it additionally satisfies for all $x \in X$

$$x \,\underline{\bullet}\, x = x \,\overline{\bullet}\, x\,.$$

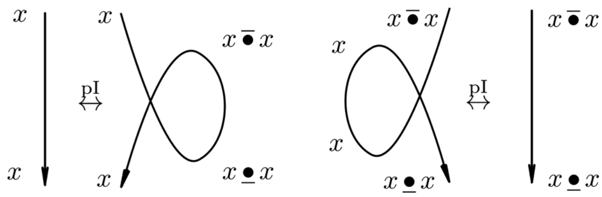
DEFINITION 2. Let X be a psyquandle (respectively, a pI-adequate psyquandle) and L an oriented singular link (respectively, oriented pseudolink) diagram. Then an X*-coloring* of L is an assignment of elements of $X$to the semiarcs in L such that every crossing we have the following:



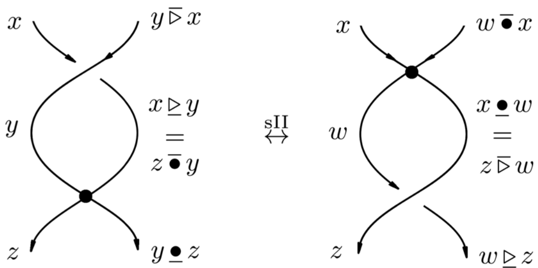
The psyquandle axioms are motivated by the moves {p/sII, p/sIII, p/sIII′} (and in the case of pI-adequate psyquandles, move pI) using the coloring rule in Definition 2 at singular crossings and precrossings. In particular, we have:

THEOREM 2. *Let L be an oriented singular link* (*respectively*, *pseudolink*) *diagram. For any finite psyquandle X, the number of X-colorings of L is preserved by Reidemeister moves and hence defines an invariant $\Phi_X^{\mathbf{Z}}(L)$ called the psyquandle counting invariant.*

PROOF. We verify for each of the moves pI, p/sII, p/sIII and p/sIII′. First, move pI requires $x \,\underline{\bullet}\, x = x \,\overline{\bullet}\, x$ for all $x \in X$:

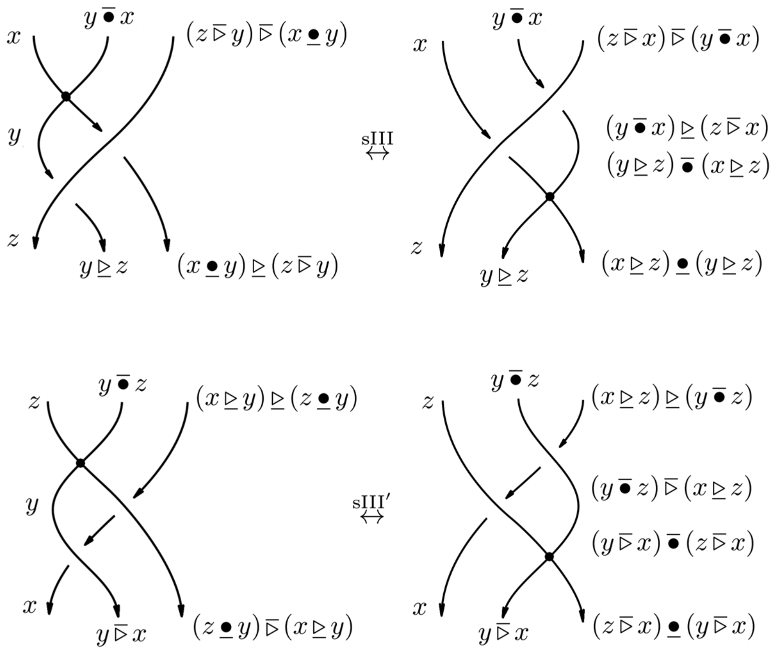


Next, let us consider the p/sII move.



We want each X-coloring of the diagram on the left to correspond to exactly one X-coloring of the diagram on the right. The fact that $\trianglerighteq$, $\overline{\triangleright}$, $\underline{\bullet}$ and $\overline{\bullet}$ satisfy the adjacent labels rule implies that the colors x, y determine all the semiarc colors in the left diagram and the requirement that colors agree on the boundary of the neighborhood of the move the implies that x, y also determine the colors in the right diagram. Then for each pair $x, y \in X$ there should be unique z, w satisfying the pictured conditions.

Finally, for the p/sIII and p/sIII′ moves, we compare semiarc labels on both sides of the moves. □



EXAMPLE 3. Let X be a biquandle. Replacing the singular/precrossing with a positive crossing shows that that setting $\overline{\bullet} = \overline{\triangleright}$ and $\underline{\bullet} = \underline{\triangleright}$ yields a pI-adequate psyquandle, and replacing it with a negative crossing shows that setting $\overline{\bullet} = \underline{\triangleright}$ and $\underline{\bullet} = \overline{\triangleright}$ yields a pI-adequate psyquandle.

DEFINITION 3. A **pure psyquandle** is a psyquandle with trivial classical operations, i.e. a psyquandle X such that

$$x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = x$$

for all $x, y \in X$. We note that the mixed exchange laws are automatically satisfied in this case, so every pair of operations $x \,\underline{\bullet}\, y$, $x \,\overline{\bullet}\, y$ satisfying (p/si) and (p/sii) is a pure psyquandle.

EXAMPLE 4. Let X be a set and $\sigma, \tau : X \to X$ bijections. Then $x \,\underline{\triangleright}\, y = x \,\overline{\triangleright}\, y = \tau(x)$ defines a biquandle operation called a *constant action biquandle*. Defining $x \,\underline{\bullet}\, y = x \,\overline{\bullet}\, y = \sigma(x)$ makes this a pI-adequate psyquandle we call a *constant action psyquandle* provided

$$\sigma^{-1}\tau = \tau^{-1}\sigma \quad \text{and} \quad \sigma\tau = \tau\sigma\,.$$

We verify the axioms:

(p/si) $\alpha' = \beta' = \sigma$ is invertible and $S'^{-1}(x, y) = (\tau^{-1}(y), \tau^{-1}(x))$,

(p/sii) Given $x, y \in X$, define $z = \tau^{-1}\sigma(x)$ and $w = \tau^{-1}\sigma(y)$. Then we have

$$\begin{array}{rcccccc}
x \,\underline{\triangleright}\, y & = & \sigma(x) & = & \tau(\tau^{-1}(\sigma(x))) & = & z \,\overline{\triangleright}\, y \\
y \,\overline{\triangleright}\, x & = & \sigma(y) & = & \tau(\tau^{-1}(\sigma(y))) & = & w \,\overline{\triangleright}\, x \\
w \,\underline{\triangleright}\, z & = & \sigma(\tau^{-1}(\sigma(y))) & = & \tau(y) & = & y \,\underline{\bullet}\, z \\
z \,\overline{\triangleright}\, w & = & \sigma(\tau^{-1}(\sigma(x))) & = & \tau(x) & = & x \,\overline{\bullet}\, w
\end{array}$$

and

(p/siii) For all $x, y, z \in X$ we have

$$\begin{array}{rcccccl}
(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\bullet}\, y) & = & \sigma^2(x) & = & \sigma^2(x) & = & (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\bullet}\, z) \\
(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\bullet}\, y) & = & \sigma^2(x) & = & \sigma^2(x) & = & (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\underline{\bullet}\, z) \\
(x \,\overline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\overline{\triangleright}\, y) & = & \sigma(\tau(x)) & = & \tau(\sigma(x)) & = & (x \,\overline{\bullet}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z) \\
(x \,\underline{\triangleright}\, y) \,\underline{\bullet}\, (z \,\underline{\triangleright}\, y) & = & \sigma(\tau(x)) & = & \tau(\sigma(x)) & = & (x \,\underline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z) \\
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(x \,\underline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\underline{\triangleright}\, y) & = & \sigma(\tau(x)) & = & \tau(\sigma(x)) & = & (x \,\overline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z)
\end{array}$$

as required.

EXAMPLE 5. We can express a psyquandle structure on a finite set $X = \{x_1, \dots, x_n\}$ with an $n \times 4n$ matrix encoding the operation tables of $\underline{\triangleright}$, $\overline{\triangleright}$, $\underline{\bullet}$, $\overline{\bullet}$ where the (j, k) entry m in the matrix satisfies

$$x_m = \begin{cases} x_j \,\underline{\triangleright}\, x_k & 1 \le k \le n \\ x_j \,\overline{\triangleright}\, x_k & n+1 \le k \le 2n \\ x_j \,\underline{\bullet}\, x_k & 2n+1 \le k \le 3n \\ x_j \,\overline{\bullet}\, x_k & 3n+1 \le k \le 4n \end{cases}$$

For instance, the constant action psyquandle on $X = \{x_1, x_2, x_3, x_4\}$ where $\sigma = (12)$ and $\tau = (34)$ has operation matrix

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3
\end{array}\right]$$

EXAMPLE 6. Let $X = \{1, 2, 3\}$. The operation matrix

$$\left[\begin{array}{ccc|ccc|ccc|ccc}
1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 1 & 3 & 1 & 1 & 1 & 1 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 1 & 3 & 2 & 3 & 3
\end{array}\right]$$

defines a pure psyquandle which is not pI-adequate.

DEFINITION 4. Let D be an oriented singular or pseudolink diagram representing an oriented singular or pseudolink L and let $G = \{g_1, \dots, g_n\}$ be a set of symbols corresponding to the semiarcs in D. We define the *fundamental psyquandle* of D is the usual universal algebraic way, namely:

- The set $W(G)$ of *psyquandle words* in G is defined recursively by the rules
  - (i) $G \subset W(G)$ and
  - (ii) $x, y \in W(G)$ implies

$$x \,\underline{\triangleright}\, y,\ x \,\overline{\triangleright}\, y,\ x \,\underline{\bullet}\, y,\ x \,\overline{\bullet}\, y,\ \alpha_y^{-1}(x),\ \alpha_y'^{-1}(x),\ \beta_y^{-1}(x),\ \beta_y'^{-1}(x)\,,$$

$$S_1(x, y),\ S_2(x, y),\ S_1'(x, y),\ S_2'(x, y),\ w(x, y),\ z(x, y) \in W(G)\,,$$

- We introduce an equivalence relation on $W(G)$ generated by relations representing the psyquandle axioms, e.g.

$$(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\bullet}\, y) \sim (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\bullet}\, z),\ \ x \,\underline{\triangleright}\, y \sim z(x, y) \,\overline{\bullet}\, y, \text{ etc.},$$

- The *free psyquandle on G* is the set of equivalence classes of $W(G)$ modulo this equivalence relation; if we include axiom (pi) we obtain the *free pI-adequate psyquandle*, and
- Including the crossing relations form Definition 2 in our equivalence relation yields the *fundamental psyquandle* of D, denoted $\mathcal{P}(D)$ or $\mathcal{P}_I(D)$ for the fundamental pI-adequate psyquandle.

THEOREM 3. *The isomorphism class $\mathcal{P}(L)$ of $\mathcal{P}(D)$ is an invariant of oriented singular links, and the isomorphism class $\mathcal{P}_I(L)$ of $\mathcal{P}_I(D)$ is an invariant of oriented pseudolinks.*

PROOF. By construction, Reidemeister moves on diagrams induce Tietze moves on presentations of $\mathcal{P}(D)$ and $\mathcal{P}_I(D)$ respectively, resulting in isomorphic psyquandles. □

Psyquandles form a category with psyquandles as objects and *psyquandle homomorphisms*, maps $f : X \to Y$ satisfying

$$f(x \,\underline{\triangleright}\, y) = f(x) \,\underline{\triangleright}\, f(y)\,, \quad f(x \,\overline{\triangleright}\, y) = f(x) \,\overline{\triangleright}\, f(y)\,,$$

$$f(x \,\underline{\bullet}\, y) = f(x) \,\underline{\bullet}\, f(y) \quad \text{and} \quad f(x \,\overline{\bullet}\, y) = f(x) \,\overline{\bullet}\, f(y)$$

as morphisms.

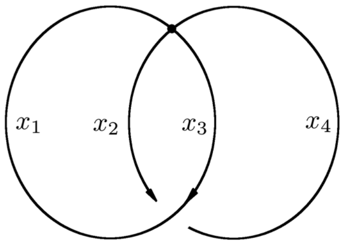
Let D be an oriented singular link or pseudolink diagram and let X be a finite psyquandle. An assignment of elements of X to the semiarcs in D defines a homomorphism $f : \mathcal{P}(D) \to X$ if and only if the coloring conditions in Definition 2 are satisfied at every crossing; we will refer to such an assignment as an *X-coloring* of D. Thus, we can compute the set of psyquandle homomorphisms $\text{Hom}(\mathcal{P}(L), X)$ for an oriented singular link or pseudolink L by computing the set of X-colorings of a diagram D representing L. More precisely, fixing an ordering of the semiarcs in D gives us a way to represent homomorphisms

$f \in \mathrm{Hom}(\mathcal{P}(L), X)$ concretely as ordered tuples of elements of X. The number of such colorings is an integer-valued invariant of singular links and pseudolinks we call the *psyquandle counting invariant*, denoted $\Phi_X^{\mathbf{Z}}(L) = |\mathrm{Hom}(\mathcal{P}(L), X)|$.

EXAMPLE 7. Consider the psyquandle X with operation matrix

$$\left[\begin{array}{cc|cc|cc|cc} 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

The 2-bouquet graph 1_1^l below

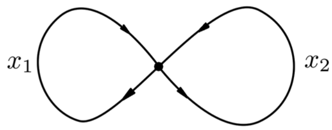


1_1^l

has 4 X-colorings, each of which we can identify explicitly as a 4-tuple $(f(x_1), f(x_2), f(x_3), f(x_4))$:

$$\mathrm{Hom}(\mathcal{P}(1_1^l), X) = \{(1,1,2,2), (1,2,2,1), (2,1,2,1), (2,2,1,1)\}.$$

This distinguishes this link from the 2-bouquet graph 0_1^k



0_1^k

which has only two X-colorings

$$\mathrm{Hom}(\mathcal{P}(0_1^k), X) = \{(1,2), (2,1)\}.$$

EXAMPLE 8. Using our custom `Python` code, we computed the counting invariant for the 2-bouquet graphs (with choices of orientation) in [18] using the psyquandle with operation matrix

$$\left[\begin{array}{cccccc|cccccc|cccccc|cccccc} 2 & 4 & 4 & 6 & 6 & 2 & 2 & 6 & 2 & 6 & 2 & 6 & 2 & 4 & 2 & 6 & 2 & 2 & 2 & 6 & 4 & 6 & 6 & 6 \\ 3 & 5 & 5 & 1 & 1 & 3 & 1 & 5 & 1 & 5 & 1 & 5 & 3 & 5 & 5 & 5 & 1 & 5 & 1 & 5 & 1 & 1 & 1 & 3 \\ 4 & 6 & 6 & 2 & 2 & 4 & 6 & 4 & 6 & 4 & 6 & 4 & 6 & 6 & 6 & 2 & 6 & 4 & 4 & 4 & 6 & 4 & 2 & 4 \\ 5 & 1 & 1 & 3 & 3 & 5 & 5 & 3 & 5 & 3 & 5 & 3 & 5 & 3 & 1 & 3 & 3 & 3 & 5 & 1 & 5 & 3 & 5 & 5 \\ 6 & 2 & 2 & 4 & 4 & 6 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 2 & 4 & 4 & 4 & 6 & 6 & 2 & 2 & 2 & 4 & 2 \\ 1 & 3 & 3 & 5 & 5 & 1 & 3 & 1 & 3 & 1 & 3 & 1 & 1 & 1 & 3 & 1 & 5 & 1 & 3 & 3 & 3 & 5 & 3 & 1 \end{array}\right].$$

The results are collected in the table.

$\Phi_X^{\mathbf{Z}}(L)$	L
6	$1_1^k, 3_1^k, 4_1^k, 4_2^k, 5_1^k, 5_4^k, 5_5^k, 5_6^k, 5_7^k, 5_8^k, 6_1^k, 6_2^k, 6_3^k, 6_4^k, 6_5^k, 6_6^k, 6_8^k, 6_9^k, 6_{10}^k, 6_{11}^k,$ $6_{12}^k, 6_{13}^k, 6_{14}^k, 6_{15}^k, 6_{18}^k$
8	$5_3^l, 6_5^l$
12	$3_1^l, 4_1^l, 5_2^l, 5_3^l, 6_1^l, 6_2^l, 6_6^l$
18	$2_1^k, 5_2^k, 5_3^k, 6_7^k, 6_{16}^k, 6_{17}^k, 6_{19}^k$
24	$1_1^l, 5_1^l, 6_3^l, 6_5^l, 6_7^l, 6_8^l, 6_9^l, 6_{10}^l, 6_{11}^l$
36	$6_4^l, 6_{12}^l.$

EXAMPLE 9. Noticing that the psyquandle in example 8 is pI-adequate since the two right blocks have the same diagonal, we computed the counting invariant for a choice of orientations for the pseudoknots in [9]. The results are collected in the table.

$\Phi_X^{\mathbf{Z}}(L)$	L
6	$3_1.2, 3_1.3, 4_1.4, 4_1.3, 4_1.4, 4_1.5, 5_1.1, 5_1.3, 5_1.4, 5_2.1, 5_2.2, 5_2.3, 5_2.4,$ $5_2.5, 5_2.6, 5_2.7, 5_2.8, 5_2.9, 5_2.10$
18	$3_1.1, 4_1.1, 5_1.2, 5_1.5.$

4. Alexander Psyquandles

Let $\Lambda = \mathbf{Z}[t^{\pm 1}, s^{\pm 1}]$. Any Λ-module X is a biquandle under the operations

$$x \underline{\triangleright} y = tx + (s-t)y \quad \text{and} \quad x \overline{\triangleright} y = sx$$

known as an *Alexander biquandle* (see [4] or [12]). Interpreting the fundamental biquandle of a knot or link as an Alexander biquandle yields invariants including the Alexander polynomials and generalizations such as the Sawollek polynomials [12, 19] and the Alexander-Gröbner invariants [3]. In this section we will extend this definition to the case of psyquandles and as an application define notions of Alexander polynomials, Alexander-Gröbner invariants and a special case we call *Jablan polynomials* for singular and pseudoknots and links.

PROPOSITION 2. *Let* $\Lambda' = \mathbf{Z}[t^{\pm 1}, s^{\pm 1}, a^{\pm 1}, b^{\pm 1}]/(s+t-a-b)$ *and let* X *be a*

Λ'-module. The operations

$$
\begin{aligned}
x \,\underline{\triangleright}\, y &= tx + (s-t)y \\
x \,\overline{\triangleright}\, y &= sx \\
x \,\underline{\bullet}\, y &= ax + (s-a)y \\
x \,\overline{\bullet}\, y &= bx + (s-b)y
\end{aligned}
$$

make X a pI-adequate psyquandle called an Alexander psyquandle.

PROOF. We verify the axioms. First, checking pI-adequacy, we have

$$x \,\overline{\bullet}\, x = bx + (s-b)x = sx = ax + (s-a)x = x \,\underline{\bullet}\, x\,.$$

Next, for axiom (p/si) if we define

$$
\begin{aligned}
\alpha'_y(x) &= bx + (s-b)y\,, \\
\beta'_y(x) &= ax + (s-a)y\,, \\
S'(x,y) &= ((s-b)x + by,\ ax + (s-a)y)
\end{aligned}
$$

then setting

$$
\begin{aligned}
\alpha'^{-1}_y(x) &= b^{-1}(x - (s-b)y), \\
\beta'^{-1}_y(x) &= a^{-1}(x - (s-a)y), \\
S'^{-1}(x,y) &= ((s^{-1} - bs^{-1}t^{-1})x + bs^{-1}t^{-1}y,\ as^{-1}t^{-1}x + (s^{-1} - as^{-1}t^{-1})y)
\end{aligned}
$$

yields the inverse maps. Let us verify:

$$
\begin{aligned}
\alpha'_y(bx + (s-b)y) &= b^{-1}(bx + (s-b)y - (s-b)y)) = x \\
\beta'^{-1}_y(ax + (s-a)y) &= a^{-1}(ax + (s-a)y - (s-a)y) = x
\end{aligned}
$$

and writing

$$S'^{-1}((s-b)x + by,\ ax + (s-a)y) = (Ax + By, Cx + Dy)$$

we compute

$$
\begin{aligned}
A &= (s^{-1} - bs^{-1}t^{-1})(s-b) + abs^{-1}t^{-1} \\
&= 1 - bs^{-1} - bt^{-1} + b^2s^{-1}t^{-1} + abs^{-1}t^{-1} \\
&= 1 - bs^{-1} - bt^{-1} + (s+t-a)bs^{-1}t^{-1} + abs^{-1}t^{-1} \\
&= 1 - bs^{-1} - bt^{-1} + bt^{-1} + bs^{-1} - abs^{-1}t^{-1} + abs^{-1}t^{-1} \\
&= 1, \\
B &= (s^{-1} - bs^{-1}t^{-1})b + bs^{-1}t^{-1}(s-a)
\end{aligned}
$$

$$\begin{aligned}
&= bs^{-1} - b^2 s^{-1}t^{-1} + bt^{-1} - abs^{-1}t^{-1}\\
&= bs^{-1} - (s+t-a)bs^{-1}t^{-1} + bt^{-1} - abs^{-1}t^{-1}\\
&= bs^{-1} - bt^{-1} + bs^{-1} + abs^{-1}t^{-1} + bt^{-1} - abs^{-1}t^{-1}\\
&= 0,
\end{aligned}$$

$$\begin{aligned}
C &= as^{-1}t^{-1}(s-b) + (s^{-1} - as^{-1}t^{-1})a\\
&= at^{-1} - abs^{-1}t^{-1} + (s^{-1} - (s+t-b)s^{-1}t^{-1})a\\
&= at^{-1} - abs^{-1}t^{-1} + as^{-1} - at^{-1} - as^{-1} + abs^{-1}t^{-1}\\
&= 0 \quad \text{and}\\
D &= abs^{-1}t^{-1} + (s^{-1} - as^{-1}t^{-1})(s-a)\\
&= abs^{-1}t^{-1} + 1 - at^{-1} - as^{-1} + a^2 s^{-1}t^{-1})\\
&= abs^{-1}t^{-1} + 1 - at^{-1} - as^{-1} + a(s+t-b)s^{-1}t^{-1})\\
&= abs^{-1}t^{-1} + 1 - at^{-1} - as^{-1} + at^{-1} + as^{-1} - abs^{-1}t^{-1}\\
&= 1
\end{aligned}$$

and axiom (p/si) is satisfied.

To verify axiom (p/sii), we observe that given x, y we can define

$$\begin{aligned}
w &= b^{-1}(b-s)x + b^{-1}sy\\
z &= b^{-1}tx + b^{-1}(s-a)y
\end{aligned}$$

and then we have

$$\begin{aligned}
bw + (s-b)x &= b(b^{-1}(b-s)x + b^{-1}sy) + (s-b)x\\
&= (b-s)x + (s-b)x + sy\\
&= sy,
\end{aligned}$$

$$\begin{aligned}
tw + (s-t)z &= t(b^{-1}(b-s)x + b^{-1}sy) + (s-t)(b^{-1}tx + b^{-1}(s-a)y)\\
&= b^{-1}(tb - ts + ts - t^2)x + b^{-1}(st + s^2 - st - as + at)y\\
&= b^{-1}t(b-t)x + b^{-1}(s^2 - as + at)y\\
&= (s-a)b^{-1}tx + b^{-1}(s^2 - 2as + a^2 + as + at - a^2)y\\
&= (s-a)b^{-1}tx + b^{-1}(s^2 - 2as + a^2 + a(s+t-a))y\\
&= (s-a)b^{-1}tx + (b^{-1}(s-a)^2 + a)y\\
&= ay + (s-a)(b^{-1}tx + b^{-1}(s-a)y)\\
&= ay + (s-a)z,
\end{aligned}$$

$$bz + (s-b)y = b(b^{-1}tx + b^{-1}(s-a)y) + (s-b)y$$

$$
\begin{aligned}
&= tx + (s-a)y + (a-t)y \\
&= tx + (s-t)y\,,
\end{aligned}
$$

$$
\begin{aligned}
ax + (s-a)w &= ax + (s-a)(b^{-1}(b-s)x + b^{-1}sy) \\
&= b^{-1}(ab + (s-a)(b-s))x + b^{-1}(s-a)sy \\
&= b^{-1}(ab + sb - ab - s^2 + as)x + b^{-1}(s-a)sy \\
&= b^{-1}(s(b-s+a)x + b^{-1}(s-a)sy \\
&= sb^{-1}tx + b^{-1}(s-a)y \\
&= sz
\end{aligned}
$$

as required.

Finally, for axiom (p/siii) we verify each of the mixed exchange laws:

$$
\begin{aligned}
(x \,\overline{\triangleright}\, y) \,\overline{\triangleright}\, (z \,\overline{\bullet}\, y) &= s(sx) \\
&= (x \,\overline{\triangleright}\, z) \,\overline{\triangleright}\, (y \,\underline{\bullet}\, z),
\end{aligned}
$$

$$
\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\underline{\triangleright}\, (z \,\overline{\bullet}\, y) &= t(tx + (s-t)y) + (s-t)((s+t-a)z + (a-t)y) \\
&= t^2x + (t(s-t) + (s-t)(a-t))y + (s-t)(s+t-a)z \\
&= t(tx + (s-t)z) + (s-t)(ay + (s-a)z) \\
&= (x \,\underline{\triangleright}\, z) \,\underline{\triangleright}\, (y \,\underline{\bullet}\, z),
\end{aligned}
$$

$$
\begin{aligned}
(x \,\overline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\overline{\triangleright}\, y) &= b(sx) + (s-b)(sz) \\
&= s(bx + (s-b)z) \\
&= (x \,\overline{\bullet}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z),
\end{aligned}
$$

$$
\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\underline{\bullet}\, (z \,\underline{\triangleright}\, y) &= a(tx + (s-t)y) + (s-a)(tz + (s-t)y) \\
&= t(ax + (s-a)z + (s-t)(sy) \\
&= (x \,\underline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z),
\end{aligned}
$$

$$
\begin{aligned}
(x \,\overline{\triangleright}\, y) \,\underline{\bullet}\, (z \,\overline{\triangleright}\, y) &= a(sx) + (s-a)(sz) \\
&= s(ax + (s-a)z) \\
&= (x \,\underline{\bullet}\, z) \,\overline{\triangleright}\, (y \,\underline{\triangleright}\, z) \quad \text{and}
\end{aligned}
$$

$$
\begin{aligned}
(x \,\underline{\triangleright}\, y) \,\overline{\bullet}\, (z \,\underline{\triangleright}\, y) &= b(tx + (s-t)y) + (s-b)(tz + (s-t)y) \\
&= t(bx + (s-b)z) + (s-t)(sy) \\
&= (x \,\overline{\bullet}\, z) \,\underline{\triangleright}\, (y \,\overline{\triangleright}\, z)
\end{aligned}
$$

as required. □

EXAMPLE 10. We can define finite psyquandles by selecting units $s, t, a, b \in \mathbf{Z}_n$ such that $s + t = a + b$. For instance, in $\mathbf{Z}_5$ we can select $s = 2$, $t = 3$, $a = 4$ and $b = 1$; then $s + t = 2 + 3 = 0 = 1 + 4$ and we have an Alexander psyquandle with operations

$$\begin{array}{rclcrcl} x \,\underline{\triangleright}\, y & = & 3x + 4y & \quad & x \,\underline{\bullet}\, y & = & 4x + 4y \\ x \,\overline{\triangleright}\, y & = & 2x & & x \,\overline{\bullet}\, y & = & x + y \end{array}$$

and operation matrix

$$\left[\begin{array}{ccccc|ccccc|ccccc|ccccc} 2 & 1 & 5 & 4 & 3 & 2 & 2 & 2 & 2 & 2 & 3 & 2 & 1 & 5 & 4 & 2 & 3 & 4 & 5 & 1 \\ 5 & 4 & 3 & 2 & 1 & 4 & 4 & 4 & 4 & 4 & 2 & 1 & 5 & 4 & 3 & 3 & 4 & 5 & 1 & 2 \\ 3 & 2 & 1 & 5 & 4 & 1 & 1 & 1 & 1 & 1 & 1 & 5 & 4 & 3 & 2 & 4 & 5 & 1 & 2 & 3 \\ 1 & 5 & 4 & 3 & 2 & 3 & 3 & 3 & 3 & 3 & 5 & 4 & 3 & 2 & 1 & 5 & 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 & 5 & 5 & 5 & 5 & 5 & 5 & 4 & 3 & 2 & 1 & 5 & 1 & 2 & 3 & 4 & 5 \end{array}\right]$$

where we use 5 as the class of zero in $\mathbf{Z}_5$.

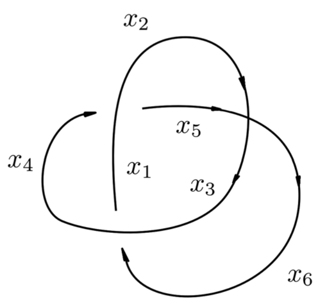
EXAMPLE 11. If X is a commutative ring with identity in which 2 is invertible, we can set $a = b = \frac{s+t}{2}$ to get pI-adequate psyquandle operations

$$x \,\underline{\bullet}\, y = \frac{s+t}{2}x + \frac{s-t}{2}y = x \,\overline{\bullet}\, y\,.$$

We can interpret these operations as averaging the two possible classical resolutions of an oriented precrossing. We call this type of psyquandle a *Jablan psyquandle* since it was originally inspired by Slavik Jablan's notion of precrossings as averages of two classical crossings. For instance, in $X = \mathbf{Z}_5$ choosing $s = 2$ and $t = 4$ yields

$$x \,\underline{\bullet}\, y = 3x + y = x \,\overline{\bullet}\, y\,.$$

EXAMPLE 12. We can compute $\Phi_X^{\mathbf{Z}}$ for a singular link or pseudolink using linear algebra when X is an Alexander psyquandle. For example, the pseudoknot



has system of coloring equations given by

$$\begin{array}{rcl} sx_1 & = & x_2 \\ tx_4+(s-t)x_1 & = & x_5 \\ sx_3 & = & x_4 \\ tx_6+(s-t)x_3 & = & x_1 \\ ax_5+(s-a)x_3 & = & x_6 \\ (s+t-a)x_3+(a-t)x_5 & = & x_2. \end{array}$$

Choosing as a coloring psyquandle $X=\mathbf{Z}_5$ with $s=3$, $t=1$, $a=2$ and $b=2$, this becomes

$$\begin{array}{rcl} 3x_1+4x_2 & = & 0 \\ 2x_1+x_4+4x_5 & = & 0 \\ 3x_3+4x_4 & = & 0 \\ 4x_1+2x_3+x_6 & = & 0 \\ x_3+2x_5+4x_6 & = & 0 \\ 4x_2+2x_3+x_5 & = & 0 \end{array}$$

which we can solve by row-reduction over $\mathbf{Z}_5$:

$$\left[\begin{array}{cccccc} 3 & 4 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 4 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 4 & 2 & 0 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccccc} 1 & 4 & 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 4 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

so $\dim(\ker(A))=0$ and $\Phi_X^{\mathbf{Z}}(L)=1$. Since the unknot has $\Phi_X^{\mathbf{Z}}(0_1)=5\neq 1$, this invariant detects the (pseudo)knottedness of L.

Let $X=\mathbf{Z}_p$ and set $s=1$ and $t=-1$ so we have

$$x \,\underline{\triangleright}\, y = -x+(1-(-1))y = 2y-x \quad \text{and} \quad x \,\overline{\triangleright}\, y = x\,.$$

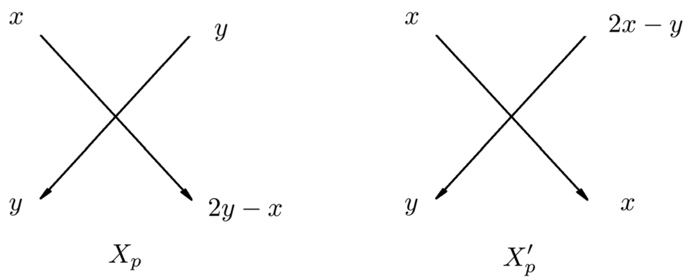
Colorings of classical knots and links by this type of biquandle are known as *p-colorings*. Let us denote by X_p and X'_p respectively the Alexander psyquandle structures on X with $s=1$, $t=-1$, $a=1$ and $b=-1$ and $s=1$, $t=-1$, $a=-1$ and $b=1$ respectively. Observe that X_p satisfies $x \,\underline{\triangleright}\, y = x \,\underline{\bullet}\, y$ and $x \,\overline{\triangleright}\, y = x \,\overline{\bullet}\, y$ while X'_p satisfies $x \,\underline{\triangleright}\, y = x \,\overline{\bullet}\, y$ and $x \,\overline{\triangleright}\, y = x \,\underline{\bullet}\, y$. In particular, we have the following observation:

OBSERVATION 1. An X_p-coloring of a pseudolink diagram D coincides with a p-coloring of the positive resolution of D, while an X'_p-coloring coincides with a p-coloring of the negative resolution of D.

In [10], two notions of p-colorability of pseudolinks were introduced. More precisely, a pseudolink L is *p-colorable* if every resolution of L is p-colorable. A *strong p-coloring* is a p-coloring at classical crossings such that at every precrossing, all four semiarcs have the same color.

LEMMA 1. *Let $p \in \mathbf{Z}$ be odd. A coloring of a pseudolink diagram which is both an X_p-coloring and an X'_p-coloring is a strong p-coloring.*

PROOF. At precrossings we have



so a coloring which satisfies both X_p and X'_p must satisfy $2x = 2y$ at every precrossing. Since p is odd, 2 is invertible in $\mathbf{Z}_p$ and we have $x = y = 2x - y = 2y - x$ as required. □

COROLLARY 1. *Let $p \in \mathbf{Z}$ be odd. A pseudolink L is strongly p-colorable if and only if*

$$\mathrm{Hom}(\mathcal{P}(L), X_p)) \cap \mathrm{Hom}(\mathcal{P}(L), X'_p)) \neq \emptyset .$$

Finally, we conclude with generalizations of the Alexander polynomial to the cases of singular links and pseudolinks.

Let D be an oriented singular link diagram or pseudolink diagram. We obtain a homogeneous system of linear equations over Λ' from the crossing relations of D, describing a presentation of the *fundamental Alexander psyquandle* of L. In fact, using our crossing labelings this presentation is given by a matrix A with entries in the polynomial ring $\hat{\Lambda} = \mathbf{Z}[t, s, a, b, t^{-1}, s^{-1}, a^{-1}, b^{-1}]$ where $t^{-1}, s^{-1}, a^{-1}, b^{-1}$ are independent variables and which has $\Lambda' = \hat{\Lambda}/(tt^{-1} - 1, ss^{-1} - 1, aa^{-1} - 1, bb^{-1} - 1, s + t - a - b)$ as a quotient. Following the same procedure described in [3] (see also Chapter 6 in [13] for a nice summary of the classical case, and note that our matrix A is the transpose of the analogous matrix in [13]), we obtain a sequence of ideals $I_k \subset \hat{\Lambda}$ which are invariants of L by setting I_k to be the ideal in $\hat{\Lambda}$ generated by the codimension k minors M_k of A together with the polynomials $\{tt^{-1} - 1, ss^{-1} - 1, aa^{-1} - 1, bb^{-1}, s + t - a - b\}$.

DEFINITION 5. Let L be an oriented singular link or pseudolink. Any generator of the smallest principal ideal P_k containing the ideal $I_k \subset \hat{\Lambda}$ generated by the codimension k minors of a presentation matrix A and the polynomials $\{tt^{-1} - 1, ss^{-1} - 1, aa^{-1} - 1, bb^{-1}, s + t - a - b\}$ is the *kth Alexander psyquandle polynomial* of L, and fixing a monomial ordering $\prec$ on

$\{t, s, a, b, t^{-1}, s^{-1}, a^{-1}, b^{-1}\}$, the reduced Gröbner basis for I_k is the *kth Alexander-Gröbner* invariant of L.

A useful special case is to use the Jablan psyquandle, i.e. set $a = b = \frac{s+t}{2}$ with coefficients in $\mathbf{Z}[\frac{1}{2}]$. More precisely, we have:

DEFINITION 6. The *Jablan Polynomial* $\Delta_J(L)$ of an oriented pseudolink or singular link L is any generator of the smallest principal ideal in $\Lambda_J = \mathbf{Z}[\frac{1}{2}, s^{\pm 1}, t^{\pm 1}, \frac{1}{s+t}]$ containing the ideal generated by the codimension 1 minors of the Jablan psyquandle matrix of L with $a = b = \frac{s+t}{2}$.

As in the case of the Alexander polynomial, the codimension 1 elementary ideal in the Jablan module is principal, so we can simply take any codimension 1 minor to compute Δ_J up to units. First, we have

LEMMA 2. *Let L be a classical link considered as a pseudolink without precrossings. Then $\Delta_J(L)$ is a homogeneous polynomial in s and t which specializes to the Alexander polynomial up to powers of* 2 *by setting $s = 1$.*

PROOF. The Jablan matrix of a classical link is equivalent by row and column moves to the block matrix

$$\left[\begin{array}{c|c} A' & 0 \\ \hline 0 & I \end{array}\right]$$

where A' is the matrix obtained from the presentation matrix A of the Alexander quandle of L by replacing every 1 with s. Then the codimension 1 minors of J equal the codimension 1 minors of A'; these are homogeneous since every entry is either $\pm s$, t or $s - t$. □

We have the following standard lemma, sometimes given as an exercise in commutative algebra courses:

LEMMA 3. *Let R be a commutative ring with identity. Then the units in $R[x^{\pm 1}]$ have the form rx^n where r is a unit in R.*

PROOF. Any Laurent polynomial $p(x) = \sum_{k=a}^{b} r_k x^k$ can be rewritten as

$$p(x) = x^a \sum_{k=a}^{b} r_k x^{k-a} = x^a q(x)$$

where $q(0) = r_a \neq 0$. Then if $p(x)$ is a unit with inverse $p'(x) = x^{a'} q'(x)$ where $q'(0) = r'_{a'} \neq 0$, we have

$$1 = pp' = qq' x^{a-a'}.$$

Evaluating at $x = 0$ yields a contradiction unless $a = a'$, so we have $pp' = qq' = 1$; then q is an invertible (non-Laurent) polynomial in x, that is to say, a unit in the ring R, and we have $p = rx^n$ as required. □

Applying the lemma 3 with $x = 2, s, t$, we see that units in $\mathbf{Z}[2^{-1}, s, t]$ are of the form $\pm 2^j s^k t^n$; then in the case of adjoining $(s+t)^{-1}$, after factoring out the minimal power of $(s+t)$ and the minimal power of $s^j t^k$ in lexicographical ordering on (j, k), evaluation at $(0, 0)$ yields the analogous result and we see that that the units in Λ_J are of the form $\pm 2^i s^j t^k (s+t)^l$. Hence, we can normalize a Jablan polynomial up to sign by clearing the denominator and canceling any common factors of 2, s, t and $(s+t)$.

EXAMPLE 13. The 2-bouquet graph 1_1^l in example 7 has Jablan psyquandle matrix

$$\begin{bmatrix} \frac{s+t}{2} & \frac{s-t}{2} & -1 & 0 \\ \frac{s-t}{2} & \frac{s+t}{2} & 0 & -1 \\ s-t & t & 0 & -1 \\ s & 0 & -1 & 0 \end{bmatrix}$$

which has codimension 1 minors

$$\left\{ -\frac{s-t}{2}, \frac{s-t}{2}, \frac{-s(s-t)}{2}, \frac{s(s-t)}{2} \right\}$$

which have gcd $s - t$ up to units in Λ_J (indeed, are equal up to units in Λ_J), so we have $\Delta_J(1_1^l) = s - t$.

EXAMPLE 14. We computed the Jablan polynomials of a choice of orientation for each of the pseudoknots and 2-bouquet graphs in [9] and [18] respectively. The results are collected in the tables.

$\Delta_J(L)$	L
1	$3_1.1, 3_1.2, 4_1.1, 4_1.2, 4_1.3,$
	$5_1.1, 5_1.2, 5_2.1, 5_2.2, 5_2.6, 5_2.9$
s^2+t^2	$3_1.3, 5_2.3, 5_2.4$
s^2-st+t^2	$5_2.5, 5_2.10$
$s^2-4st+t^2$	$4_1.5$
$s^2-6st+t^2$	$4_1.4$
$3s^2-2st+3t^2$	$5_2.7$
$3s^2-4st+3t^2$	$5_2.8$
$s^4+2s^3t+2s^2t^2+2st^3+t^4$	$5_1.3$
$s^4+s^3t+st^3+t^4$	$5_1.4$
s^4+t^4	$5_1.5$

$\Delta_J(L)$	L
1	0^k_1
s^2+t^2	2^k_1
$s^2-4st+t^2$	3^k_1
s^2-st+t^2	4^k_2
$s^2-3st+t^2$	5^k_1
$2s^2-5st+2t^2$	6^k_{19}
$3s^2-4st+3t^2$	6^k_2
$3s^2-5st+3t^2$	6^k_6
$3s^2-8st+3t^2$	5^k_2
$5s^2-8st+5t^2$	$6^k_3,\ 6^k_7$
s^4+t^4	$4^k_1, 4^k_3$
$s^4+s^3t-2s^2t^2+st^3+t^4$	5^k_8
$s^4-s^3t+s^2t^2-st^3+t^4$	$6^k_{10}, 6^k_{13}$
$s^4-s^3t-2s^2t^2-st^3+t^4$	6^k_{18}
$s^4-4s^3t+4s^2t^2-4st^3+t^4$	5^k_3

$\Delta_J(L)$	L
$s^4-3s^3t+2s^2t^2-3st^3+t^4$	5^k_4
$s^4-2s^3t-2st^3+t^4$	5^k_5
$s^4-2s^3t+4s^2t^2-2st^3+t^4$	5^k_6
$s^4-3s^3t+6s^2t^2-3st^3+t^4$	5^k_7
$s^4-4s^3t+8s^2t^2-4st^3+t^4$	6^k_{16}
$s^4-5s^3t+6s^2t^2-5st^3+t^4$	6^k_{14}
$s^4-5s^3t+10s^2t^2-5st^3+t^4$	6^k_{17}
$s^4-6s^3t+8s^2t^2-6st^3+t^4$	6^k_{11}
$s^4-6s^3t+12s^2t^2-6st^3+t^4$	6^k_{15}
$s^4-7s^3t+10s^2t^2-7st^3+t^4$	6^k_{12}
$2s^4-s^3t-st^3+2t^4$	6^k_5
$2s^4-3s^3t+4s^2t^2-3st^3+2t^4$	6^k_8
$3s^4-4s^3t+4s^2t^2-4st^3+3t^4$	6^k_4
$3s^4-5s^3t+6s^2t^2-5st^3+3t^4$	6^k_9
s^6+t^6	6^k_1

$\Delta_J(L)$	L
$s-t$	$1_1^l, 6_{11}^l$
$5s-5t$	5_1^l
s^3-t^3	3_1^l
$s^3-2s^2t+2st^2-t^3$	$4_1^l, 5_3^l$
$s^3-4s^2t+4st^2-t^3$	6_2^l
$s^3-8s^2t+8st^2-t^3$	6_{12}^l
$2s^3-s^2t+st^2-2t^3$	5_2^l
$2s^3-3s^2t+3st^2-2t^3$	6_1^l
$2s^3-5s^2t+5st^2-2t^3$	6_6^l
$s^5-2s^3t^2+2s^2t^3-t^5$	6_{10}^l
$s^5-2s^4t+2s^3t^2-2s^2t^3+2st^4-t^5$	6_5^l
$s^5-2s^4t+4s^3t^2-4s^2t^3+2st^4-t^5$	$6_3^l, 6_4^l$
$s^5-3s^4t+5s^3t^2-5s^2t^3+3st^4-t^5$	6_4^l
$s^5-4s^4t+6s^3t^2-6s^2t^3+4st^4-t^5$	6_7^l
$s^5-4s^4t+8s^3t^2-8s^2t^3+4st^4-t^5$	6_9^l

In light of example 14, we make a few observations in the following remarks:

REMARK 1. The polynomials in Example 14 are all homogeneous as previously noted and symmetric in the sense that the coefficients of $s^{n-k}t^k$ and s^kt^{n-k} are equal. In the case of classical knots and links, symmetry in t and s follows from the fact that the upper and lower biquandles are isomorphic and in our notation, the resulting polynomials are related by switching s and t.

REMARK 2. One alternative idea for an Alexander-style polynomial for a pseudoknot or pseudolink would be to take a weighted average of Alexander polynomials of the classical resolutions of the pseudoknot or pseudolink with weights from the WeRe set. Indeed, at the level of Jablan matrix this is effectively what we are doing.

However, it is not clear in general how to take a weighted average of Alexander polynomials since the Alexander polynomial is only defined up to multiplication by units: should an average of t and t be $\frac{t+t}{2}=t$ or $\frac{t+(-1)t}{2}=0$ or even $\frac{(t^{-1})t+(t)t}{2}=\frac{1+t^2}{2}$? We observe that in the cases above, the $s=1$ specialization of the Jablan polynomial of a pseudolink does in fact agree with a weighted sum of some choice of normalizations of Alexander polynomials of the classical resolutions: for example, pseudoknot $3_1.1$ has Jablan polynomial $s^2+2st+t^2$, specializing to $1+2t+t^2$. If we symmetrize this in t, we obtain $t^{-1}+2+t$. Then $3_1.1$ has WeRe set

$$\left\{\left(0_1,\frac{3}{4}\right),\left(3_1,\frac{1}{4}\right)\right\}$$

and taking a weighted sum of symmetric normalizations with positive leading coefficient of the Alexander polynomials of 0_1 and 3_1 and clearing the denominator, we have

$$3(1) + 1(t^{-1} - 1 + t) = t^{-1} + 2 + t.$$

However, the pseudoknot $4_1.4$ has Jablan polynomial $\Delta_J(4_1.4) = s^2 - 6st + t^2$ and WeRe set

$$\left\{\left(0_1, \frac{3}{4}\right), \left(4_1 \frac{1}{4}\right)\right\}$$

with positive symmetric normalized Alexander polynomials

$$\Delta(0_1) = 1 \quad \Delta(4_1) = t^{-1} - 3 + t;$$

taking the weighted sum and clearing the denominator, we have

$$3\Delta(0_1) + 1\Delta(4_1) = 3 + (t^{-1} - 3 + t) = t^{-1} + t \neq t^{-1} - 6 + t.$$

But, if we multiply the first polynomial by the unit -1, we obtain

$$3(-1)\Delta(0_1) + 1\Delta(4_1) = -3 + (t^{-1} - 3 + t) = t^{-1} - 6 + t,$$

coinciding with the specialization of $\Delta_J(4_1.4)$ as desired.

In light of these remarks, we propose the following conjecture:

CONJECTURE 1. *There exists a choice of normalization rule for the Jablan polynomial such that for every pseudoknot K with WeRe set $S = \{(\alpha_1, K_1), \dots, (\alpha_n, K_n)\}$ we have*

$$\Delta_J(K) = \sum_{j=1}^{n} \alpha_j \Delta_J(K_j).$$

5. Questions

We conclude with some questions for future research.

The main question, of course, is conjecture 1 true? More precisely, what normalization rule makes

$$\Delta_J(K) = \sum_{j=1}^{n} \alpha_j \Delta_J(K_j)$$

for pseudoknots with WeRe set $S = \{(\alpha_1, K_1), \dots, (\alpha_n, K_n)\}$?

What enhancements of psyquandle counting invariants can be defined? Enhancements of psyquandle counting invariants will be the topics of future papers.

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