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Asymptotic Behavior of Solutions to the One-dimensional Keller-Segel System with Small Chemotaxis

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Abstract. In this paper, the one-dimensional Keller-Segel system defined on a bounded interval with the Neumann boundary conditions is considered. The system describes the phenomenon such that the cellular slime molds form an aggregation by the chemotaxis movement. In the case of small chemotaxis, the asymptotic behavior of solutions to the system are analyzed, as the time development, by using the Fourier series. Some of numerical examples are also given.

1. Introduction

We consider the Keller-Segel system which has been posed as a mathematical and biological model for the cellular slime molds by E.F. Keller and S.A. Segel [7] in 1970s. N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler [1] gives a general survey of the Keller-Segel system. In T. Hillen and K.J. Painter [6], there is a detailed exploration of variations of the Keller-Segel model. Here, we analyze an original model dealt in [6]. As has been considered in K. Osaki and A. Yagi [10], we investigate the one-dimensional Keller-Segel system defined on a bounded interval with the Neumann boundary conditions, which is given as (KS) in Section 3. We focus on the case where the chemotaxis is small, and we analyze the asymptotic behavior, as the time development, of the cellular slime molds and chemotactic substance. In Theorem 1, which we describe in Section 5, we show that the solutions of (KS) converge to some constants, as time tends to infinity, in the case of small chemotaxis. For the case of one-dimension, in Z. Wang and K. Zhao [11] and M. Winkler [12], analogous results on the asymptotic behavior of the solutions to corresponding Keller-Segel systems, which are similar to but not same as the present (KS), are derived. In X. Cao [2] and T. Cieślak, P. Laurençot and C. Morales-Rodrigo [3], the higher-dimensional analogue of the present (KS) are considered, and the corresponding results of the asymptotic behavior of the solutions are investigated. Nevertheless as is indicated in [6], the behavior of the solutions of (KS) depends strongly on the space dimensions. In this paper to prove Theorem 1, we adopt a discussion

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FIGURE 1. The life cycle of the cellular slime molds
(From the homepage of Japanese Society for the Study of Cellular Slime Molds)

through the Fourier series, which is efficient for considering the problems of one-dimension.

This paper is organized as follows. In Sections 2 and 3, we give the brief explanations about the cellular slime molds and the Keller-Segel system, respectively. In Section 4, we introduce some propositions which will play an important role in this paper. Section 5 is the main section. In Theorem 1, we show that both the solutions of (KS) converge to some constants in the case of sufficiently small chemotaxis, as time tends to infinity. Section 6 is devoted to introducing some results by means of the numerical calculation, and a conjecture on our results is presented. Finally in Section 7, we give the proofs of main propositions given in Section 4.

2. The cellular slime molds

We describe the life cycle of the cellular slime molds as follows. The cellular slime mold forms the structure like the plant called a fruit body finally. Then the spore released from a fruit body germinates, and it eats bacteria inhabit in the soil as feed, and increases in the state of the amoeba. After it eats whole of feed of bacteria in the surrounding area, it falls into starvation. Then it begins to release a chemical substance , that is acrasin, which attracts other cells. Hence they are gathering. Then a cell body moves to the lightning place, and it grows to a fruit body (see Figure 1). The Keller-Segel system is the biological model which describes the movement until a cellular slime mold falls in the hunger state and forms an aggregate.

3. The Keller-Segel system

As we described previously, the mathematical formulation which describes the phenomenon that cellular slime molds form an aggregation by chemotaxis movement was firstly posed by E.F. Keller and L.A. Segel [7]. Nowadays, it has been extensively studied as the Keller-Segel system. In this paper, we deal with the following one-dimensional Keller-Segel system (1.1), (1.2) with the Neumann boundary conditions (1.3). The Neumann boundary conditions are also referred to as reflection boundary conditions, and they mean that there is no out of cells and chemotactic substance through the boundary $\partial I = \{a, b\}$.

$$
(\mathrm{KS})\begin{cases}
u_t = u_{xx} - \chi(uv_x)_x & \text{in } I \times (0,\infty), \quad (1.1)\\
v_t = v_{xx} - \gamma v + \alpha u & \text{in } I \times (0,\infty), \quad (1.2)\\
u_x(a,t) = u_x(b,t) = v_x(a,t) = v_x(b,t) = 0 & \text{in } (0,\infty), \quad (1.3)\\
u(x,0) = \overline{u}(x),\ v(x,0) = \overline{v}(x) & \text{in } I,
\end{cases}
$$

where $I = (a, b)$ with some given a and b such that $-\infty < a < b < \infty$ is a bounded open interval, and χ, α, γ are some given positive constants. The solutions $u = u(x, t)$ and $v = v(x, t)$ represent the cell density of the cellular slime molds and the cell concentration of the chemical substance that released by the cellular slime molds at the position x, and at time t, respectively. We review the following: $u_t = u_{xx}$ is the heat equation, and $u_t = -(u \cdot \chi v_x)_x$ is the equation of continuity by Euler. It follows that the first term on the righthand side of (1.1) means the diffusion phenomenon, and the second term means the concentration phenomenon by which the cells move around randomly. Thus (KS) has the terms expressing both of the concentration phenomenon and diffusion phenomenon. Remark that the system given by (KS) is understood as a particular model where it is not affected by the concentration of a chemical substance, but is affected by the gradient of the concentration of a chemical substance. There exists an intensive consideration on the existence of unique solution of (KS) and its asymptotic behavior (cf. K. Osaki and A. Yagi [10]). The purpose of this paper is to show that both the solutions u and v of (KS) converge to some constants depending only on the initial data of u, as the time development, in the case of sufficiently small chemotaxis.

4. Existing results

Throughout this paper, we denote $L^r \equiv L^r(I)$, the usual Lebesgue space on I with the norm $\|u\|_{L^r} \equiv (\int_I |u(x)|^r\, dx)^{\frac{1}{r}}$ for $1 \le r < \infty$, and $\|u\|_{L^\infty} \equiv \operatorname{ess\,sup}_{x\in I} |u(x)|$. The Sobolev space $W^{m,r}(I)$, $m = 1, 2, \ldots$, $1 < r < \infty$ is the space of all functions u on I such that $\|u\|_{W^{m,r}} \equiv \sum_{i=1}^{m} \|D^i u\|_{L^r} < \infty$, with the derivative D with respect to the variable x. In this section, we give some propositions that will play an important role in our main results given in the next section. It is known that (KS) has a unique global-in-time classical solution (u, v) under suitable initial conditions (cf. Section 7 in K. Osaki and A. Yagi [10]):

PROPOSITION 1 (K. Osaki and A. Yagi [10]). *Suppose that the initial data $\overline{u}, \overline{v}$ satisfy the following conditions*:

$$\overline{u}, \overline{v} \in W^{1,2}(I), \quad \inf_{x\in I} \overline{u} > 0, \quad \inf_{x\in I} \overline{v} > 0\,.$$

Then, there exists a unique global-in-time classical solution (u, v) *of* (*KS*).

Proposition 2, below, is derived easily by using integration by parts and the Neumann boundary conditions.

PROPOSITION 2. *Assume that* (u, v) *is the solution of* (*KS*). *Then the following identity holds*:

$$\int_I u(x,t)\,dx = \int_I \overline{u}(x)\,dx\,. \tag{2}$$

The identity (2) is called as "mass conservation law", and it tells us that the amounts of the cellular slime molds do not change in time. In the probability theory, functions that possess the mass conservation law can be interpreted as probability densities (cf. Y. Yahagi [13]).

Then, we consider the following non-homogeneous heat equation (3.1) with the Neumann boundary conditions (3.2).

$$(\mathrm{Heat})\quad \begin{cases} w_t = w_{xx} + z & \text{in } I \times (0,\infty)\,, \qquad (3.1)\\ w_x(a,t) = w_x(b,t) = 0, & \text{in } (0,\infty)\,, \qquad (3.2)\\ w(x,0) = \overline{w}(x) & \text{in } I\,, \end{cases}$$

where, $I = (a, b)$, $\overline{w} \in L^2(I)$ and $z \in C([0,\infty); L^2(I))$.

PROPOSITION 3. *The solution* w *of* $(Heat)$ *is given by the following formula*:

$$w(x,t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 t}\Big(T_n(0) + \int_0^t z_n(\tau) e^{\lambda_n^2 \tau}\,d\tau\Big)\cos\lambda_n(x-a)\,, \tag{4}$$

where,

$$\lambda_n = \frac{\pi n}{b-a} \quad (n \ge 0)\,,$$

$$T_n(0) = \frac{2}{b-a}\int_a^b \overline{w}(x)\cos\lambda_n(x-a)\,dx \quad (n \ge 1)\,,$$

$$T_0(0) = \frac{1}{b-a}\int_b^a \overline{w}(x)\,dx\,,$$

$$z_n(t) = \frac{2}{b-a}\int_a^b z(x,t)\cos\lambda_n(x-a)\,dx \quad (n \ge 1)\,,$$

$$z_0(t) = \frac{1}{b-a}\int_a^b z(x,t)\,dx\,.$$

We will give the proof of Proposition 3 in Section 7.

To prove our main theorems given by Section 5, we efficiently use the following two propositions, known as the Poincaré inequalities (cf. e.g. Section 7 in D. Gilbarg and N.S.Trudinger [5], Chapter 3 in S. Mizohata [9]), and the Gronwall inequality(cf. eg. Appendix B in L.C. Evans [4]), respectively .

PROPOSITION 4. *Let $I = (a,b)$ be the given open interval.*
(i) *There exists a positive constant $C = C(a,b)$ such that for $u \in W^{1,2}(I)$ the following inequality holds*:

$$\|u - M_u\|_{L^2} \le C\|Du\|_{L^2}\,, \tag{5}$$

where, $M_u := \dfrac{1}{b-a}\displaystyle\int_I u(y)\,dy$.

(ii) *For $u \in W_0^{1,2}(I)$, the following inequality holds*:

$$\|u\|_{L^2} \le \frac{b-a}{\sqrt{2}}\|Du\|_{L^2}\,, \tag{6}$$

where $W_0^{1,2}(I)$ is a subspace of $W^{1,2}(I)$, composed with the functions u satisfying $supp[u] \subset (a,b)$.

PROPOSITION 5. *Let $J = (t_0, t_1)$ be the open interval. Suppose that $u \in C^1(J)$ and $\beta \in C(J)$ that satisfy*

$$u'(t) \le \beta(t)\,u(t), \quad (t \in J)\,.$$

Then, *the following inequality holds*:

$$u(t) \le u(c)\ \exp\Big(\int_c^t \beta(s)\,ds\Big)\,,$$

where $t_0 < c \le t < t_1$.

Finally in this section, we prepare the following proposition which is an application of Proposition 5.

PROPOSITION 6. *Let $F \in C^1(0,\infty)$ be a nonnegative function*, *and let $G \in C(0,\infty)$ be a nonnegative function such that $\lim_{t\to\infty} G(t) = 0$. Assume that the following differential inequality holds*:

$$F'(t) \le -kF(t) + lG(t)\,,$$

where k and l are some given positive constants. Then, it holds that

$$\lim_{t\to\infty} F(t) = 0\,.$$

In Section 7, we will give the proof of Proposition 6.

5. Main Results and their proofs

Recall that in this paper, we focus on sufficiently small chemotaxis, and that the second term on the righthand side of (1.1) means that the cells move with the speed χv_x. Intuitively, it is expected that if χv_x is small enough, then the diffusion phenomenon is stronger than the concentration phenomenon on (KS). As a result, it is expectable that the solution u of (KS) will converge to a constant as the time goes by. In fact, we obtain the following main result.

THEOREM 1. *Let $\overline{u}, \overline{v} \in W^{1,2}(I)$ and $\inf_{x\in I} \overline{u} > 0$, $\inf_{x\in I} \overline{v} > 0$, and let (u, v) be the global-in-time classical solution of (KS). Assume that there exists $t_* > 0$ such that for any $t \geq t_*$, the following inequality holds:*

$$\chi C \|v_x(\cdot, t)\|_{L^\infty} < 1\,, \tag{7}$$

where $C = C(a, b)$ is a constant appearing in (5). Furthermore, assume that

$$\lim_{t\to\infty} \|v_{xx}(\cdot, t)\|_{L^2} = 0\,.$$

Then, it holds that

$$\lim_{t\to\infty} \|u(\cdot, t) - M\|_{L^2} = 0\,, \tag{8}$$

$$\lim_{t\to\infty} \left\| v(\cdot, t) - \frac{\alpha M}{\gamma} \right\|_{L^2} = 0\,, \tag{9}$$

where $M := \dfrac{1}{b-a} \displaystyle\int_a^b \overline{u}(x)\, dx$.

Note that by the "mass conservation law" (2), if the solution u of (KS) converges to a constant C, then it must hold that $C = \frac{1}{b-a} \int_a^b \overline{u}(x)\, dx$. Before we prove this main theorem, we prepare the following system (KS*) which is obtained by substituting $\chi = 0$ in (KS).

$$(\mathrm{KS}^*)\ \begin{cases} \tilde{u}_t = \tilde{u}_{xx} & \text{in } I \times (0, \infty), \quad (10.1) \\ \tilde{v}_t = \tilde{v}_{xx} - \gamma \tilde{v} + \alpha \tilde{u} & \text{in } I \times (0, \infty), \quad (10.2) \\ \tilde{u}_x(a, t) = \tilde{u}_x(b, t) = \tilde{v}_x(a, t) = \tilde{v}_x(b, t) = 0, & \text{in } (0, \infty), \quad (10.3) \\ \tilde{u}(x, 0) = \overline{u}(x),\ \tilde{v}(x, 0) = \overline{v}(x) & \text{in } I. \quad (10.4) \end{cases}$$

Note that (KS*) is linear, although (KS) is nonlinear. At first, we discuss the asymptotic behavior of $(\tilde{u}, \tilde{v})$, which is the solution of (KS*), and we proceed to the discussion about (KS). Fortunately, it is possible to solve (KS*). Indeed, we have the following proposition.

PROPOSITION 7. *Let* $\overline{u}, \overline{v} \in W^{1,2}(I)$ *and* $\inf_{x\in I} \overline{u} > 0$, $\inf_{x\in I} \overline{v} > 0$. *Then the classical solution* $(\tilde{u}, \tilde{v})$ *of* (KS*) *is given as follows*:

$$\tilde{u}(x,t) = M + \sum_{n=1}^{\infty} A_n \cos \lambda_n (x-a) e^{-\lambda_n^2 t}, \tag{11}$$

$$\tilde{v}(x,t) = e^{-\gamma t} \sum_{n=0}^{\infty} e^{-\lambda_n^2 t} \Big(T_n(0) + \alpha \int_0^t \tilde{u}_n(\tau) e^{(\gamma+\lambda_n^2)\tau}\, d\tau \Big) \cos \lambda_n (x-a), \tag{12}$$

where λ_n *is the number given in Proposition* 3, *and*

$$A_n = \frac{2}{b-a} \int_a^b \overline{u}(x) \cos \lambda_n (x-a)\, dx \ (n \geq 1),$$

$$T_n(0) = \frac{2}{b-a} \int_a^b \overline{v}(x) \cos \lambda_n (x-a)\, dx \ \ (n \geq 1), \tag{13}$$

$$T_0(0) = \frac{1}{b-a} \int_a^b \overline{v}(x)\, dx =: N, \tag{14}$$

$$\tilde{u}_n(t) = \frac{2}{b-a} \int_a^b \tilde{u}(x,t) \cos \lambda_n (x-a)\, dx \ (n \geq 1), \tag{15}$$

$$\tilde{u}_0(t) = \frac{1}{b-a} \int_a^b \tilde{u}(x,t)\, dx = \frac{1}{b-a} \int_a^b \overline{u}(x)\, dx = M.$$

By using Proposition 7, we have the following theorem.

THEOREM 2. *For the solution* $(\tilde{u}, \tilde{v})$ *of* (KS*) *given by* (11), (12), *the following two hold*:

$$\lim_{t\to\infty} \|\tilde{u}(\cdot,t) - M\|_{L^\infty} = 0, \tag{16}$$

$$\lim_{t\to\infty} \Big\| \tilde{v}(\cdot,t) - \frac{\alpha M}{\gamma} \Big\|_{L^\infty} = 0. \tag{17}$$

PROOF OF PROPOSITION 7. It is easy to solve (10.1) with (10.3) and (10.4) (cf. e.g. Section 12.3 in E. Kreyszig [8]). Here, we give the proof of (12) only. By transformation of variable $\tilde{v}(x,t) = e^{-\gamma t}\tilde{w}(x,t)$, it holds that

$$\tilde{v}_t = -\gamma e^{-\gamma t}\tilde{w} + e^{-\gamma t}\tilde{w}_t, \quad \tilde{v}_x = e^{-\gamma t}\tilde{w}_x, \quad \tilde{v}_{xx} = e^{-\gamma t}\tilde{w}_{xx}.$$

Therefore we have the following differential equation instead of (10.2),

$$\tilde{w}_t = \tilde{w}_{xx} + \alpha\tilde{u}e^{\gamma t}.$$

Also, we have

$$\tilde{w}(x,0) = \overline{v}(x).$$

Then, by substituting $w = \tilde{w}$, $z = \alpha\tilde{u}e^{\gamma t}$ in Proposition 3, it follows that

$$\tilde{w}(x,t) = \sum_{n=0}^{\infty} e^{-\lambda_n^2 t}\Big(T_n(0) + \int_0^t z_n(\tau)e^{\lambda_n^2\tau}\, d\tau\Big)\cos\lambda_n(x-a),$$

where, $T_n(0)$ and $u_n(t)$ are the same as (13), (14), (15) and $z_n(t) = \alpha e^{\gamma t}\tilde{u}_n(t)$. Finally, we obtain the required equation:

$$\tilde{v}(x,t) = e^{-\gamma t}\tilde{w}(x,t) = e^{-\gamma t}\sum_{n=0}^{\infty} e^{-\lambda_n^2 t}\Big(T_n(0) + \alpha\int_0^t \tilde{u}_n(\tau)e^{(\gamma+\lambda_n^2)\tau}\, d\tau\Big)\cos\lambda_n(x-a).$$

□

PROOF OF THEOREM 2. Firstly, we show (18) given below.
Since $e^{-\lambda_n^2 t} = \dfrac{1}{e^{\lambda_n^2 t}} \le \dfrac{1}{\lambda_n^2 t + 1} \le \dfrac{1}{\lambda_n^2 t}$ for any $\lambda_n^2 t > 0$, it holds that

$$\sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \le \frac{1}{t}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} = \frac{(b-a)^2}{t\pi^2}\frac{\pi^2}{6} = \frac{(b-a)^2}{6t}. \tag{18}$$

Now, by using (18), we shall show the first requirement. By (11), we have

$$\begin{aligned}|\tilde{u}(x,t) - M| &= \Big|\sum_{n=1}^{\infty} A_n\cos\lambda_n(x-a)e^{-\lambda_n^2 t}\Big| \\ &\le \sum_{n=1}^{\infty}|A_n|e^{-\lambda_n^2 t} \\ &\le \frac{2M}{t}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^2} = \frac{2M(b-a)^2}{t\pi^2}\cdot\frac{\pi^2}{6} = \frac{M(b-a)^2}{3t}.\end{aligned}$$

It follows that

$$\|\tilde{u}(\cdot,t) - M\|_{L^\infty} \le \frac{M(b-a)^2}{3t} \to 0 \quad (t\to\infty).$$

Then by (12), we have

$$\begin{aligned}\tilde{v}(x,t) &= e^{-\gamma t}\sum_{n=0}^{\infty} e^{-\lambda_n^2 t}\Big(T_n(0) + \alpha\int_0^t \tilde{u}_n(\tau)e^{(\gamma+\lambda_n^2)\tau}\,d\tau\Big)\cos\lambda_n(x-a)\\ &= e^{-\gamma t}\sum_{n=0}^{\infty} e^{-\lambda_n^2 t}T_n(0)\cos\lambda_n(x-a) + \alpha e^{-\gamma t}\int_0^t \tilde{u}_0(\tau)e^{\gamma\tau}\,d\tau\\ &\qquad + e^{-\gamma t}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\alpha\int_0^t \tilde{u}_n(\tau)e^{(\gamma+\lambda_n^2)\tau}\,d\tau\cos\lambda_n(x-a)\\ &=: I_1(x,t) + I_2(t) + I_3(x,t)\,.\end{aligned}$$

We see that

$$|I_1(x,t)| \le e^{-\gamma t}\sum_{n=0}^{\infty} e^{-\lambda_n^2 t}|T_n(0)||\cos\lambda_n(x-a)| \le 2Ne^{-\gamma t}\sum_{n=0}^{\infty} e^{-\lambda_n^2 t}\,. \tag{19}$$

By using (18) again, it follows that

$$\sum_{n=0}^{\infty} e^{-\lambda_n^2 t} = 1 + \sum_{n=1}^{\infty} e^{-\lambda_n^2 t} \le 1 + \frac{(b-a)^2}{6t} \quad (t>0)\,. \tag{20}$$

From (19) and (20), we have

$$\sup_I |I_1(x,t)| \le 2Ne^{-\gamma t}(1 + \frac{(b-a)^2}{6t}) \;\to 0 \;\; (t\to\infty)\,.$$

Next, we consider $I_2(t)$.

$$\begin{aligned}I_2(t) &= \alpha e^{-\gamma t}\int_0^t \tilde{u}_0(\tau)e^{\gamma\tau}\,d\tau = \alpha e^{-\gamma t}\int_0^t Me^{\gamma\tau}d\tau\\ &= \alpha e^{-\gamma t}M\cdot\frac{1}{\gamma}(e^{\gamma t}-1) \to \frac{\alpha M}{\gamma}\;(t\to\infty)\,.\end{aligned}$$

Finally let us evaluate $I_3(x,t)$. From (11) and (15), for $n\ge 1$, we have

$$\begin{aligned}\tilde{u}_n(t) &= \frac{2}{b-a}\int_a^b \tilde{u}(x,t)\cos\lambda_n(x-a)\,dx\\ &= \frac{2}{b-a}\int_a^b \Big(M + \sum_{m=1}^{\infty} A_m\cos\lambda_m(x-a)e^{-\lambda_m^2 t}\Big)\cos\lambda_n(x-a)\,dx\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{b-a}\sum_{m=1}^{\infty} A_m e^{-\lambda_m^2 t}\int_a^b \cos\lambda_m(x-a)\cos\lambda_n(x-a)\,dx \\
&= \frac{2}{b-a} A_n e^{-\lambda_n^2 t}\int_a^b \cos\lambda_n^2(x-a)\,dx \\
&= A_n e^{-\lambda_n^2 t}\,.
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
I_3(x,t) &= e^{-\gamma t}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\alpha\int_0^t \tilde{u}_n(\tau)e^{(\gamma+\lambda_n^2)\tau}\,d\tau\cos\lambda_n(x-a) \\
&= e^{-\gamma t}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}\alpha\int_0^t A_n e^{\gamma\tau}\,d\tau\cos\lambda_n(x-a) \\
&= \alpha e^{-\gamma t}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}A_n\cos\lambda_n(x-a)\int_0^t e^{\gamma\tau}\,d\tau \\
&= \alpha e^{-\gamma t}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}A_n\cos\lambda_n(x-a)\frac{1}{\gamma}(e^{\gamma t}-1) \\
&= \frac{\alpha}{\gamma}\sum_{n=1}^{\infty} e^{-\lambda_n^2 t}A_n\cos\lambda_n(x-a)(1-e^{-\gamma t}) \to 0\ (t\to\infty)\,.
\end{aligned}
$$

Hence, we obtain the following result:

$$
\left\|\tilde{v}(\cdot,t)-\frac{\alpha M}{\gamma}\right\|_{L^\infty} \le \|I_1(\cdot,t)\|_{L^\infty} + \left|I_2(t)-\frac{\alpha M}{\gamma}\right| + \|I_3(\cdot,t)\|_{L^\infty} \to 0\ (t\to\infty)\,.
$$

Thus (17) has been proven. □

At the end of this section, we give the proof of Theorem 1.

PROOF OF THEOREM 1. First of all, we shall show (8). Multiplying the first equation (1.1) of (KS) by u and integrating the product on the interval I, we have

$$
\frac{1}{2}\frac{d}{dt}\int_I u^2\,dx = \int_I u\,u_{xx}\,dx - \chi\int_I u(uv_x)_x\,dx\,. \tag{21}
$$

By integration by parts formula, from (21) it follows that

$$
\frac{1}{2}\frac{d}{dt}\int_I (u-M^2)\,dx = -\int_I (u_x)^2\,dx - \chi\int_I (u-M)(uv_x)_x\,dx =: J_1(t)+J_2(t)\,. \tag{22}
$$

Because of the "mass conservation law" (2), it holds that $\frac{d}{dt}\int_I u\,dx = 0$. Thus we have

$$\frac{d}{dt}\|u(\cdot,t)-M\|_{L^2}^2 = \frac{d}{dt}\int_I (u^2 - 2Mu + M^2)\,dx = \frac{d}{dt}\int_I u^2\,dx = \frac{d}{dt}\|u(\cdot,t)\|_{L^2}^2\,.$$

Also it holds that

$$J_1(t) = -\int_I (u_x)^2\,dx = -\|u_x(\cdot,t)\|_{L^2}^2\,.$$

Then, by the Hölder inequality and the Poincaré inequality (6), we see

$$\begin{aligned}
J_2(t) &= -\chi\int_I (u-M)(uv_x)_x\,dx \\
&= -\chi\int_I (u-M)\big[((u-M)v_x)_x + Mv_{xx}\big]\,dx \\
&= \chi\int_I (u-M)_x(u-M)v_x\,dx - \chi\int_I (u-M)Mv_{xx}\,dx \\
&\le \chi\|v_x(\cdot,t)\|_{L^\infty}\|u_x(\cdot,t)\|_{L^2}\|u(\cdot,t)-M\|_{L^2} + \chi M\|v_{xx}(\cdot,t)\|_{L^2}\|u(\cdot,t)-M\|_{L^2}\,.
\end{aligned}$$

Suppose that there exists $t_* > 0$ such that the following property holds (cf. (7)):

$$p := 1 - \chi C\|v_x(\cdot,t)\|_{L^\infty} > 0 \quad (t \ge t_*)\,, \tag{23}$$

where $C = C(a,b) > 0$. Furthermore, suppose that $\lim_{t\to\infty}\|v_{xx}(\cdot,t)\|_{L^2} = 0$. Then, from (23), it follows that

$$\frac{d}{dt}\|u(\cdot,t)-M\|_{L^2}^2 \le -\frac{p}{C^2}\|u(\cdot,t)-M\|_{L^2}^2 + \frac{\chi^2M^2}{2}\|v_{xx}(\cdot,t)\|_{L^2}\,.$$

By Proposition 6, we find

$$\|u(\cdot,t)-M\|_{L^2}^2 \to 0 \quad (t\to\infty)\,,$$

and thus we have proved (8).

Nextly, we shall prove (9). We denote by $(\tilde{u},\tilde{v})$ the solution of (KS*), and let

$$w(x,t) := u(x,t) - \tilde{u}(x,t),\quad z(x,t) := v(x,t) - \tilde{v}(x,t)\,.$$

Then, we have easily the following system $(*)$.

$$(*)\begin{cases}
w_t = w_{xx} - \chi(u_xv_x + uv_{xx}) & \text{in } I\times(0,\infty), \quad (24.1)\\
z_t = z_{xx} - \gamma z + \alpha w & \text{in } I\times(0,\infty), \quad (24.2)\\
w_x(a,t) = w_x(b,t) = z_x(a,t) = z_x(b,t) = 0 & \text{in } (0,\infty),\\
w(x,0) = \overline{w}(x) = 0,\ z(x,0) = \overline{z}(x) = 0 & \text{in } I\,.
\end{cases}$$

Here, we are paying our attention to (24.2). As well as (21) and (22), we have

$$\frac{1}{2}\frac{d}{dt}\int_I z^2\,dx = -\int_I z_x^2\,dx - \gamma\int_I z^2\,dx + \alpha\int_I zw\,dx\,. \tag{25}$$

We define

$$F(t) := (\int_I z^2\,dx)^{\frac{1}{2}} = \|z(\cdot,t)\|_{L^2}\,.$$

Then the lefthand side of (25) is written as follows:

$$\frac{1}{2}\frac{d}{dt}\int_I z^2\,dx = \frac{1}{2}\frac{d}{dt}\{F(t)\}^2 = F(t)F'(t)\,.$$

Furthermore, we set

$$G(t) := (\int_I w^2\,dx)^{\frac{1}{2}} = \|w(\cdot,t)\|_{L^2}\,.$$

By the Hölder inequality and the Poincaré inequality (6), the righthand side of (25) becomes

$$\begin{aligned} -\int_I z_x^2\,dx - \gamma\int_I z^2\,dx + \alpha\int_I zw\,dx& \\ &\le -\gamma\|z(\cdot,t)\|_{L^2}^2 + \alpha\|z(\cdot,t)\|_{L^2}\,\|w(\cdot,t)\|_{L^2} \\ &\le -\gamma\{F(t)\}^2 + \alpha F(t)\,G(t)\,. \end{aligned}$$

That is,

$$F(t)F'(t) \le -\gamma\{F(t)\}^2 + \alpha F(t)\,G(t)\,.$$

Thus we obtain the following inequality:

$$F'(t) \le -\gamma F(t) + \alpha G(t)\,. \tag{26}$$

Note that (cf. (8), (16))

$$G(t) = \|w(\cdot,t)\|_{L^2} = \|(u-\tilde{u})(\cdot,t)\|_{L^2} \le \|u(\cdot,t)-M\|_{L^2} + \|\tilde{u}(\cdot,t)-M\|_{L^2} \to 0\ (t\to\infty)\,,$$

hence by Proposition 6, we have

$$F(t) = \|z(\cdot,t)\|_{L^2} = \|(v-\tilde{v})(\cdot,t)\|_{L^2} \to 0\ \ (t\to\infty)\,.$$

Moreover, since for bounded interval I, L^∞ is continuously embedded in L^2, by (17) it holds that

$$\left\|\tilde{v}(\cdot,t) - \frac{\alpha M}{\gamma}\right\|_{L^2} \to 0\ \ (t\to\infty)\,.$$

By combining the above two, we get

$$\left\| v(\cdot,t) - \frac{\alpha M}{\gamma} \right\|_{L^2} \le \|(v-\tilde{v})(\cdot,t)\|_{L^2} + \left\| \tilde{v}(\cdot,t) - \frac{\alpha M}{\gamma} \right\|_{L^2} \to 0 \quad (t \to \infty)\,.$$

We have completed the proof of Theorem 1. □

6. Examples of numerical calculation

As we showed in Theorem 1, the classical solutions u and v of (KS) with sufficiently small chemotaxis converge to the constants. Here, we introduce some examples of numerical calculations. From these examples, heuristic however, we give the following conjecture.

CONJECTURE 1. *Let $\overline{u}, \overline{v} \in W^{1,2}(I)$ and $\inf_{x\in I} \overline{u} > 0$, $\inf_{x\in I} \overline{v} > 0$, and (u,v) be the classical solution of (KS). Assume that χ is small enough. Then, for any $x \in I$, the following hold*:

$$\lim_{t\to\infty} u(x,t) = M\,,$$

$$\lim_{t\to\infty} v(x,t) = \frac{\alpha M}{\gamma}\,.$$

EXAMPLE 1. Let $\alpha = 2$, $\gamma = 3$, $a = 0$, $b = \pi$, $\overline{u}(x) = 3 - \cos 2x$, $\overline{v}(x) = 3$, $\chi = 1$. As we see in Figure 2, the solutions u and v converge to some constants, respectively.

EXAMPLE 2. Let $\alpha, \gamma, a, \overline{u}, \overline{v}(x)$ be same as Example 1, but let $\chi = \frac{5}{4}$. As we see in Figure 3, the solutions u and v do not converge to some constants.

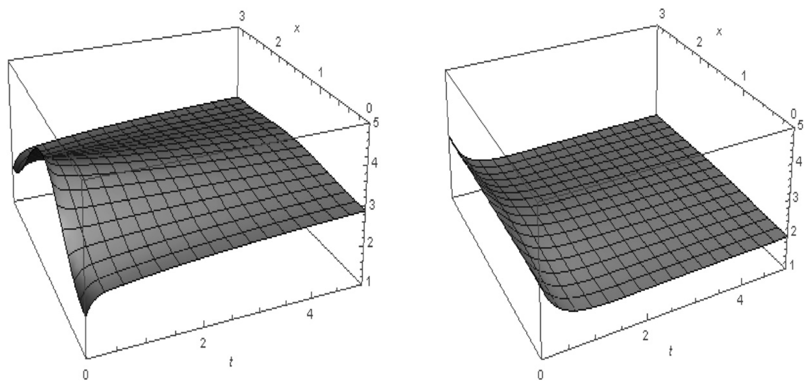
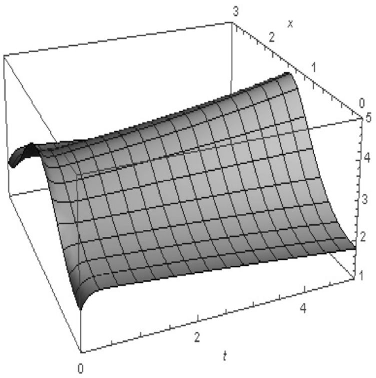


FIGURE 2. Result of the numerical computation of Example 1
The figure of the lefthand side is the graph of $u = u(x,t)$, and the righthand side is the graph of $v = v(x,t)$.



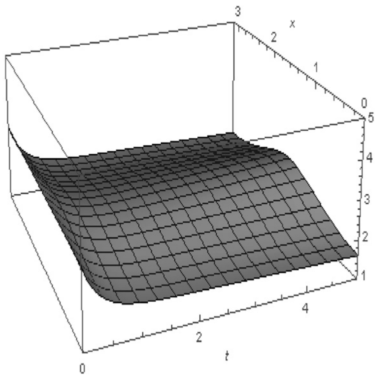


FIGURE 3. Result of the numerical computation of Example 2
The figure of the lefthand side is the graph of $u = u(x, t)$, and the righthand side is the graph of $v = v(x, t)$.

7. Appendix

As the final section of this paper, we shall present the proofs of Propositions 3 and 6.

7.1. Proof of Proposition 3. We use an eigenfunction expansion method. We set

$$w(x,t) := \sum_{n=0}^{\infty} w_n(x,t) = \sum_{n=0}^{\infty} T_n(t) X_n(x)\,, \tag{27}$$

where

$$X_n(x) = \cos\lambda_n(x-a), \quad \lambda_n = \frac{n\pi}{b-a}\,. \tag{28}$$

Step 1. We set

$$z(x,t) := \sum_{n=0}^{\infty} z_n(t) X_n(x)\,. \tag{29}$$

From (29), we have the following equation:

$$\int_a^b z(x,t)\cos\lambda_m(x-a)\,dx = \int_a^b \sum_{n=0}^{\infty} z_n(t)\cos\lambda_n(x-a)\cos\lambda_m(x-a)\,dx\,. \tag{30}$$

Note that if $n \neq m$, then it holds that

$$\int_a^b \cos\lambda_n(x-a)\cos\lambda_m(x-a)\,dx = 0\,,$$

and in the case $n = m \neq 0$, then it holds that

$$\int_a^b \cos\lambda_n(x-a)\cos\lambda_n(x-a)\,dx = \frac{b-a}{2}\,.$$

By substituting these formulas into (30), we obtain

$$\int_a^b z(x,t)\cos\lambda_m(x-a)\,dx = \frac{b-a}{2} z_n(t) \quad (n \geq 1)\,,$$

namely,

$$z_n(t) = \frac{2}{b-a}\int_a^b z(x,t)\cos\lambda_n(x-a)\,dx \quad (n \geq 1)\,.$$

In the case $n = m = 0$, it holds that $\int_a^b \cos\lambda_0(x-a)\cos\lambda_0(x-a)\,dx = b-a$,

thus it follows that

$$z_0(t) = \frac{1}{b-a}\int_a^b z(x,t)\,dx\,.$$

Step 2. From (27) and (29), we know that (3.1) given in Proposition 3 means

$$T_n'(t)X_n(x) = T_n(t)X_n''(x) + z_n(t)X_n(x)\,. \tag{31}$$

For $n \geq 1$, (31) is transformed to

$$T_n'(t) + \lambda_n^2 T_n(t) = z_n(t)\,. \tag{32}$$

On the other hand, from (27) and (28), we notice that

$$\overline{w}(x) = \sum_{n=0}^{\infty} T_n(0)\cos\lambda_n(x-a)\,.$$

With the Fourier series expansion of $\overline{w}(x)$, we have

$$T_n(0) = \frac{2}{b-a}\int_a^b \overline{w}(x)\cos\lambda_n(x-a)\,dx \quad (n \geq 1)\,,$$

and

$$T_0(0) = \frac{1}{b-a}\int_a^b \overline{w}(x)\,dx\,.$$

For $n = 0$, (31) means

$$T_0'(t) = z_0(t)\,. \tag{33}$$

Step 3. Let us solve the differential equations (32) and (33) . By a standard method, we have the following solutions:

$$T_n(t) = e^{-\lambda_n^2 t}\Big(T_n(0) + \int_0^t z_n(\tau)e^{\lambda_n^2 \tau}\, d\tau\Big) \ \ (n \geq 1)\,, \tag{34}$$

and for $n = 0$,

$$T_0(t) = T_0(0) + \int_0^t z_n(\tau)\, d\tau\,.$$

Notice that (34) is correct in the case $n = 0$.

Step 4. Combined (27), (28) with (34), we have proved Proposition 3.

7.2. Proof of Proposition 6. We will show that for any $\epsilon > 0$, there exists $t_0 > 0$ such that $F(t) < \epsilon$ for all $t \geq t_0$. We set $\delta = \dfrac{k}{2l}\epsilon$. On the other hand, because it is assumed that $\lim\limits_{t\to\infty} G(t) = 0$, we know that there exists $t_1 > 0$ such that $G(t) < \delta$ for all $t \geq t_1$. From the given assumption, we have

$$F'(t) \leq -kF(t) + l\delta = -k\{F(t) - \frac{l\delta}{k}\}\,.$$

We set $X(t) = F(t) - \dfrac{l\delta}{k}$. Then it follows that

$$X'(t) \leq -kX(t)\,.$$

By Proposition 5, we obtain the following inequality:

$$X(t) \leq X(t_1)e^{-k(t-t_1)}\,.$$

Therefore, we have

$$F(t) \leq \{F(t_1) - \frac{l\delta}{k}\}e^{-k(t-t_1)} + \frac{l\delta}{k} \leq F(t_1)e^{-k(t-t_1)} + \frac{l\delta}{k}\,. \tag{35}$$

Since we can choose $t_0 (> t_1)$ such that

$$F(t_1)e^{-k(t_0-t_1)} < \frac{\epsilon}{2}\,,$$

the inequality (35) means $F(t) < \epsilon$ for all $t > t_0$. □

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References

[1] N. BELLOMO, A. BELLOUQUID, Y. TAO and M. WINKLER, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, Mathematical Models and Methods in Applied Sciences **25** (2015), 1663–1763.

[2] X. CAO, Global bounded solutions of the higher-dimensional Keller-Segel system under smallness conditions in optimal spaces, Discrete and Continuous Dynamical Systems **35** (2014), 1891–1904.

[3] T. CIEŚLAK, P. LAURENÇOT and C. MORALES-RODRIGO, Global existence and convergence to steady states in a chemorepulsion system, Parabolic and Navier-Stokes Equations. Part1 **81** (2008), 105–117.

[4] L. C. EVANS, *Parial Differential Equations*, American Mathematical Society (1998).

[5] D. GILBARG and N. S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer (1998).

[6] T. HILLEN and K. J. PAINTER, A user's guide to PDE models for chemotaxis, Journal of Mathematical Biology **58** (2009), 183–217.

[7] E. F. KELLER and L. A. SEGEL, Initiation of slime mold aggregation viewed as instability, Journal of Theoretical Biology **26** (1970), 399–415.

[8] E. KREYSZIG, *Advanced Engineering Mathematics*, 10*th Edition*, Wiley (2011).

[9] S. MIZOHATA, *The Theory of Partial Differential Equations*, Cambridge University Press (1979).

[10] K. OSAKI and A. YAGI, Finite dimensional attactor for one-dimensional Keller-Segel equations, Funkcialaj Ekvacioj **44** (2001), 441–469.

[11] Z. WANG and K. ZHAO, Global dynamics and diffusion limit of a one-dimensional repulsive chemotaxis model, Communcation on Pure and Applied Analysis **12** (2013), 3027–3046.

[12] M. WINKLER, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, Journal of Differential Equations **257** (2014), 1056–1077.

[13] Y. YAHAGI, A probabilistic consideration on one dimensional Keller Segel system, Neural, Parallel, and Scientific Computations **24** (2016), 15–28.

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