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A Function Determined by a Hypersurface of Positive Characteristic

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Abstract. Let $R = k[\![X_1, \ldots, X_{n+1}]\!]$ be a formal power series ring over a perfect field k of characteristic $p > 0$, and let $\mathfrak{m} = (X_1, \ldots, X_{n+1})$ be the maximal ideal of R. Suppose $0 \neq f \in \mathfrak{m}$. In this paper, we introduce a function $\xi_f(x)$ associated with a hypersurface $R/(f)$ defined on the closed interval [0, 1] in $\mathbf{R}$. The Hilbert-Kunz multiplicity and the F-signature of $R/(f)$ appear as the values of our function $\xi_f(x)$ on the interval's endpoints. The F-signature of the pair, denoted by $s(R, f^t)$, was defined by Blickle, Schwede and Tucker. Our function $\xi_f(x)$ is integrable, and the integral $\int_t^1 \xi_f(x)dx$ is just $s(R, f^t)$ for any $t \in [0, 1]$.

1. Introduction

For Noetherian local rings of characteristic $p > 0$, some important invariants can be defined using the Frobenius endomorphism as follows.

The Hibert-Kunz multiplicity $e_{HK}(R)$ of a d-dimensional Noetherian local ring $(R, \mathfrak{n}, k)$ of characteristic $p > 0$ is defined by Kunz [9] to be

$$e_{HK}(R) = \lim_{e \to \infty} \frac{\ell(R/\mathfrak{n}^{[p^e]})}{p^{ed}},$$

where $\ell(R/\mathfrak{n}^{[p^e]})$ is the length of $R/\mathfrak{n}^{[p^e]}$, and $\mathfrak{n}^{[p^e]}$ is the ideal generated by all the p^e-th powers of elements of $\mathfrak{n}$. Monsky [11] showed that this limit always exists. The Hibert-Kunz multiplicity $e_{HK}(R)$ gives a measure of the singularity of R. In fact, for an unmixed local ring of characteristic $p > 0$, Watanabe and Yoshida [14] proved that $e_{HK}(R) = 1$ if and only if R is regular.

Huneke and Leuschke [7] defined the F-signature $s(R)$ of a d-dimensional reduced Noetherian local ring of characteristic $p > 0$ to be

$$s(R) = \lim_{e \to \infty} \frac{a_e}{p^{ed}},$$

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where a_e is the e-th Frobenius splitting number of R, that is the largest integer such that $R^{\oplus a_e}$ is a direct summand of $R^{\frac{1}{p^e}}$. Tucker [13] proved that this limit always exists. Huneke and Leuschke [7] proved that $0 \leq s(R) \leq 1$, and $s(R) = 1$ if and only if R is regular. Therefore, F-signature $s(R)$ gives a measure of the singularity of R, as well as Hibert-Kunz multiplicity. Aberbach and Leuschke [2] proved that $s(R) > 0$ if and only if R is strongly F-regular.

The F-pure threshold $\mathrm{fpt}(f)$ for an element f in R was defined by Takagi and Watanabe [12] to be

$$\mathrm{fpt}(f) = \lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e},$$

where $\mu_f(p^e) = \min\{t \geq 1 \mid f^t \in \mathfrak{m}^{[p^e]}\}$ for each integer $e > 0$. This limit exists because the sequence $\left\{\frac{\mu_f(p^e)}{p^e}\right\}_{e>0}$ is decreasing and $\frac{\mu_f(p^e)}{p^e} \geq 0$ for any $e > 0$.

Blickle, Schwede and Tucker [4] defined the F-signature

$$s(R, f^t) = \lim_{e\to\infty} \frac{1}{p^{e(n+1)}} \ell_R \left(\frac{R}{\mathfrak{n}^{[p^e]} : f^{\lceil t(p^e-1)\rceil}} \right)$$

of a pair (R, f^t) for an F-finite regular local ring $(R, \mathfrak{n})$, $0 \neq f \in \mathfrak{n}$ and a real number $t \in [0, 1]$. They proved the following. The right derivative of $s(R, f^t)$ exists at $t = 0$ and equals to the negative of the Hilbert-Kunz multiplicity of $R/(f)$. The left derivative of $s(R, f^t)$ exists at $t = 1$ and equals to the negative of the F-signature of $R/(f)$.

The purpose of this paper is to introduce a function $\xi_f(x)$ associated with a hypersurface $R/(f)$ defined on the closed interval $[0, 1]$ in $\mathbf{R}$. The function $\xi_f(x)$ is decreasing and Riemann integrable. Important invariants for Noetherian local rings of characteristic $p > 0$ appears in this function $\xi_f(x)$. In fact, the Hilbert-Kunz multiplicity $e_{HK}(R/(f))$ equals to $\xi_f(0)$, and the F-signature $s(R/(f))$ equals to $\xi_f(1)$. We shall prove that $\xi'_f(0) = 0$ if $R/(f)$ is normal. The F-pure threshold $\mathrm{fpt}(f)$ satisfies $\xi_f(\mathrm{fpt}(f) + \delta) = 0$ and $\xi_f(\mathrm{fpt}(f) - \delta) > 0$ for any small real number $\delta > 0$. We show

$$\int_t^1 \xi_f(x)dx = s(R, f^t)$$

for $t \in [0, 1]$, and

$$\int_0^1 \xi_f(x)dx = 1.$$

In Section 2, we define this function $\xi_f(x)$ and state our main theorem. We investigate the basic behavior of $\xi_f(x)$ here. Considering this function, we prove

$$e_{HK}(R/(f)) \times \mathrm{fpt}(f) \geq 1$$

in Corollary 1. In Section 3, we calculate this function $\xi_f(x)$ for a monomial f. We obtain an example of $\xi_f(x)$ which is continuous on $[0, 1]$. Furthermore, we know that $\xi_f(x)$ is discontinuous in almost all cases.

2. The main theorem

The aim of this section is to state the main theorem and prove it.

In the rest of this paper, let $n \geq 1$ be an integer. Let $R = k[[X_1, \dots, X_{n+1}]]$ be a formal power series ring over a perfect field k of characteristic $p > 0$, and let $\mathfrak{m} = (X_1, \dots, X_{n+1})$ be the maximal ideal of R. Suppose $0 \neq f \in \mathfrak{m}$. Rings of the form $R/(f)$ are called "n-dimensional hypersurfaces".

DEFINITION 1. We define

$$M_{e,t} = \frac{(f^t) + \mathfrak{m}^{[p^e]}}{(f^{t+1}) + \mathfrak{m}^{[p^e]}} \simeq \frac{R}{\left((f^{t+1}) + \mathfrak{m}^{[p^e]}\right) : f^t} = \frac{R}{(f) + (\mathfrak{m}^{[p^e]} : f^t)},$$

where $e \geq 0$ and $t \geq 0$ are integers.

Since $(f) + (\mathfrak{m}^{[p^e]} : f^t) \subset (f) + (\mathfrak{m}^{[p^e]} : f^{t+1})$, the natural surjection $M_{e,t} \to M_{e,t+1}$ exists. Let $\overline{R} = R/\mathfrak{m}^{[p^e]}$. Then, remark that $M_{e,t} = f^t\overline{R}/f^{t+1}\overline{R}$.

DEFINITION 2. We define

$$C_{e,t} = \frac{\ell_R(M_{e,t})}{p^{en}},$$

where $\ell_R(M_{e,t})$ is the length as an R-module.

Then we have

$$p^e \geq C_{e,0} \geq C_{e,1} \geq C_{e,2} \geq \cdots \geq C_{e,p^e-1} \geq C_{e,p^e} = C_{e,p^e+1} = \cdots = 0. \tag{2.1}$$

A sequence of functions $\{\xi_{f,e} : [0, 1] \to \mathbf{R}\}_{e \geq 0}$ is defined by

$$\xi_{f,e}(x) = \begin{cases} C_{e,\lfloor xp^e \rfloor} & (0 \leq x < 1), \\ C_{e,p^e-1} & (x = 1), \end{cases}$$

where $\lfloor xp^e \rfloor = \max\{a \in \mathbf{Z} | xp^e \geq a\}$ is the floor function. By the definition, we have $\int_0^1 \xi_{f,e}(x)dx = 1$ because

$$\begin{aligned} \int_0^1 \xi_{f,e}(x)dx &= \frac{1}{p^e}\left(C_{e,0} + C_{e,1} + C_{e,2} + \cdots + C_{e,p^e-1}\right) \\ &= \frac{1}{p^e} \times \frac{1}{p^{en}}\left(\ell_R(M_{e,0}) + \ell_R(M_{e,1}) + \cdots + \ell_R(M_{e,p^e-1})\right) \\ &= \frac{1}{p^{e(n+1)}}\ell_R(R/\mathfrak{m}^{[p^e]}) \end{aligned}$$

$$= \frac{1}{p^{e(n+1)}} \times p^{e(n+1)}$$
$$= 1\,.$$

DEFINITION 3. We define the function $\xi_f(x)$ by

$$\xi_f(x) = \limsup_{e\to\infty} \xi_{f,e}(x)$$

for $x \in [0, 1]$.

By Eq. (2.1), $\xi_f(x)$ is decreasing on $[0, 1]$. If $\lim_{e\to\infty} \xi_{f,e}(\alpha)$ exists, then $\xi_f(\alpha) = \lim_{e\to\infty} \xi_{f,e}(\alpha)$. The sequence $\{C_{e,0}\}_e$ is increasing by Lemma 1 in this section.

$$\lim_{e\to\infty} C_{e,0} = \lim_{e\to\infty} \frac{\ell_R(M_{e,0})}{p^{en}} = \lim_{e\to\infty} \frac{\ell_R(R/(f) + \mathfrak{m}^{[p^e]})}{p^{en}}\,.$$

This limit exists and is called the Hilbert-Kunz multiplicity of $R/(f)$, denoted by $e_{HK}(R/(f))$. Therefore, by (2.1), $\limsup_{e\to\infty} \xi_{f,e}(\alpha)$ is not $+\infty$ for any $\alpha \in [0, 1]$. We shall give an example that $\lim_{e\to\infty} \xi_{f,e}(\alpha)$ does not exist for some $f \in R$ and $\alpha \in [0, 1]$ in Section 3. We have

$$\xi_f(0) = e_{HK}(R/(f))\,.$$

Therefore, $\xi_f(x)$ is a bounded and decreasing function on $[0, 1]$. In particular, $\xi_f(x)$ is integrable, and has at most countably many points of discontinuity on $[0, 1]$.

The main theorem of this paper is the following:

THEOREM 1. 1) *The function $\xi_f(x)$ is decreasing. There exists a countable subset C of the interval $[0, 1]$ such that $\xi_f(x)$ is continuous at any $\alpha \in [0, 1]-C$. Moreover, $\xi_f(x)$ is continuous at 0 and 1.*

2) *If $\xi_f(x)$ is continuous at $\alpha \in [0, 1]$, then $\lim_{e\to\infty} \xi_{f,e}(\alpha) = \xi_f(\alpha)$.*
3) *We have $\xi_f(0) = e_{HK}(R/(f))$, and also $\xi_f(1) = s(R/(f))$.*
4) *Suppose that $\xi_f(1) = 0$, then $\mathrm{fpt}(f) = \inf\{\alpha \in [0, 1] \mid \xi_f(\alpha) = 0\}$ holds.*
5) *The function $\xi_f(x)$ is integrable, and we have $\int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_f(x)dx = \frac{\ell_R(M_{e,a})}{p^{e(n+1)}}$ for integers $0 \le a < p^e$. In particular, $\int_0^1 \xi_f(x)dx = 1$ holds.*
6) *If $R/(f)$ is normal then $\xi'_f(0) = 0$, where ξ'_f is the derivative of ξ_f.*

REMARK 1. By Theorem 1.1 and Proposition 3.2 (i) in [3], we know that above $\mathrm{fpt}(f)$ is a positive rational number. Note that F-pure thresholds are defined as the smallest F-jumping exponents in [3].

REMARK 2. We define the function $\varphi_f(x)$ on [0, 1] as follows;

$$\varphi_f(x) = \int_0^x \xi_f(t)dt\,.$$

Actually, we have

$$\varphi_f(x) = \lim_{e\to\infty} \frac{1}{p^e}\left(C_{e,\,0} + C_{e,\,1} + \cdots + C_{e,\,\lfloor xp^e\rfloor-1}\right)\,.$$

Since $\xi_f(x)$ is bounded and integrable on [0, 1], $\varphi_f(x)$ is Lipschitz continuous on [0, 1]. In particular, $\varphi_f(x)$ is continuous on [0, 1]. We can rewrite 3) and 4) in Theorem 1 as follows;

3′) The function $\varphi_f(x)$ is differentiable at $x = 0$ and 1, and $\varphi'_f(0) = e_{HK}(R/(f))$ and $\varphi'_f(1) = s(R/(f))$.

4′) Suppose that $s(R/(f)) = 0$, then

$$\mathrm{fpt}(f) = \inf\{\alpha \in [0, 1] \mid \varphi_f(\alpha) = 1\}$$

holds.

Using 5) in Theorem 1, we know

$$1 - \varphi_f(x) = \int_t^1 \xi_f(x)dx = s(R, f^t)$$

for $t \in [0, 1]$. Moreover, if we know that $\xi_f(x)$ is continuous at 0 and 1 (see Theorem 1 1)), we obtain 3) in Theorem 1 immediately from Theorem 4.4 in [4].

In this section, we shall prove Theorem 1. The following corollary immediately follows from Theorem 1 3) and 5).

COROLLARY 1. $e_{HK}(R/(f)) \times \mathrm{fpt}(f) \geq 1$.

EXAMPLE 1. Suppose $R = k[\![X_1, X_2, \dots, X_{n+1}]\!]$ and $\alpha > 0$. Then $e_{HK}(R/(X_1^\alpha)) = \alpha$ and $\mathrm{fpt}(X_1^\alpha) = \dfrac{1}{\alpha}$. Therefore, if $\tau(f) = X_1^\alpha$ for a linear transformation τ (for example, $f = X_1 + X_2$), then $e_{HK}(R/(f)) \times \mathrm{fpt}(f) = 1$ and $s(R/(f)) = 1$ (see Section 3). We do not know another example that the equality holds in Corollary 1.

REMARK 3. By Theorem 1 1), 3) and 5), we immediately know that $e_{HK}(R/(f)) = 1$ if and only if $s(R/(f)) = 1$. These conditions are equivalent to that $R/(f)$ is regular by the following results.

1) Let S be an unmixed local ring of positive characteristic. Then $e_{HK}(S) = 1$ if and only if S is regular ([14], Theorem 1.5).
2) Let S be a reduced F-finite Cohen-Macaulay local ring of positive characteristic. Then $s(S) = 1$ if and only if S is regular ([7], Corollary 16).

REMARK 4. Let $m < n = \dim R/(f)$, and set $a_e = \ell(M_{e,\,p^e-1})$. Assume that $a_e = \alpha p^{em} + o(p^{em})$, that is $\lim_{e\to\infty} \frac{a_e}{p^{em}} = \alpha$. Let $g_e = a_e - \alpha p^{em}$. Then

$$\begin{aligned}\varphi_f(1) - \varphi_f\left(\frac{p^e-1}{p^e}\right) &= \sum_{i=0}^{p^e-1} \frac{\ell(M_{e,\,i})}{p^{e(n+1)}} - \sum_{i=0}^{p^e-2} \frac{\ell(M_{e,\,i})}{p^{e(n+1)}}\\ &= \frac{\ell(M_{e,\,p^e-1})}{p^{e(n+1)}}\\ &= \frac{\alpha}{p^{e(n-m+1)}} + \frac{g_e}{p^{e(n+1)}}\end{aligned}$$

holds. Let $x = \frac{p^e-1}{p^e}$. Since $x - 1 = -\frac{1}{p^e}$, we know

$$\varphi_f(x) = \varphi_f(1) + (-1)^{n-m}\alpha(x-1)^{n-m+1} + o((x-1)^{n-m+1})\,. \tag{2.2}$$

Since $\varphi_f(x)$ is continuous on [0, 1] from Remark 1, $\varphi_f(x)$ has the form of Eq. (2.2) around the point $x = 1$. Therefore, if $\varphi_f(x)$ is equal to its Taylor series around the point $x = 1$, we obtain that

$$\varphi_f^{(i)}(1) = \begin{cases} 0 & (i = 1, 2, \ldots, n-m)\,,\\ (-1)^{n-m}(n-m+1)!\alpha & (i = n-m+1)\,,\end{cases}$$

$$\xi_f^{(i)}(x) = \begin{cases} 0 & (i = 1, 2, \ldots, n-m-1)\,,\\ (-1)^{n-m}(n-m+1)!\alpha & (i = n-m)\,.\end{cases}$$

Let $F : R \to R$ be the Frobenius map $a \mapsto a^p$. Since k is perfect, we have $F_*R \simeq R^{\oplus p^{n+1}}$, where F_*R stands for $F_*^1 R$. Therefore,

$$(M_{e,\,t})^{\oplus p^{n+1}} \simeq M_{e,\,t} \otimes_R F_*R = \frac{((f^t) + \mathfrak{m}^{[p^e]})F_*R}{((f^{t+1}) + \mathfrak{m}^{[p^e]})F_*R} = F_*\left(\frac{(f^{pt}) + \mathfrak{m}^{[p^{e+1}]}}{(f^{pt+p}) + \mathfrak{m}^{[p^{e+1}]}}\right)$$

for all $e, t \geq 0$. Consequently,

$$p \times C_{e,\,t} = C_{e+1,\,pt} + C_{e+1,\,pt+1} + \cdots + C_{e+1,\,pt+p-1}, \tag{2.3}$$

where the sum on the right-hand side of Eq. (2.3) has p-terms. That is, $C_{e,\,t}$ is the mean of $C_{e+1,\,pt}, C_{e+1,\,pt+1}, \ldots, C_{e+1,\,pt+p-1}$. Therefore, by Eq. (2.1) and Eq. (2.3), we obtain the following inequalities immediately.

LEMMA 1. $C_{e+1,\,pt} \geq C_{e,\,t} \geq C_{e+1,\,pt+p-1}$.

Hence, by Eq. (2.1) and Lemma 1, we have

$$\begin{array}{ccccc}
C_{e,\,\lfloor xp^e\rfloor-1} & \underset{\text{by Lemma 1}}{\geq} & C_{e+1,\,(\lfloor xp^e\rfloor-1)p+(p-1)} & \geq & C_{e+1,\,\lfloor xp^{e+1}\rfloor-1} \\
\rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\
C_{e,\,\lfloor xp^e\rfloor} & & & & C_{e+1,\,\lfloor xp^{e+1}\rfloor} \\
\rotatebox{90}{$\leq$} & & & & \rotatebox{90}{$\leq$} \\
C_{e,\,\lceil xp^e\rceil} & \underset{\text{by Lemma 1}}{\leq} & C_{e+1,\,\lceil xp^e\rceil p} & \leq & C_{e+1,\,\lceil xp^{e+1}\rceil}
\end{array}$$

and here, we note that $\lfloor xp^e\rfloor p \leq \lfloor xp^{e+1}\rfloor$ and $\lceil xp^e\rceil p \geq \lceil xp^{e+1}\rceil$. Therefore, the sequence $\{C_{e,\,\lfloor xp^e\rfloor-1}\}_e$ is decreasing, the sequence $\{C_{e,\,\lceil xp^e\rceil}\}_e$ is increasing, and $C_{e,\,\lfloor xp^e\rfloor-1} \geq C_{e,\,\lceil xp^e\rceil}$ for all $e \geq 0$ by Eq. (2.1). Consequently, the limits $\lim_{e\to\infty} C_{e,\,\lfloor xp^e\rfloor-1}$ and $\lim_{e\to\infty} C_{e,\,\lceil xp^e\rceil}$ exist in $\mathbf{R}$. In particular,

$$C_{e,\,\lfloor \alpha p^e\rfloor-1} \geq \lim_{e\to\infty} C_{e,\,\lfloor \alpha p^e\rfloor-1} \geq \xi_f(\alpha) \geq \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil} \geq C_{e,\,\lceil \alpha p^e\rceil} \geq 0 \tag{2.4}$$

holds for any $\alpha \in (0, 1]$ and e satisfying $\lfloor \alpha p^e\rfloor - 1 \geq 0$.

LEMMA 2. *We set* $\overline{C}(\alpha) = \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil}$ *for* $\alpha \in [0, 1]$ *and* $\underline{C}(\beta) = \lim_{e\to\infty} C_{e,\,\lfloor \beta p^e\rfloor-1}$ *for* $\beta \in (0, 1]$.

1) *For* $\alpha \in [0, 1]$ *and any integer* $i \geq 0$, $\{C_{e+1,\,\lceil \alpha p^e\rceil p+i}\}_e$ *is an increasing sequence. The limits* $\lim_{e\to\infty} C_{e+1,\,\lceil \alpha p^e\rceil p+i}$ *and* $\lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil+k}$ *exist for any non-negative integers* $i, k \geq 0$. *Furthermore,*
$$\overline{C}(\alpha) = \lim_{e\to\infty} C_{e+1,\,\lceil \alpha p^e\rceil p+i} = \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil+k} \tag{2.5}$$
holds.
2) *For* $\beta \in (0, 1]$ *and any integer* $i > 0$, $\{C_{e+1,\,\lfloor \beta p^e\rfloor p-i}\}_e$ *is a decreasing sequence. The limits* $\lim_{e\to\infty} C_{e+1,\,\lfloor \beta p^e\rfloor p-i}$ *and* $\lim_{e\to\infty} C_{e,\,\lfloor \beta p^e\rfloor-k}$ *exist for any positive integers* $i, k > 0$. *Furthermore,*
$$\underline{C}(\beta) = \lim_{e\to\infty} C_{e+1,\,\lfloor \beta p^e\rfloor p-i} = \lim_{e\to\infty} C_{e,\,\lfloor \beta p^e\rfloor-k} \tag{2.6}$$
holds.

PROOF. Let $\alpha \in [0, 1]$ and $\beta \in (0, 1]$, and let $k \geq 0$ and $\ell > 0$ be integers. We know

$$\begin{cases} (\lceil \alpha p^e\rceil p + k)p = \lceil \alpha p^e\rceil p^2 + kp \geq \lceil \alpha p^{e+1}\rceil p + kp \geq \lceil \alpha p^{e+1}\rceil p + k\,, \\ (\lfloor \beta p^e\rfloor p - \ell)p + (p-1) \leq \lfloor \beta p^e\rfloor p^2 - \ell p + (p-1)\ell \leq \lfloor \beta p^{e+1}\rfloor p - \ell\,, \end{cases}$$

and therefore

$$\begin{cases} C_{e+1,\,\lceil \alpha p^e\rceil p+k} \leq C_{e+2,\,(\lceil \alpha p^e\rceil p+k)p} \leq C_{e+2,\,\lceil \alpha p^{e+1}\rceil p+k} \leq \lim_{e\to\infty} C_{e,\,0}\,, \\ C_{e+1,\,\lfloor \beta p^e\rfloor p-\ell} \geq C_{e+2,\,(\lfloor \beta p^e\rfloor p-\ell)p+(p-1)} \geq C_{e+2,\,\lfloor \beta p^{e+1}\rfloor p-\ell} \geq 0\,, \end{cases}$$

by Eq. (2.1) and Lemma 1. Hence, $\{C_{e+1,\, \lceil \alpha p^e \rceil p+k}\}_e$ is increasing and bounded. $\{C_{e+1,\, \lfloor \beta p^e \rfloor p-\ell}\}_e$ is decreasing and bounded. Therefore, $\lim_{e\to\infty} C_{e+1,\, \lceil \alpha p^e \rceil p+k}$ and $\lim_{e\to\infty} C_{e+1,\, \lfloor \beta p^e \rfloor p-\ell}$ exist.

Next, we shall show that

$$\overline{C}(\alpha) = \lim_{e\to\infty} C_{e+1,\, \lceil \alpha p^e \rceil p+i} \tag{2.7}$$

and

$$\underline{C}(\beta) = \lim_{e\to\infty} C_{e+1,\, \lfloor \beta p^e \rfloor p-j} \tag{2.8}$$

hold for any integers $0 \le i \le p-1$ and $1 \le j \le p$. We have

$$\begin{cases} p \times C_{e,\, \lceil \alpha p^e \rceil} = C_{e+1,\, \lceil \alpha p^e \rceil p} + C_{e+1,\, \lceil \alpha p^e \rceil p+1} + \cdots + C_{e+1,\, \lceil \alpha p^e \rceil p+p-1}\,, \\ p \times C_{e,\, \lfloor \beta p^e \rfloor -1} = C_{e+1,\, \lfloor \beta p^e \rfloor p-p} + C_{e+1,\, \lfloor \beta p^e \rfloor p-(p-1)} + \cdots + C_{e+1,\, \lfloor \beta p^e \rfloor p-1}\,, \end{cases}$$

by Eq. (2.3). Thus, it holds that

$$\begin{cases} p \times \lim\limits_{e\to\infty} C_{e,\, \lceil \alpha p^e \rceil} = \lim\limits_{e\to\infty} C_{e+1,\, \lceil \alpha p^e \rceil p} + \cdots + \lim\limits_{e\to\infty} C_{e+1,\, \lceil \alpha p^e \rceil p+p-1}\,, \\ p \times \lim\limits_{e\to\infty} C_{e,\, \lfloor \beta p^e \rfloor -1} = \lim\limits_{e\to\infty} C_{e+1,\, \lfloor \beta p^e \rfloor p-p} + \cdots + \lim\limits_{e\to\infty} C_{e+1,\, \lfloor \beta p^e \rfloor p-1}\,. \end{cases}$$

On the other hand, we have

$$\begin{cases} \lim\limits_{e\to\infty} C_{e,\, \lceil \alpha p^e \rceil} = \lim\limits_{e\to\infty} C_{e+1, \lceil \alpha p^e \rceil p} \ge \lim\limits_{e\to\infty} C_{e+1, \lceil \alpha p^e \rceil p+1} \\ \qquad \ge \cdots \ge \lim_{e\to\infty} C_{e+1, \lceil \alpha p^e \rceil p+p-1}\,, \\ \lim\limits_{e\to\infty} C_{e,\, \lfloor \beta p^e \rfloor -1} = \lim\limits_{e\to\infty} C_{e+1, \lfloor \beta p^e \rfloor p-1} \le \lim\limits_{e\to\infty} C_{e+1, \lfloor \beta p^e \rfloor p-2} \\ \qquad \le \cdots \le \lim_{e\to\infty} C_{e+1, \lfloor \beta p^e \rfloor p-p}\,, \end{cases}$$

since $C_{e,\, \lceil \alpha p^e \rceil} \le C_{e+1,\, \lceil \alpha p^e \rceil p} \le C_{e+1,\, \lceil \alpha p^{e+1} \rceil}$ and $C_{e,\, \lfloor \beta p^e \rfloor -1} \ge C_{e+1,\, \lfloor \beta p^e \rfloor p-1} \ge C_{e+1,\, \lfloor \beta p^{e+1} \rfloor -1}$. Consequently, we have Eq. (2.7) and Eq. (2.8).

In order to complete the proof of the assertion 1), we have the inequalities

$$\begin{aligned} C_{e,\, \lceil \alpha p^e \rceil +k} &\le C_{e+1,\, (\lceil \alpha p^e \rceil +k)p} \\ &= C_{e+1,\, \lceil \alpha p^e \rceil p+kp} \\ &\le C_{e+1,\, \lceil \alpha p^e \rceil p+k} \\ &\le C_{e+1,\, \lceil \alpha p^{e+1} \rceil +k} \end{aligned}$$

for any $k \ge 1$. Hence,

$$\lim_{e\to\infty} C_{e,\, \lceil \alpha p^e \rceil +k} = \lim_{e\to\infty} C_{e+1,\, \lceil \alpha p^e \rceil p+k}$$

holds. Therefore, we obtain Eq. (2.5).

In order to complete the proof of the assertion 2), we have the inequalities

$$\begin{aligned} C_{e,\,\lfloor \beta p^e\rfloor-k} &\geq C_{e+1,\,(\lfloor \beta p^e\rfloor-k)p+p-1} \\ &= C_{e+1,\,\lfloor \beta p^e\rfloor p-(k-1)p-1} \\ &\geq C_{e+1,\,\lfloor \beta p^e\rfloor p-k} \\ &\geq C_{e+1,\,\lfloor \beta p^{e+1}\rfloor-k} \end{aligned}$$

for any $k \geq 2$. Hence,

$$\lim_{e\to\infty} C_{e,\,\lfloor \beta p^e\rfloor-k} = \lim_{e\to\infty} C_{e+1,\,\lfloor \beta p^e\rfloor p-k}$$

holds. Therefore, we obtain Eq. (2.6). □

PROPOSITION 1. 1) *For $\alpha \in [0, 1)$, $\lim_{x\to\alpha+0} \xi_f(x) = \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil}$ holds.*

2) *For $\beta \in (0, 1]$, $\lim_{x\to\beta-0} \xi_f(x) = \lim_{e\to\infty} C_{e,\,\lfloor \beta p^e\rfloor-1}$ holds.*

In particular, we have

$$\begin{cases} \lim\limits_{x\to+0} \xi_f(x) = \lim\limits_{e\to\infty} C_{e,\,0} = \xi_f(0)\,, \\ \lim\limits_{x\to 1-0} \xi_f(x) = \lim\limits_{e\to\infty} C_{e,\,p^e-1} = \xi_f(1)\,, \end{cases}$$

that is to say that $\xi_f(x)$ is continuous at $x = 0$ and 1.

PROOF. 1) First, we show $\lim_{x\to\alpha+0} \xi_f(x) \leq \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil}$. Take $x_0 > \alpha$. For a large enough number e', we may assume that $\alpha p^{e'} \leq x_0 p^{e'} - 2$ holds. Then, $\lceil \alpha p^{e'}\rceil \leq \lfloor x_0 p^{e'}\rfloor - 1$. Hence, by Eq. (2.1) and Eq. (2.4),

$$\xi_f(x_0) \leq C_{e',\,\lfloor x_0 p^{e'}\rfloor-1} \leq C_{e',\,\lceil \alpha p^{e'}\rceil} \leq \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil}\,,$$

as desired.

Next, we shall show the opposite inequality. By Lemma 2 1), we have only to show that

$$\lim_{x\to\alpha+0} \xi_f(x) \geq \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil+1}\,.$$

For any $e \geq 0$, $\alpha < \frac{\lceil \alpha p^e\rceil+1}{p^e}$. Hence, there exists a real number $x_1 \in \mathbf{R}$ such that $\alpha < x_1 < \frac{\lceil \alpha p^e\rceil+1}{p^e}$. Then $\lceil x_1 p^e\rceil \leq \lceil \alpha p^e\rceil + 1$, and therefore

$$\lim_{x\to\alpha+0} \xi_f(x) \geq \xi_f(x_1) \geq C_{e,\,\lceil x_1 p^e\rceil} \geq C_{e,\,\lceil \alpha p^e\rceil+1}$$

for any $e \geq 0$ because we have Eq. (2.1) and Eq. (2.4), and $\xi_f(x)$ is decreasing. Consequently,

$$\lim_{x\to\alpha+0} \xi_f(x) \geq \lim_{e\to\infty} C_{e,\,\lceil \alpha p^e\rceil+1}\,,$$

as desired.

2) It is proved in the same way as 1). □

REMARK 5. From Eq. (2.1), we have

$$C_{e,\lfloor \alpha p^e \rfloor -1} \geq \xi_{f,e}(\alpha) = C_{e,\lfloor \alpha p^e \rfloor} \geq C_{e,\lceil \alpha p^e \rceil}$$

for any $\alpha \in [0, 1]$. Hence, if

$$\lim_{e\to\infty} C_{e,\lfloor \alpha p^e \rfloor -1} = \lim_{e\to\infty} C_{e,\lceil \alpha p^e \rceil},$$

there exists $\lim_{e\to\infty} \xi_{f,e}(\alpha)$ in $\mathbf{R}$, and it is equal to $\xi_f(\alpha)$.

COROLLARY 2. *If $\xi_f(x)$ is continuous at $\alpha \in [0, 1]$ then $\lim_{e\to\infty} \xi_{f,e}(\alpha)$ exists, so that it is equal to $\xi_f(\alpha)$.*

PROOF. The proof is obtained from Remark 5 immediately. □

We have just shown Theorem 1 1).

We obtain the following Corollary 3 immediately from Proposition 1.

COROLLARY 3. *We define $\varphi_f(x)$ by*

$$\varphi_f(x) = \int_0^x \xi_f(t)dt$$

for $x \in [0, 1]$. Then we have the followings.

1) *$\varphi_f(x)$ is differentiable at* 0, *and* $\varphi'_f(0) = \xi_f(0) = \lim_{e\to\infty} C_{e,0} = e_{HK}(R/(f))$.
2) *$\varphi_f(x)$ is differentiable at* 1, *and* $\varphi'_f(1) = \xi_f(1) = \lim_{e\to\infty} C_{e,p^e-1}$.

Set $\mu_f(p^e) = \min\{t \geq 0 \mid f^t \in \mathfrak{m}^{[p^e]}\}$ for each $e \geq 0$. Since $f^{\mu_f(p^e)} \in \mathfrak{m}^{[p^e]}$, $f^{\mu_f(p^e)p} \in \mathfrak{m}^{[p^{e+1}]}$. Hence $\mu_f(p^e)p \geq \mu_f(p^{e+1})$, and so

$$1 \geq \frac{\mu_f(p^e)}{p^e} \geq \frac{\mu_f(p^{e+1})}{p^{e+1}} \geq 0\,.$$

Since $\left\{\frac{\mu_f(p^e)}{p^e}\right\}_{e\geq 0}$ is decreasing and bounded below, the limit $\lim_{e\to\infty} \frac{\mu_f(p^e)}{p^e}$ exists in $\mathbf{R}$, and it is called the F-pure threshold of f, denoted by $\mathrm{fpt}(f)$. It is easy to see that $\mathrm{fpt}(f) \in (0, 1]$, and $\mathrm{fpt}(f) = 1$ if and only if $\mu_f(p^e) = p^e$ for any $e \geq 1$.

LEMMA 3. *$C_{e,t} = 0$ if and only if $t \geq \mu_f(p^e)$.*

PROOF. If $M_{e,t} = 0$, then $M_{e,t} = M_{e,t+1} = M_{e,t+2} = \cdots = M_{e,p^e} = 0$. Hence, $f^t \in \mathfrak{m}^{[p^e]}$, and so $t \geq \mu_f(p^e)$. Conversely if $t \geq \mu_f(p^e)$, then $f^t \in \mathfrak{m}^{[p^e]}$ holds. □

We start to prove Theorem 1. The assertion 1) follows from Proposition 1. The assertion 2) follows from Corollary 2. The first half of 3) follows from the definition of $C_{e,0}$. Now, we shall show 4).

PROOF. First, we check that

$$\inf\{\alpha \in [0,1] \mid \xi_f(\alpha) = 0\} \le \mathrm{fpt}(f)\,.$$

If $\mathrm{fpt}(f) = 1$, then the assertion is easy. Assume $\mathrm{fpt}(f) < 1$. Let $1 > \alpha > \mathrm{fpt}(f)$. Since $\mathrm{fpt}(f) = \inf_{e \ge 0}\left\{\frac{\mu_f(p^e)}{p^e}\right\}$,

$$\mathrm{fpt}(f) \le \frac{\mu_f(p^{e_1})}{p^{e_1}} < \alpha$$

holds for $e_1 \gg 0$. Then, it holds that

$$\begin{aligned}\xi_f(\alpha) &\le \xi_f\left(\frac{\mu_f(p^{e_1})}{p^{e_1}}\right)\\ &= \limsup_{e\to\infty} C_{e,\left\lfloor \frac{\mu_f(p^{e_1})}{p^{e_1}} p^e \right\rfloor}\\ &= 0\end{aligned}$$

because, by Lemma 3,

$$C_{e_1+s,\ \mu_f(p^{e_1})p^s} \le C_{e_1+s,\ \mu_f(p^{e_1+s})} = 0$$

for any integers $s \ge 0$. Therefore, $\xi_f(\alpha) = 0$ for all $\alpha > \mathrm{fpt}(f)$, as desired. Conversely, suppose $\alpha < \mathrm{fpt}(f)$. Hence, we have $(\mathrm{fpt}(f) - \alpha)p^{e'} \ge 1$ for $e' \gg 0$, and therefore $\alpha p^{e'} \le \mathrm{fpt}(f)p^{e'} - 1$. Then, since we have

$$\alpha \le \frac{\mathrm{fpt}(f)p^{e'} - 1}{p^{e'}} < \frac{\mathrm{fpt}(f)p^{e'}}{p^{e'}} = \mathrm{fpt}(f) \le \frac{\mu_f(p^{e'})}{p^{e'}}\,,$$

we obtain

$$\alpha \le \frac{\mu_f(p^{e'}) - 1}{p^{e'}}\,.$$

Therefore,

$$\xi_f(\alpha) \ge \xi_f\left(\frac{\mu_f(p^{e'}) - 1}{p^{e'}}\right) \underset{\text{by Eq. (2.4)}}{\ge} \lim_{e\to\infty} C_{e,\left\lceil \frac{\mu_f(p^{e'})-1}{p^{e'}} p^e \right\rceil}$$

holds. We have $C_{e',\,\mu_f(p^{e'})-1} \ne 0$ by Lemma 3. Since $\left\{C_{e,\left\lceil \frac{\mu_f(p^{e'})-1}{p^{e'}} p^e \right\rceil}\right\}_{e\ge 0}$ is an increasing sequence, we obtain $\lim_{e\to\infty} C_{e,\left\lceil \frac{\mu_f(p^{e'})-1}{p^{e'}} p^e \right\rceil} > 0$. Therefore, $\xi_f(\alpha) > 0$ for all α such that $\alpha < \mathrm{fpt}(f)$, as desired. □

Next, we shall show 5).

PROOF. Let $F = \left\{\alpha \in \left[\frac{a}{p^e}, \frac{a+1}{p^e}\right] \;\middle|\; \alpha \text{ is a discontinuity for } \xi_f(x)\right\}$ and $\Omega = \left[\frac{a}{p^e}, \frac{a+1}{p^e}\right] - F$. Recall that F is a countable set, and $\lim_{s\to\infty} \xi_{f,s}(\alpha) = \xi_f(\alpha)$ for any $\alpha \in \Omega$ by Theorem 1 1), 2). Then, we have

$$\begin{aligned}\int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_f(x)dx &= \int_\Omega \xi_f(x)dx \\ &= \int_\Omega \lim_{s\to\infty} \xi_{f,s}(x)dx \\ &= \lim_{s\to\infty} \int_\Omega \xi_{f,s}(x)dx \\ &= \lim_{s\to\infty} \int_{\frac{a}{p^e}}^{\frac{a+1}{p^e}} \xi_{f,s}(x)dx \\ &= \frac{1}{p^e} C_{e,\,a}\end{aligned}$$

by Lebegue's dominated convergence theorem, as desired. □

We shall show 6).

PROOF. Let $g, h : \mathbf{N} \to \mathbf{R}$ be functions. If there exists a positive constant C such that $|h(n)| \leq Cg(n)$ for $n \gg 0$, then we write $h(n) = O(g(n))$. If $R/(f)$ is normal, then there exists $\beta(R/(f)) \in \mathbf{R}$ such that

$$e_{HK}(R/(f))p^{ne} + \beta(R/(f))p^{(n-1)e} = \ell_R(M_{e,\,0}) + O(p^{(n-2)e})$$

by Huneke-McDermott-Monsky [8]. Since a hypersurface is Gorenstein, $\beta(R/(f)) = 0$ follows from Corollary 1.4 in Kurano [10]. Therefore, we have

$$e_{HK}(R/(f))p^{ne} = \ell_R(M_{e,\,0}) + O(p^{(n-2)e})\,. \tag{2.9}$$

First, we shall show that

$$\left|\frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}}\right| \longrightarrow 0 \ \ (s \to \infty).$$

Since the sequence $\left\{C_{s+i,\, p^i}\right\}_{i\geq 0}$ is increasing, we have

$$\xi_f\left(\frac{1}{p^s}\right) = \limsup_{e\to\infty} C_{e,\,\lfloor p^{e-s}\rfloor} \geq C_{s,\,1}\,.$$

Hence, we obtain

$$\left| \frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}} \right| = \frac{\xi_f(0) - \xi_f\left(\frac{1}{p^s}\right)}{\frac{1}{p^s}} \le \frac{\xi_f(0) - C_{s,1}}{\frac{1}{p^s}} .$$

Set $\lambda_i(e) = e_{HK}(R/(f))p^{en} - \ell_R(M_{e,i})$ for each $e \ge 0$ and $0 \le i \le p-1$. Note that

$$0 \le \lambda_0(e) \le \lambda_1(e) \le \cdots \le \lambda_{p-1}(e) .$$

Since we have,

$$p \times \frac{\ell_R(M_{s-1,0})}{p^{(s-1)n}} = \frac{\ell_R(M_{s,0})}{p^{sn}} + \frac{\ell_R(M_{s,1})}{p^{sn}} + \cdots + \frac{\ell_R(M_{s,p-1})}{p^{sn}}$$

for any $s \ge 1$ by Eq. (2.3), then we obtain

$$p \times \frac{\lambda_0(s-1)}{p^{(s-1)n}} = \frac{\lambda_0(s)}{p^{sn}} + \frac{\lambda_1(s)}{p^{sn}} + \cdots + \frac{\lambda_{p-1}(s)}{p^{sn}} .$$

Hence, since

$$p \times \frac{\lambda_0(s-1)}{p^{(s-1)n}} \ge \frac{\lambda_1(s)}{p^{sn}} ,$$

it holds that

$$p^2 \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-1)}} \ge \frac{\lambda_1(s)}{p^{s(n-1)}} \ge 0 .$$

Therefore,

$$\begin{aligned}
\frac{\xi_f(0) - C_{s,1}}{\frac{1}{p^s}} &= \frac{p^s}{p^{sn}}\left(e_{HK}(R/(f))p^{sn} - C_{s,1} \times p^{sn}\right) \\
&= \frac{\lambda_1(s)}{p^{s(n-1)}} \\
&\le p^2 \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-1)}} \\
&= \frac{p^2}{p^{s-1}} \times \frac{\lambda_0(s-1)}{p^{(s-1)(n-2)}} \\
&\to 0 \quad (s \to \infty)
\end{aligned}$$

by Eq. (2.9). Consequently, for any positive real number $\varepsilon > 0$, there exists a natural number $s_0 \in \mathbf{N}$ such that $s \ge s_0$ implies that

$$\left| \frac{\xi_f\left(\frac{1}{p^s}\right) - \xi_f(0)}{\frac{1}{p^s}} \right| < \frac{\varepsilon}{p} .$$

Let $\delta = \dfrac{1}{p^{s_0}}$. If $0 < x < \delta$, then there exists $s \in \mathbf{N}$ such that

$$\frac{1}{p^{s+1}} < x < \frac{1}{p^{s}} \leq \frac{1}{p^{s_0}}\,.$$

Therefore,

$$\begin{aligned}\left|\frac{\xi_f(x)-\xi_f(0)}{x}\right| &= \frac{\xi_f(0)-\xi_f(x)}{x}\\ &\leq \frac{\xi_f(0)-\xi_f\left(\frac{1}{p^s}\right)}{\frac{1}{p^{s+1}}}\\ &\leq p \times \frac{\varepsilon}{p}\\ &= \varepsilon,\end{aligned}$$

as desired. □

Finally, we shall prove the last half of 3).

DEFINITION 4. Let $(S, \mathfrak{n})$ be a $(d+1)$-dimensional regular local ring. Let $0 \neq \alpha \in \mathfrak{n}$. The pair (ρ, σ) is called a *matrix factorization* of the element α if all of the following conditions are satisfied:

(1) $\rho : G \to F$ and $\sigma : F \to G$ are S-homomorphisms, where F and G are finitely generated S-free modules, and $\mathrm{rank}_S F = \mathrm{rank}_S G$.
(2) $\rho \circ \sigma = \alpha \cdot id_F$.
(3) $\sigma \circ \rho = \alpha \cdot id_G$.

Actually, if either (2) or (3) is satisfied, the other is satisfied.

DEFINITION 5. Let $(S, \mathfrak{n})$ be a $(d+1)$-dimensional regular local ring, and let $0 \neq \alpha \in \mathfrak{n}$. Let (ρ, σ) and (ρ', σ') be matrix factorizations of α. We regard ρ and σ as $r \times r$ matrices with entries in S, and ρ' and σ' as $r' \times r'$ matrices with entries in S. Then, we write

$$(\rho, \sigma) \oplus (\rho', \sigma') = \left(\begin{pmatrix} \rho & 0 \\ 0 & \rho' \end{pmatrix}, \begin{pmatrix} \sigma & 0 \\ 0 & \sigma' \end{pmatrix} \right)$$

which is a matrix factorization of α.

DEFINITION 6. Let $(S, \mathfrak{n})$ be a $(d+1)$-dimensional regular local ring, and let $0 \neq \alpha \in \mathfrak{n}$. A matrix factorization (ρ, σ) of α is called *reduced* if all the entries of ρ and σ are in $\mathfrak{n}$.

REMARK 6. Let $(S, \mathfrak{n})$ be a $(d+1)$-dimensional regular local ring, and let $0 \neq \alpha \in \mathfrak{n}$. Let the map $\alpha : S \to S$ be multiplication by $\alpha \in \mathfrak{n}$ on S. If (ρ, σ) is a matrix factorization of $\alpha \in \mathfrak{n}$, then we can write

$$(\rho, \sigma) \simeq (\alpha, id_S)^{\oplus v} \oplus (id_S, \alpha)^{\oplus u} \oplus (\gamma_1, \gamma_2)\,,$$

where v and u are some integers, and (γ_1, γ_2) is reduced. Therefore,

$$\begin{aligned}\operatorname{cok}(\rho) &\simeq \operatorname{cok}(\alpha)^{\oplus v} \oplus \operatorname{cok}(id_S)^{\oplus u} \oplus \operatorname{cok}(\gamma_1)\\ &\simeq (S/(\alpha))^{\oplus v} \oplus \operatorname{cok}(\gamma_1)\,.\end{aligned}$$

It is known that $\operatorname{cok}(\gamma_1)$ has no free direct summands if (γ_1, γ_2) is reduced ([6], Corollary 6.3). Consequently, v is equal to the largest rank of a free $S/(\alpha)$-module appearing as a direct summand of $\operatorname{cok}(\rho)$.

Let $F^e : R \to F^e_* R$ be the e-th Frobenius map. Consider the map $f : F^e_* R \to F^e_* R$. We have $f = F^e_*(f^{p^e}) = F^e_*(f) \cdot F^e_*(f^{p^e-1}) = F^e_*(f^{p^e-1}) \cdot F^e_*(f)$. Therefore, $(F^e_*(f), F^e_*(f^{p^e-1}))$ is a matrix factorization. We put

$$(F^e_*(f), F^e_*(f^{p^e-1})) = (f, id_R)^{\oplus v_e} \oplus (id_R, f)^{\oplus u_e} \oplus (\text{reduced})\,.$$

By Remark 6 this implies that v_e is the number of $R/(f)$ appearing as the direct summand of $\frac{F^e_* R}{F^e_*(f)(F^e_* R)} = F^e_*(R/(f))$. That is, $lim_{e\to\infty} \frac{v_e}{p^{en}}$ is the F-signature of $R/(f)$, denoted by $s(R/(f))$.

PROPOSITION 2. $v_e = \ell_R(M_{e,\, p^e-1})$.

PROOF. We can regard the map $F^e_*(f^{p^e-1}) : F^e_* R \longrightarrow F^e_* R$ as a $p^{(n+1)e} \times p^{(n+1)e}$ matrix A with entries in R;

$$A = \left(\begin{array}{c|ccc|c} I_{v_e} & & & & \\ \hline & f & & & \\ & & \ddots & & \\ & & & f & \\ \hline & & & & B \end{array}\right),$$

where I_{v_e} is the identity matrix of size v_e, and B is a matrix with entries in $\mathfrak{m}$. Therefore, we have

$$v_e = \dim_{R/\mathfrak{m}}\Big(\operatorname{Im}\big(R/\mathfrak{m} \otimes F^e_*(f^{p^e-1})\big)\Big) = \dim_{R/\mathfrak{m}}\left(\frac{(f^{p^e-1}) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}}\right) = \ell_R(M_{e,\, p^e-1})\,.$$

$\square$

We completed a proof of Theorem 1.

REMARK 7. Let $(S, \mathfrak{n}, k)$ be a complete regular local ring of characteristic $p > 0$. Suppose that k is perfect. Let I be an ideal of S, and put $\overline{S} = S/I$. Suppose that a_e is equal to the largest rank of a free $\overline{S}$-module appearing in a direct summand of $F_*^e\overline{S}$. Then it is known that

$$a_e = \dim_k \frac{(I^{[p^e]} : I) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}}$$

by Fedder's lemma (see [1]). If $I = (f)$, then

$$a_e = \dim_k \frac{(f^{p^e-1}) + \mathfrak{m}^{[p^e]}}{\mathfrak{m}^{[p^e]}} .$$

3. Examples

Let $f = X_1^{\alpha_1} X_2^{\alpha_2} \ldots X_{n+1}^{\alpha_{n+1}}$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1}$. We set $(\underline{X}^{p^e}) = (X_1^{p^e}, X_2^{p^e}, \ldots, X_{n+1}^{p^e})$ for $e \geq 0$. By Theorem 2.1 in Conca [5], we know that there exists a polynomial $P(y) \in \mathbf{Z}[y]$ such that

$$\ell_R\left(\frac{R}{(f^t) + (\underline{X}^{p^e})}\right) = P(p^e)$$

for all $p^e \geq \alpha_{n+1}$. In fact, since the sequence

$$0 \longrightarrow \frac{(f^t) + (\underline{X}^{p^e})}{(\underline{X}^{p^e})} \longrightarrow \frac{R}{(\underline{X}^{p^e})} \longrightarrow \frac{R}{(f^t) + (\underline{X}^{p^e})} \longrightarrow 0$$

is exact, we have

$$\ell_R\left(\frac{R}{(f^t) + (\underline{X}^{p^e})}\right) = \begin{cases} p^{e(n+1)} - \displaystyle\prod_{j=1}^{n+1}(p^e - t\alpha_j) & (\text{if } t\alpha_{n+1} < p^e)\,, \\ p^{e(n+1)} & (\text{otherwise})\,. \end{cases}$$

On the other hand, we have an exact sequence

$$0 \longrightarrow M_{e,t} \longrightarrow \frac{R}{(f^{t+1}) + (\underline{X}^{p^e})} \longrightarrow \frac{R}{(f^t) + (\underline{X}^{p^e})} \longrightarrow 0$$

for any $t \geq 0$. Therefore, we have

$$\ell_R(M_{e,t}) = \begin{cases} 0 & \left(\dfrac{p^e}{\alpha_{n+1}} \leq t\right), \\ \displaystyle\prod_{j=1}^{n+1}(p^e - t\alpha_j) & \left(\dfrac{p^e}{\alpha_{n+1}} - 1 \leq t < \dfrac{p^e}{\alpha_{n+1}}\right), \\ \displaystyle\prod_{j=1}^{n+1}(p^e - t\alpha_j) - \prod_{j=1}^{n+1}(p^e - (t+1)\alpha_j) & \left(t < \dfrac{p^e}{\alpha_{n+1}} - 1\right). \end{cases} \tag{3.1}$$

If $t < \dfrac{p^e}{\alpha_{n+1}} - 1$,

$$\begin{aligned} \ell_R(M_{e,t}) &= \prod_{j=1}^{n+1}(p^e - t\alpha_j) - \prod_{j=1}^{n+1}(p^e - (t+1)\alpha_j) \\ &= \sum_{j=1}^{n+1}(-1)^j t^j \beta_j p^{e(n+1-j)} - \sum_{j=1}^{n+1}(-1)^j (t+1)^j \beta_j p^{e(n+1-j)} \\ &= \sum_{j=1}^{n+1}(-1)^{j+1}\left(\sum_{i=0}^{j-1}\binom{j}{i}t^i\right)\beta_j p^{e(n+1-j)}, \end{aligned}$$

where β_j denotes the elementary symmetric polynomial of degree j in $\alpha_1, \alpha_2, \ldots, \alpha_{n+1}$. Hence

$$C_{e,t} = \frac{\ell_R(M_{e,t})}{p^{en}} = \sum_{j=1}^{n+1}(-1)^{j+1}\left(\sum_{i=0}^{j-1}\binom{j}{i}\frac{t^i}{p^{e(j-1)}}\right)\beta_j$$

holds. We shall calculate $\xi_f(x)$. If $x < \dfrac{1}{\alpha_{n+1}}$, then $\lfloor xp^e \rfloor < \dfrac{p^e}{\alpha_{n+1}} - 1$ for $e \gg 0$. Then,

$$C_{e,\lfloor xp^e \rfloor} = \sum_{j=1}^{n+1}(-1)^{j+1}\left(\sum_{i=0}^{j-1}\binom{j}{i}\frac{\lfloor xp^e \rfloor^i}{p^{e(j-1)}}\right)\beta_j .$$

Since $xp^e - 1 \leq \lfloor xp^e \rfloor \leq xp^e$, we have

$$\lim_{e \to \infty} \frac{\lfloor xp^e \rfloor^a}{p^{eb}} = \begin{cases} x^a & (\text{if } a = b), \\ 0 & (\text{if } a < b). \end{cases}$$

Consequently,

$$\xi_f(x) = \beta_1 - 2\beta_2 x + 3\beta_3 x^2 - \cdots + (-1)^n(n+1)\beta_{n+1}x^n \tag{3.2}$$

holds for $0 \le x < \dfrac{1}{\alpha_{n+1}}$. In particular, $e_{HK}(R/(f)) = \xi_f(0) = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}$. By Eq. (3.1), we have

$$\xi_f(x) = 0 \tag{3.3}$$

if $x > \dfrac{1}{\alpha_{n+1}}$.

Next, we shall calculate $\xi_f\left(\frac{1}{\alpha_{n+1}}\right)$. Since $\frac{p^e}{\alpha_{n+1}} - 1 \le \left\lfloor \frac{p^e}{\alpha_{n+1}} \right\rfloor \le \frac{p^e}{\alpha_{n+1}}$ for any $e \ge 0$,

$$\begin{aligned}
&\ell\Big(M_{e,\left\lfloor \frac{1}{\alpha_{n+1}} p^e \right\rfloor}\Big) \\
&\quad = \prod_{j=1}^{n+1} \left\{ p^e - \left\lfloor \frac{p^e}{\alpha_{n+1}} \right\rfloor \alpha_j \right\} \\
&\quad = \varepsilon_e \prod_{j=1}^{n} \left\{ p^e - (p^e - \varepsilon_e) \frac{\alpha_j}{\alpha_{n+1}} \right\} \\
&\quad = \varepsilon_e \prod_{j=1}^{n} \left\{ \left(1 - \frac{\alpha_j}{\alpha_{n+1}}\right) p^e + \frac{\varepsilon_e}{\alpha_{n+1}} \alpha_j \right\} \\
&\quad = \varepsilon_e \left(\frac{1}{\alpha_{n+1}}\right)^n \prod_{j=1}^{n} \left\{ (\alpha_{n+1} - \alpha_j) p^e + \varepsilon_e \alpha_j \right\} \\
&\quad = \varepsilon_e \left(\frac{1}{\alpha_{n+1}}\right)^n \left\{ p^{en} \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^{n} \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \delta_{\underline{i}}\, p^{e(n-k)} \varepsilon_e^k \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_k} \right\} \\
&\quad = \varepsilon_e \left(\frac{1}{\alpha_{n+1}}\right)^n p^{en} \left\{ \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^{n} \delta_k \left(\frac{\varepsilon_e}{p^e}\right)^k \right\},
\end{aligned}$$

where $\varepsilon_e \equiv p^e \pmod{\alpha_{n+1}}$ such that $0 \le \varepsilon_e < \alpha_{n+1}$, and

$$\delta_{\underline{i}} = \prod_{j \ne i_1, i_2, \ldots, i_k} (\alpha_{n+1} - \alpha_j),$$

$$\delta_k = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} \delta_{\underline{i}}\, \alpha_{i_1} \alpha_{i_2} \ldots \alpha_{i_k}.$$

Hence,

$$C_{e,\left\lfloor \frac{1}{\alpha_{n+1}} p^e \right\rfloor} = \varepsilon_e \left(\frac{1}{\alpha_{n+1}}\right)^n \left\{ \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) + \sum_{k=1}^{n} \delta_k \left(\frac{\varepsilon_e}{p^e}\right)^k \right\},$$

and therefore

$$\limsup_{e\to\infty} C_{e,\left\lfloor \frac{1}{\alpha_{n+1}} p^e \right\rfloor} = \left(\limsup_{e\to\infty} \varepsilon_e\right) \left(\frac{1}{\alpha_{n+1}}\right)^n \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) . \tag{3.4}$$

We shall examine whether $\lim_{e\to\infty} \varepsilon_e$ exists. Let $\alpha_{n+1} = p^s q$, where q is coprime to p, and s is a non-negative integer. If $p \equiv 1 \pmod q$, then we can find that ε_e is constant for any $e \geq s$ by the Chinese remainder theorem. If $p \not\equiv 1 \pmod q$, then ε_e is eventually periodic with period more than 1.

From the following Proposition 3, we get to know the function $\xi_f(x)$.

PROPOSITION 3. *Let $f = X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_{n+1}^{\alpha_{n+1}}$ with $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{n+1}$.*

1) *We have* $\operatorname{fpt}(f) = \frac{1}{\alpha_{n+1}}$. *If $\alpha_{n+1} \geq 2$, we have $s\big(R/(f)\big) = 0$.*
2) $\lim_{x\to \frac{1}{\alpha_{n+1}}-0} \xi_f(x) = \left(\frac{1}{\alpha_{n+1}}\right)^{n-1} \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) \geq 0$.
3) *The function $\xi_f(x)$ is continuous on* $[0, 1]$ *if and only if $\alpha_{n+1} = \alpha_n$ holds.*
4) *Let $\alpha_{n+1} = p^s q$, where q is coprime to p, and s is a non-negative integer. The limit* $\lim_{e\to\infty} \xi_{f,e}\left(\frac{1}{\alpha_{n+1}}\right)$ *exists if and only if it satisfies that $\alpha_{n+1} = \alpha_n$ or $p \equiv 1$* $\pmod q$.

PROOF. By Eq. (3.2) and Eq. (3.3), we obtain 1) immediately.

Next we shall prove 2). We set

$$g(x) = \beta_1 - 2\beta_2 x + 3\beta_3 x^2 - \cdots + (-1)^n (n+1)\beta_{n+1} x^n$$

and

$$h(x) = (x - \alpha_1)(x - \alpha_2)\ldots(x - \alpha_{n+1}) .$$

Now, since $h(x) = x^{n+1} - \beta_1 x^n + \beta_2 x^{n-1} - \cdots + (-1)^{n+1}\beta_{n+1}$,

$$x^{n+1} h\left(\frac{1}{x}\right) = 1 - \beta_1 x + \beta_2 x^2 - \cdots + (-1)^{n+1} \beta_{n+1} x^{n+1} .$$

Hence, we have the following equation

$$g(x) = -\left\{x^{n+1} h\left(\frac{1}{x}\right)\right\}' = -(n+1)x^n h\left(\frac{1}{x}\right) + x^{n-1} h'\left(\frac{1}{x}\right) .$$

Since $h(\alpha_{n+1}) = 0$,

$$\begin{aligned}\lim_{x\to \frac{1}{\alpha_{n+1}}-0} \xi_f(x) &= g\left(\frac{1}{\alpha_{n+1}}\right) \\ &= \left(\frac{1}{\alpha_{n+1}}\right)^{n-1} h'(\alpha_{n+1})\end{aligned}$$

$$= \left(\frac{1}{\alpha_{n+1}}\right)^{n-1} \prod_{j=1}^{n} (\alpha_{n+1} - \alpha_j) \geq 0\,.$$

The assertion 3) follows from Eq. (3.1), Eq. (3.2) and 2) as above. The assertion 4) follows from Eq. (3.4). □

EXAMPLE 2. If $\alpha_1 = \alpha_2 = \cdots = \alpha_{n-2} = 0$ and $\alpha_{n-1} \neq 0$, the derivative

$$g'(x) = -2(\alpha_{n+1}\alpha_n + \alpha_{n+1}\alpha_{n-1} + \alpha_n\alpha_{n-1}) + 6\alpha_{n+1}\alpha_n\alpha_{n-1}x\,.$$

Let α be the root of $g'(x) = 0$, that is,

$$\alpha = \frac{1}{3} \times \frac{\alpha_{n+1}\alpha_n + \alpha_{n+1}\alpha_{n-1} + \alpha_n\alpha_{n-1}}{\alpha_{n+1}\alpha_n\alpha_{n-1}}\,.$$

Then, we have

$$\alpha - \frac{1}{\alpha_{n+1}} = \frac{1}{\alpha_{n+1}} \left\{ \frac{1}{3} \left(\frac{\alpha_{n+1}}{\alpha_{n-1}} + \frac{\alpha_{n+1}}{\alpha_n} + 1 \right) - 1 \right\} \geq 0\,,$$

and so $g'(x) < 0$ for any $x < \frac{1}{\alpha_{n+1}}$. Moreover, if $\alpha_{n+1} \neq \alpha_n$ we obtain $g'\left(\frac{1}{\alpha_{n+1}}\right) < 0$. The second derivative $g''(x)$ is positive for any $x \in \mathbf{R}$. In fact, $g''(x) = 6\alpha_{n-1}\alpha_n\alpha_{n+1} > 0$.

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