TOKYO J. MATH.
VOL. 40, NO. 1, 2017

The Defects of Power Series in the Unit Disk with Hadamard Gaps

Narufumi TSUBOI

Tokyo Institute of Techonology

(Communicated by K. Matsuzaki)

Abstract. We show a sufficient condition for the defect $\delta(0, f)$ of an analytic function $f(z) = 1+\sum_{k=1}^{\infty} c_k z^{n_k}$ in the unit disk with Hadamard gaps to vanish. As a consequence, we find that such $f(z)$ whose characteristic function is sufficiently large has no finite defective value.

1. Introduction

Let

$$f(z) = 1 + \sum_{k=1}^{\infty} c_k z^{n_k} \tag{1.1}$$

be a power series convergent in the open disk $\{|z| < R\}$ $(0 < R \leq +\infty)$ with gaps, i.e. the sequence $n_1 < n_2 < \cdots < n_k < \cdots$ diverges rapidly as $k \to \infty$. The study of value distribution of gap series (1.1) has a long history. Let $f(z)$ given by (1.1) be an entire function. Fejér ([2]) proved that if $\{n_k\}$ satisfies

$$\sum_{k=1}^{\infty} \frac{1}{n_k} < +\infty\,, \tag{1.2}$$

then the image $f(\mathbf{C})$ equals $\mathbf{C}$. A strictly increasing sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers with (1.2) is called a *Fejér gap sequence*. Biernacki ([1]) improved this theorem: $f(z)$ given by (1.1) with Fejèr gaps (1.2) has no finite Picard exceptional value, i.e. $f(z)$ assumes every finite complex value $a \in \mathbf{C}$ infinitely often. Then detailed studies of value distribution of gap series have been done in terms of Nevanlinna theory.

According to [6], we introduce the notations of Nevanlinna theory. Let $f(z)$ given by (1.1) be analytic in $\{|z| < R\}$ $(0 < R \leq +\infty)$. We define the *characteristic function* $T(r, f)$ by

$$T(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \quad (0 \leq r < R)\,,$$

Received October 21, 2015; revised December 3, 2015

where

$$\log^+ x = \max\{\log x, 0\}.$$

We define the *proximity function* $m(r,a) = m(r,a,f)$ by

$$m(r,a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|} d\theta \quad (0 \le r < R,\ a \in \mathbf{C}).$$

If $T(r,f) \to +\infty$ as $r \to R$, then the *defect* $\delta(a,f)$ of $f(z)$ at a is defined by

$$\delta(a,f) = \liminf_{r \to R} \frac{m(r,a)}{T(r,f)}.$$

If $a \in \mathbf{C}$ satisfies $\delta(a,f) > 0$, then a is called a finite *defective value* of $f(z)$.

Let $n(r,a) = n(r,a,f)$ be the number of a-point of $f(z)$ in the open disk $\{|z| < r\}$ counting multiplicity. We define the *counting function* $N(r,a) = N(r,a,f)$ by

$$N(r,a) = \int_0^r \frac{n(t,a)}{t} dt \quad (0 \le r < R).$$

The *first main theorem of Nevanlinna* states that

$$T(r,f) = m(r,a) + N(r,a) + O(1),$$

so that we have

$$\delta(a,f) = 1 - \limsup_{r \to R} \frac{N(r,a)}{T(r,f)}.$$

It has to be mentioned particularly that Murai ([12]) showed that an entire function $f(z)$ given by (1.1) with Fejér gaps (1.2) has no finite defective value, i.e. the Nevanlinna defect $\delta(a,f)$ of $f(z)$ vanishes for arbitrary $a \in \mathbf{C}$. Since there are, of course, many entire functions having finite defective value whose Taylor expansions are not Fejér gap series (e.g. $\exp z$), the problems of value distribution of entire functions with gaps were solved in a sense.

We shall be concerned with only the case where the convergent radius of $f(z)$ given by (1.1) equals 1 in the present paper. Unlike the case of entire functions, no relationship between the value distribution of $f(z)$ in the unit disk $\mathbf{D} = \{|z| < 1\}$ and Fejér gap condition (1.2) has been ever known. However, if $\{n_k\}_{k=1}^{\infty}$ satisfies

$$n_{k+1}/n_k \ge q \tag{1.3}$$

for some $q > 1$, then several results about the value distribution of $f(z)$ have been established. A sequence $\{n_k\}_{k=1}^{\infty}$ of positive integers satisfying (1.3) is called a *Hadamard gap sequence*. It is obvious that a Hadamard gap sequence is a Fejér gap sequence. The Hadamard gap condition (1.3) was introduced in [5] and Hadamard there proved that $f(z)$ given by (1.1) with (1.3) whose convergent radius is 1 has the unit circle $\{|z| = 1\}$ as its natural boundary. Fuchs ([3]) proved that if an analytic function $f(z)$ in $\mathbf{D}$ given by (1.1) with Hadamard gaps

(1.3) satisfies

$$\limsup_{k\to\infty} |c_k| > 0\,, \tag{1.4}$$

then $f(z)$ assumes zero infinitely often in $\mathbf{D}$. Murai ([10]) improved this theorem: under the same conditions, the Nevanlinna defect $\delta(0, f)$ of $f(z)$ at 0 vanishes. More precisely he showed that if (and only if)

$$\sum_{k=1}^{\infty} |c_k|^2 = +\infty\,, \tag{1.5}$$

then the Nevanlinna characteristic function $T(r, f)$ diverges as $r \to 1$ and if we assume (1.4), then the proximity function $m(r, 0)$ is bounded as $r \to 1$ through a suitable sequence of r. Remark that these results yield that $f(z)$ given by (1.1) satisfying (1.3) and (1.4) has no finite defective value, that is, $\delta(a, f)$ vanishes for arbitrary $a \in \mathbf{C}$. (See Corollary of this paper.)

Now we turn to consider the case where

$$\lim_{k\to\infty} c_k = 0\,. \tag{1.6}$$

Murai ([11]) also showed that if an analytic function $f(z)$ in $\mathbf{D}$ given by (1.1) with (1.3) and (1.6) is unbounded in $\mathbf{D}$, then $f(z)$ assumes zero infinitely often in $\mathbf{D}$. It is well known (Sidon [15]) that such $f(z)$ is unbounded in $\mathbf{D}$ if and only if

$$\sum_{k=1}^{\infty} |c_k| = +\infty\,. \tag{1.7}$$

Therefore it is natural to ask whether for $f(z)$ given by (1.1) satisfying (1.3), (1.5) and (1.6), $\delta(0, f) = 0$ holds or not. (Note that the conditions (1.5) and (1.6) imply (1.7), and the convergent radius of $f(z)$ given by (1.1) satisfying (1.3), (1.5) and (1.6) must be 1.) We shall study this problem and show a sufficient condition for $\delta(0, f) = 0$ in the present paper. In particular, our main theorem and its corollary will show that if the coefficients $\{c_k\}$ of $f(z)$ satisfy

$$\log K / \log \sum_{k=1}^{K} |c_k|^2 = O(1)$$

as $K \to \infty$, then $\delta(a, f) = 0$ for any $a \in \mathbf{C}$.

ACKNOWLEDGMENT. The author would like to thank the referee for helpful comments on the presentation of this paper.

2. Notation and statement of results

We assume that $f(z)$ given by (1.1) satisfies (1.3), (1.5) and (1.6). Throughout the present paper 'const.' and $C(f)$ denote an absolute positive constant and a constant depending only on f respectively.

Before stating our theorems, we first show the existence of a certain sequence $0 < R_1 < R_2 < \cdots < 1$ of radii for the function $f(z) = 1 + \sum c_k z^{n_k}$. We shall estimate $m(r, 0)$ on the circle $\{|z| = R_l\}$. The following lemma is an analogue of Lemma 9 in Murai [11].

LEMMA 1. *For the sequence* $\{c_k\}$ *with* (1.5) *and* (1.6), Γ *denotes the set of positive integers* k *satisfying* $|c_j|n_j^{1/2} \le |c_k|n_k^{1/2}$ *for any* $j \le k$ *and* $|c_k|n_k^{-1/2} \ge |c_j|n_j^{-1/2}$ *for any* $j \ge k$. *Then*

$$\sum_{k\in\Gamma} |c_k| = +\infty\,.$$

PROOF. Note that (1.5) and (1.6) imply

$$\sum_{k=1}^{\infty} |c_k| = +\infty\,.$$

Since many indices will be used, it is convenient to write $c(k) = c_k$ and $n(k) = n_k$. Let $\{k_m\}_{m=1}^{\infty}$ be the strictly increasing sequence of all positive integers satisfying $k_1 = 1$ and

$$|c(k)|n(k)^{1/2} \le |c(k_m)|n(k_m)^{1/2}$$

for any $k \le k_m$. For any $k \in [k_m, k_{m+1})$, we have

$$|c(k_m)|n(k_m)^{1/2} \ge |c(k)|n(k)^{1/2}\,,$$

so that we obtain

$$|c(k)| \le (n(k_m)/n(k))^{1/2}|c(k_m)| \le q^{(k_m-k)/2}|c(k_m)|\,.$$

Therefore we deduce that

$$\begin{aligned}
\sum_{k=1}^{k_M-1} |c(k)| &= \sum_{m=1}^{M-1}\sum_{k=k_m}^{k_{m+1}-1} |c(k)| \\
&\le \sum_{m=1}^{M-1}\sum_{k=k_m}^{k_{m+1}-1} q^{(k_m-k)/2}|c(k_m)| \\
&= \sum_{m=1}^{M-1} |c(k_m)| \sum_{k=k_m}^{k_{m+1}-1} q^{(k_m-k)/2}
\end{aligned}$$

$$\leq \frac{1}{1-q^{-1/2}} \sum_{m=1}^{M-1} |c(k_m)|$$

$$= \frac{q^{1/2}}{q^{1/2}-1} \sum_{m=1}^{M-1} |c(k_m)| \,.$$

Let $\{k_{m_l}\}_{l=1}^{\infty}$ be the strictly increasing subsequence of $\{k_m\}_{m=1}^{\infty}$ consisting of all positive integers satisfying

$$|c(k_{m_l})| n(k_{m_l})^{-1/2} \geq |c(k_m)| n(k_m)^{-1/2}$$

for any $k_m \geq k_{m_l}$. It is trivial that $\sum_{k\in\Gamma} |c_k| = \sum_{l=1}^{\infty} |c(k_{m_l})|$. For any $k_m \in (k_{m_l}, k_{m_{l+1}}]$, we have

$$|c(k_m)| n(k_m)^{-1/2} \leq |c(k_{m_{l+1}})| n(k_{m_{l+1}})^{-1/2} \,,$$

so that we obtain

$$|c(k_m)| \leq (n(k_m)/n(k_{m_{l+1}}))^{1/2} |c(k_{m_{l+1}})| \leq q^{(k_m - k_{m_{l+1}})/2} |c(k_{m_{l+1}})| \,.$$

Therefore we deduce that, with $m_0 = 0$,

$$\begin{aligned}
\sum_{m=1}^{m_L} |c(k_m)| &= \sum_{l=0}^{L-1} \sum_{m=m_l+1}^{m_{l+1}} |c(k_m)| \\
&\leq \sum_{l=0}^{L-1} \sum_{m=m_l+1}^{m_{l+1}} q^{(k_m - k_{m_{l+1}})/2} |c(k_{m_{l+1}})| \\
&= \sum_{l=0}^{L-1} |c(k_{m_{l+1}})| \sum_{m=m_l+1}^{m_{l+1}} q^{(k_m - k_{m_{l+1}})/2} \\
&\leq \frac{1}{1-q^{-1/2}} \sum_{l=1}^{L} |c(k_{m_l})| \\
&= \frac{q^{1/2}}{q^{1/2}-1} \sum_{l=1}^{L} |c(k_{m_l})| \,.
\end{aligned}$$

In the sequel,

$$\sum_{k\in\Gamma} |c_k| = \sum_{l=1}^{\infty} |c(k_{m_l})| \geq \lim_{L\to\infty} \left(\frac{q^{1/2}-1}{q^{1/2}}\right)^2 \sum_{k=1}^{k(m_L)} |c_k| = +\infty \,.$$

We complete the proof. □

Here is an example for Lemma 1. Suppose that $|c_k| = 1/k^p$ $(0 < p \leq 1/2)$. Then it is easy to see that, if K is sufficiently large,

$$|c_K| \geq |c_k|$$

for any $k \geq K$ and

$$|c_k| n_k^{1/2} \leq |c_K| n_K^{1/2}$$

for any $k \leq K$, so that Γ is the set of positive integers which is obtained by excluding a finite number of elements from the set of positive integers $\mathbf{N}$.

For the sake of simplicity, we write $\Gamma = \{k_l\}_{l=1}^{\infty}$ $(k_l < k_{l+1})$. It holds that

$$\begin{aligned} |c_k| n_k^{1/2} &\leq |c_{k_l}| n_{k_l}^{1/2} \quad (k \leq k_l)\,, \\ |c_{k_l}| n_{k_l}^{-1/2} &\geq |c_k| n_k^{-1/2} \quad (k_l \leq k)\,. \end{aligned} \tag{2.1}$$

Let $R_l \in (0, 1)$ be defined by

$$R_l = 1 - \frac{1}{n_{k_l}}\,.$$

As an immediate consequence, we have the following:

LEMMA 2.

$$\left| \frac{\partial}{\partial \theta} f(R_l e^{i\theta}) \right| \leq C(f) |c_{k_l}| n_{k_l}. \tag{2.2}$$

PROOF. We obtain, by (2.1), that

$$\begin{aligned} \left| \frac{\partial}{\partial \theta} f(R_l e^{i\theta}) \right| &\leq \sum_{k=1}^{\infty} |c_k| n_k R_l^{n_k} \\ &= \sum_{k=1}^{k_l - 1} |c_k| n_k R_l^{n_k} + |c_{k_l}| n_{k_l} R_l^{n_k} + \sum_{k=k_l+1}^{\infty} |c_k| n_k R_l^{n_k} \\ &= \sum_{k=1}^{k_l - 1} (|c_k| n_k^{1/2}) n_k^{1/2} R_l^{n_k} + |c_{k_l}| n_{k_l} R_l^{n_k} + \sum_{k=k_l+1}^{\infty} (|c_k| n_k^{-1/2}) n_k^{3/2} R_l^{n_k} \\ &\leq |c_{k_l}| n_{k_l}^{1/2} \sum_{k=1}^{k_l - 1} n_k^{1/2} + |c_{k_l}| n_{k_l} + |c_{k_l}| n_{k_l}^{-1/2} \sum_{k=k_l+1}^{\infty} n_k^{3/2} R_l^{n_k}\,. \end{aligned}$$

Hadamard gap condition (1.3) implies

$$|c_{k_l}| n_{k_l}^{1/2} \sum_{k=1}^{k_l - 1} n_k^{1/2} = |c_{k_l}| n_{k_l} \sum_{k=1}^{k_l - 1} \left(\frac{n_k}{n_{k_l}} \right)^{1/2} \leq C(f) |c_{k_l}| n_{k_l}$$

and

$$
\begin{aligned}
|c_{k_l}| n_{k_l}^{-1/2} \sum_{k=k_l+1}^{\infty} n_k^{3/2} R_l^{n_k} &= |c_{k_l}| n_{k_l} \sum_{k=k_l+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{3/2} \left\{ \left(1 - \frac{1}{n_{k_l}}\right)^{n_{k_l}} \right\}^{\frac{n_k}{n_{k_l}}} \\
&\le |c_{k_l}| n_{k_l} \sum_{k=k_l+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{3/2} e^{-\frac{n_k}{n_{k_l}}} \\
&\le |c_{k_l}| n_{k_l} \sum_{k=k_l+1}^{\infty} \left(\frac{n_{k_l}}{n_k}\right)^{1/2} \le C(f) |c_{k_l}| n_{k_l} ,
\end{aligned}
$$

so that we have the required inequality. □

To estimate

$$
m(R_l, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \frac{1}{|f(R_l e^{i\theta})|} d\theta ,
$$

we shall use the classical central limit theorem for Hadamard gap series, due to R. Salem and A. Zygmund ([14]). The author wishes to express his thanks to Prof. T. Murai, who suggested to use the central limit theorem to study the value-distribution of Hadamard gap series. For any Lebesgue measurable set $E \subset [0, 2\pi)$, $|E|$ denotes its Lebesgue measure.

LEMMA 3 ([14]). *Suppose that $f(z)$ given by* (1.1) *satisfies* (1.3), (1.5) *and* (1.6). *Then, for any $y > 0$, we have*

$$
\frac{1}{2\pi} |\{\theta \in [0, 2\pi) : \ |f(re^{i\theta})| \le y V(r)\}| \to 1 - e^{-y^2/2} \ (r \to 1),
$$

where

$$
V(r) = \left\{ \frac{1}{2} \left(1 + \sum_{k=1}^{\infty} |c_k|^2 r^{2n_k} \right) \right\}^{1/2} .
$$

This lemma exhibits that the measure of the set

$$
\{\theta \in [0, 2\pi) : \ \log^+ 1/|f(R_l e^{i\theta})| > 0\} = \{\theta \in [0, 2\pi) : \ |f(R_l e^{i\theta})| < 1\}
$$

is small for all sufficiently large l (for the sake of simplicity, we shall omit the phrase ‘for all sufficiently large l’).

We write

$$
E_l = \{\theta \in [0, 2\pi) : \ |f(R_l e^{i\theta})| \le V(R_l)/ \log V(R_l)\} .
$$

The set E_l is represented as a finite disjoint union of closed intervals,

$$
E_l = \bigsqcup_j I_j \sqcup \bigsqcup_{j'} I_{j'} ,
$$

where each I_j contains a point z satisfying $|f(z)| = 1$ and $I_{j'}$ does not. We see, by Lemma 2, that the inequality

$$\min_j |I_j| \geq 2\pi/|c_{k_l}|n_{k_l} > 2\pi/n_{k_l} \tag{2.3}$$

holds.

It is obvious that

$$m(R_l, 0) = \sum_j \frac{1}{2\pi} \int_{I_j} \log^+ \frac{1}{|f(R_l e^{i\theta})|} d\theta\,,$$

so that we would like to calculate the 'localized' mean value

$$\frac{1}{|I_j|} \int_{I_j} \log^+ \frac{1}{|f(R_l e^{i\theta})|} d\theta\,.$$

In fact, the size of this value determines the defect $\delta(0, f)$.

We find, by (2.3), that there exists a positive integer α_l satisfying

$$2\pi/n_{k_l} \leq 2\pi/\alpha_l \leq \min_j |I_j| \tag{2.4}$$

and define the set A_l by

$$A_l = \{\alpha_l \in \mathbf{N} :\ 2\pi/n_{k_l} \leq 2\pi/\alpha_l \leq \min_j |I_j|\}\,.$$

For an $\alpha_l \in A_l$, $C_{j,l}$ denotes the set

$$C_{j,l} = \{n \in \mathbf{N} :\ I_j \cap [2(n-1)\pi/\alpha_l, 2n\pi/\alpha_l] \neq \emptyset\}\,. \tag{2.5}$$

Remark that (2.4) implies

$$\left| \bigcup_{n \in C_{j,l}} [2(n-1)\pi/\alpha_l, 2n\pi/\alpha_l] \right| \leq 3|I_j|\,. \tag{2.6}$$

We can now state the following proposition, which is interesting in itself.

PROPOSITION 1. *Take a positive integer $\alpha_l \in A_l$. Suppose that n is a positive integer of $C_{j,l}$ and $S(\theta; r_1, r_2)$ denotes the segment*

$$S(\theta; r_1, r_2) = \{z \in \mathbf{D} :\ \arg z = \theta,\ r_1 \leq |z| \leq r_2\}\,.$$

Then we obtain the following inequalities;

$$
\begin{aligned}
&\frac{\alpha_l}{2\pi}\int_{2(n-1)\pi/\alpha_l}^{2n\pi/\alpha_l}\log^{+}1/|f(R_l e^{i\theta})|d\theta \\
&\le \text{const.}\frac{\alpha_l}{4\pi}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l}\log^{+}|f(R_l e^{i\theta})|d\theta \\
&\quad + \text{const.}\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l}\log^{+}|f(re^{i\theta})|\,\alpha_l^2 r^{\alpha_l/2-1}d\theta dr \\
&\quad + \text{const.}\min\{\log 1/|f(z)| :\ z\in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\}
\end{aligned}
\tag{2.7}
$$

and

$$
\sum_{n\in C_{j,l}}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l}\log^{+}|f(R_l e^{i\theta})|d\theta \le \text{const.}|I_j|\log V(R_l)\,, \tag{2.8}
$$

where $r_l^1 = 1-3/\alpha_l$ *and* $r_l^2 = 1-2/\alpha_l$.

We will give a proof of Proposition 1 in the section 3. By this proposition, we can derive the following Proposition.

PROPOSITION 2. *Suppose that there exist infinitely many* $l\in\mathbf{N}$ *such that, for an* $\alpha_l \in A_l$, *the inequalities*

$$
\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l}\log^{+}|f(re^{i\theta})|\,\alpha_l^2 r^{\alpha_l/2-1}d\theta dr \le C(f)\log V(R_l) \tag{2.9}
$$

and

$$
\min\{\log 1/|f(z)| :\ z\in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\} \le C(f)\log V(R_l) \tag{2.10}
$$

hold for all $n\in\bigcup_j C_{j,l}$. *Then* $\delta(0, f) = 0$.

PROOF. Let l be a positive integer such that, for an $\alpha_l \in A_l$, the inequalities (2.9) and (2.10) hold for all $n\in\bigcup_j C_{j,l}$. (2.6), (2.9) and (2.10) imply that

$$
\sum_{n\in C_{j,l}}\frac{2\pi}{\alpha_l}\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l}\log^{+}|f(re^{i\theta})|\,\alpha_l^2 r^{\alpha_l/2-1}d\theta dr \le C(f)|I_j|\log V(R_l)
$$

and

$$
\sum_{n\in C_{j,l}}\frac{2\pi}{\alpha_l}\min\{\log 1/|f(z)| : z\in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\} \le C(f)|I_j|\log V(R_l)\,,
$$

so that we have, by (2.7) and (2.8), that

$$\int_{I_j} \log^+ 1/|f(R_l e^{i\theta})| d\theta \le \sum_{n \in C_{j,l}} \int_{2(n-1)\pi/\alpha_l}^{2n\pi/\alpha_l} \log^+ 1/|f(R_l e^{i\theta})| d\theta$$
$$\le C(f)|I_j| \log V(R_l).$$

Therefore we obtain that

$$m(R_l, 0) = \sum_j \frac{1}{2\pi} \int_{I_j} \log^+ 1/|f(R_l e^{i\theta})| d\theta \le C(f)|E_l| \log V(R_l). \tag{2.11}$$

Lemma 3 yields that, for any $\varepsilon > 0$, the inequality

$$|E_l| \le 2\pi\varepsilon \tag{2.12}$$

holds. We also know that

$$T(r, f) \ge C(f) \log V(r) \tag{2.13}$$

holds for all sufficiently large $r \in [0, 1)$ (Murai [10]).

We deduce, by (2.11), (2.12) and (2.13), that

$$m(R_l, 0)/T(R_l, f) \le C(f)\varepsilon.$$

Therefore we have

$$\liminf_{l \to \infty} \frac{m(R_l, 0)}{T(R_l, f)} \le C(f)\varepsilon,$$

which proves our proposition. □

Fortunately, Hadamard gap condition (1.3) gives a certain upper bound for $\min\{\log 1/|f(z)| : z \in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\}$, which we shall show below.

PROPOSITION 3. *Suppose that $\alpha_l = n_{k_l}$. Then there exists an absolute positive constant l_0 such that, for $l \ge l_0$,*

$$\min\{\log 1/|f(z)| : z \in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\} \le \log^+ 1/|c_{k_l}| + C(f) \tag{2.14}$$

holds for all $n \in \bigcup_j C_{j,l}$.

We will give a proof of Proposition 3 in the section 4. By this proposition, we can derive the following theorem.

THEOREM. *Suppose that $f(z)$ given by* (1.1) *satisfies* (1.3), (1.6) *and*

$$\log K / \log \sum_{k=1}^{K} |c_k|^2 = O(1) \tag{2.15}$$

as $K \to \infty$. Then $\delta(0, f) = 0$.

PROOF. We shall show that there exist infinitely many $l \in \mathbf{N}$ such that (2.9) and (2.10) of Proposition 2 hold for all $n \in \bigcup_j C_{j,l}$ with $\alpha_l = n_{k_l}$. Note that

$$
\begin{aligned}
\sum_{k=1}^{\infty} |c_k| R_l^{n_k} &= \sum_{k=1}^{k_l} |c_k| R_l^{n_k} + \sum_{k=k_l+1}^{\infty} |c_k| n_k^{-1/2} n_k^{1/2} R_l^{n_k} \\
&\le \sum_{k=1}^{k_l} |c_k| + \sum_{k=k_l+1}^{\infty} |c_{k_l}| n_{k_l}^{-1/2} n_k^{1/2} R_l^{n_k} \\
&= \sum_{k=1}^{k_l} |c_k| + |c_{k_l}| \sum_{k=k_l+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{1/2} \left\{\left(1 - \frac{1}{n_{k_l}}\right)^{n_{k_l}}\right\}^{\frac{n_k}{n_{k_l}}} \qquad (2.16) \\
&\le \sum_{k=1}^{k_l} |c_k| + |c_{k_l}| \sum_{k=k_l+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{1/2} \exp\left(-\frac{n_k}{n_{k_l}}\right) \\
&\le \sum_{k=1}^{k_l} |c_k| + C(f)\,.
\end{aligned}
$$

It holds similarly that

$$
V(R_l)^2 \le \sum_{k=1}^{k_l} |c_k|^2 + C(f)\,. \tag{2.17}
$$

(2.15) and (2.17) yield that

$$
\frac{\log k_l}{\log V(R_l)} = \frac{\log k_l}{\log \sum_{k=1}^{k_l} |c_k|^2} \frac{\log \sum_{k=1}^{k_l} |c_k|^2}{\log V(R_l)} = O(1)
$$

as $l \to \infty$, so that we have

$$
\log V(R_l) \ge C(f) \log k_l\,. \tag{2.18}
$$

We obtain, by (2.16), that

$$
\log \sum_{k=1}^{\infty} |c_k| R_l^{n_k} \le \log k_l + C(f)\,,
$$

so that we have, by (2.18),

$$\begin{aligned}
&\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} \log^+|f(re^{i\theta})|\,\alpha_l^2 r^{\alpha_l/2-1}d\theta dr\\
&\quad\le \int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} \log\left(1+\sum_{k=1}^{\infty}|c_k|R_l^{n_k}\right)\,\alpha_l^2 r^{\alpha_l/2-1}d\theta dr\\
&\quad\le (\log k_l + C(f))\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} \alpha_l^2 r^{\alpha_l/2-1}d\theta dr\\
&\quad\le C(f)\log V(R_l)\,.
\end{aligned}$$

By Lemma 1, we find that there exist infinitely many $l \in \mathbf{N}$ such that

$$|c_{k_l}| \ge 1/k_l^2\,. \tag{2.19}$$

Let l be a positive integer satisfying (2.19) and $l \ge l_0$, where l_0 is an absolute positive constant defined in the proof of Proposition 3. Then we deduce, by (2.18), that

$$\begin{aligned}
\min\{\log 1/|f(z)| : z \in S((2n-1)\pi/\alpha_l; r_l^1, r_l^2)\} &\le \log^+ 1/|c_{k_l}| + C(f)\\
&\le 2\log k_l + C(f)\\
&\le C(f)\log V(R_l)\,.
\end{aligned}$$

By Proposition 2, we complete the proof. □

We apply our theorem to an example. Suppose that $|c_k| = 1/k^p$ $(0 < p < 1/2)$. It is easy to see that these c_k satisfy the conditions of Theorem. In this situation, we have

$$T(R_l) \ge \text{const.} \log V(R_l) \ge C(f)\log k_l,$$

$$\begin{aligned}
&\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} \log^+|f(re^{i\theta})|\,{\alpha_l}^2 r^{\alpha_l/2-1}d\theta dr\\
&\quad\le \log\left(1+\sum_{k=1}^{\infty}|c_k|R_l^{n_k}\right)\int_0^{R_l}\int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} {\alpha_l}^2 r^{\alpha_l/2-1}d\theta dr\\
&\quad\le C(f)\log k_l
\end{aligned}$$

and

$$\log^+ 1/|c_{k_l}| + C(f) \le p\log k_l + C(f) \le C(f)\log k_l\,.$$

Therefore we deduce, by our theorem, that $\delta(0, f) = 0$.

COROLLARY. *Suppose that* $f(z)$ *given by* (1.1) *satisfies* (1.3), (1.6) *and* (2.15). *Then* $f(z)$ *has no finite defective value.*

PROOF. Let $a \in \mathbf{C}$. We define $f_a(z)$ by

$$f_a(z) = \begin{cases} (f(z) - a)/c_1 z^{n_1} & \text{if } a = 1 \\ (f(z) - a)/(1 - a) & \text{otherwise.} \end{cases}$$

It is obvious that $f_a(z)$ satisfies Hadamard gap condition (1.3) and $f_a(0) = 1$. The coefficients of $f_a(z)$ satisfy (1.5), (1.6) and (2.15). Therefore our theorem implies $\delta(0, f_a) = 0$, which yields $\delta(a, f) = 0$. $\square$

3. Proof of Proposition 1

Our proof of Proposition 1 will be based on an extension of Poisson-Jensen formula, due to W. H. J. Fuchs ([4]) and V. P. Petrenko ([13]):

LEMMA 4. *Suppose that $g(z)$ is analytic in the closed sector*

$$\{z \in \mathbf{C} : |\arg z| \le \pi/\alpha,\ |z| \le R\}\ (\alpha > 1)\,.$$

Let $t \in (0, R)$ be a point on the real axis, where $g(t) \ne 0$. For $z \ne t, 1/t$, define

$$\Phi(R, t, z) = \log\left|\frac{R^2 - tz}{R(z - t)}\right| - \log\frac{R^2 + t|z|}{R(|z| + t)}\,.$$

If we write

$$I_1 = I_1(R, t, \alpha) = \int_0^R \Big(\int_{-\pi/\alpha}^{\pi/\alpha} \log|g(re^{i\theta})| d\theta\Big) K_1(R, r, t, \alpha) dr\,,$$

$$I_2 = I_2(R, t, \alpha) = \int_{-\pi/\alpha}^{\pi/\alpha} \log|g(Re^{i\theta})| K_2(R, \theta, t, \alpha) d\theta\,,$$

where

$$K_1(R, r, t, \alpha) = \frac{\alpha^2}{2\pi}\frac{r^{\alpha-1}t^\alpha(R^{2\alpha} - t^{2\alpha})(R^{2\alpha} - r^{2\alpha})}{(r^\alpha + t^\alpha)^2(R^{2\alpha} + r^\alpha t^\alpha)^2},$$

$$K_2(R, \theta, t, \alpha) = \frac{\alpha}{\pi}\frac{R^\alpha t^\alpha(R^\alpha - t^\alpha)(1 + \cos\alpha\theta)}{(R^\alpha + t^\alpha)(R^{2\alpha} + t^{2\alpha} - 2R^\alpha t^\alpha\cos\alpha\theta)},$$

then

$$\log|g(t)| = I_1 + I_2 - \sum_{a_i} \Phi(R^\alpha, t^\alpha, a_i{}^\alpha)\,, \tag{3.1}$$

where the summation is taken over the zeros $\{a_i\}$ of g which lie in the interior of the sector.

PROOF OF PROPOSITION 1. We put $f_n(z) = f(e^{i2(n-1)/\alpha_l} z)$. Let t_n be a maximal point of $\log 1/|f_n(t)|$ in $S(0; r_l^1, r_l^2)$. We now apply the above formula for the sector $\{z \in \mathbf{C} :$

$|\arg z| \le 2\pi/\alpha_l,\ |z| \le R_l\}$. Elementary calculus gives us $K_1 \ge 0$, $K_2 \ge 0$ and $\Phi \ge 0$, so that we deduce, by (3.1), that

$$\begin{aligned}
\log|f_n(t_n)| \le & \int_0^{R_l}\left(\int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+|f_n(re^{i\theta})|d\theta\right)K_1(R_l,r,t_n,\alpha_l/2)dr \\
& + \int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+|f_n(R_le^{i\theta})|K_2(R_l,\theta,t_n,\alpha_l/2)d\theta \\
& - \int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+1/|f_n(R_le^{i\theta})|K_2(R_l,\theta,t_n,\alpha_l/2)d\theta \\
\le & \int_0^{R_l}\left(\int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+|f_n(re^{i\theta})|d\theta\right)K_1(R_l,r,t_n,\alpha_l/2)dr \\
& + \int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+|f_n(R_le^{i\theta})|K_2(R_l,\theta,t_n,\alpha_l/2)d\theta \\
& - \int_{-\pi/\alpha_l}^{\pi/\alpha_l}\log^+1/|f_n(R_le^{i\theta})|K_2(R_l,\theta,t_n,\alpha_l/2)d\theta\,.
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
&K_1(R_l,r,t_n,\alpha_l/2) \le \text{const.}\ {\alpha_l}^2 r^{\alpha_l/2-1}\,,\\
&K_2(R_l,\theta,t_n,\alpha_l/2) \le \text{const.}\ \frac{\alpha_l}{4\pi}\,,
\end{aligned}$$

and

$$\min\{K_2(R_l,\theta,t_n,\alpha_l/2):\ \theta\in[-\pi/\alpha_l,\pi/\alpha_l]\} \ge \text{const.}\ \frac{\alpha_l}{2\pi}\,,$$

so that we obtain

$$\begin{aligned}
&\frac{\alpha_l}{2\pi}\int_{-\pi/\alpha_l}^{\pi/\alpha_l}\log^+1/|f_n(R_le^{i\theta})|d\theta \\
&\quad\le \text{const.}\frac{\alpha_l}{4\pi}\int_{-2\pi/\alpha_l}^{2\pi/\alpha_l}\log^+|f_n(R_le^{i\theta})|d\theta \\
&\quad\quad + \text{const.}\int_0^{R_l}\int_{-\pi/\alpha_l}^{\pi/\alpha_l}\log^+|f_n(re^{i\theta})|\,{\alpha_l}^2 r^{\alpha_l/2-1}d\theta dr \\
&\quad\quad + \min\{\log 1/|f_n(z)|:\ z\in S(0;r_l^1,r_l^2)\}\,,
\end{aligned}$$

which is equivalent to (2.7).

We proceed to show (2.8). We write $I_j = [\theta_j^-,\theta_j^+]$, $\theta_j = (\theta_j^+ + \theta_j^-)/2$ and let $\tilde{I}_j$ be the set

$$\tilde{I}_j = \{\theta\in[0,2\pi):\ |\theta-\theta_j| < 2|I_j|\}\,. \tag{3.2}$$

Then we deduce, by (2.4), (2.5) and (3.2), that

$$\sum_{n\in C_{j,l}} \int_{(2n-3)\pi/\alpha_l}^{(2n+1)\pi/\alpha_l} \log^+|f(R_l e^{i\theta})|d\theta \le 2\int_{\tilde{I}_j} \log^+|f(R_l e^{i\theta})|d\theta\,.$$

Since $\log x$ is a convex function, we have, by Jensen's inequality, that

$$\begin{aligned}\frac{1}{|\tilde{I}_j|}\int_{\tilde{I}_j} \log^+|f(R_l e^{i\theta})|d\theta &\le \frac{1}{|\tilde{I}_j|}\int_{\tilde{I}_j} \log(1+|f(R_l e^{i\theta})|)d\theta \\ &\le \log\left\{\frac{1}{|\tilde{I}_j|}\int_{\tilde{I}_j} 1+|f(R_l e^{i\theta})|d\theta\right\}.\end{aligned}$$

Regard $f(R_l e^{i\theta})$ as a periodic function on $\mathbf{R}$. It is well known (Kochneff-Sagher-Zhou [8]) that

$$||f(R_l e^{i\theta})||_{BMO(\mathbf{R})} \le C(f)V(R_l)\,,$$

so that

$$||1+|f(R_l e^{i\theta})|||_{BMO(\mathbf{R})} \le C(f)V(R_l)\,.$$

If we assume that

$$M_{j,l} = \frac{1}{|\tilde{I}_j|}\int_{\tilde{I}_j} 1+|f(R_l e^{i\theta})|d\theta > V(R_l)^3$$

holds for infinitely many l, then we obtain, by (3.2),

$$\frac{1}{|\tilde{I}_j|}|\{\theta\in\tilde{I}_j\,:\,|(1+|(R_l e^{i\theta})|)-M_{j,l}|>V(R_l)^2\}| > \frac{|I_j|}{|\tilde{I}_j|} = 1/4\,.$$

On the other hand, the John-Nirenberg inequality ([7]) implies that

$$\begin{aligned}&\frac{1}{|\tilde{I}_j|}|\{\theta\in\tilde{I}_j\,:\,|(1+|f(R_l e^{i\theta})|)-M_{j,l}|>V(R_l)^2\}| \\ &\le \text{const.}\exp\{-\text{const.}V(R_l)^2/||1+|f(R_l e^{i\theta})|||_{BMO(\mathbf{R})}\} \\ &\le \text{const.}\exp\{-C(f)V(R_l)\}\,.\end{aligned}$$

These inequalities lead a contradiction, so that we have $M_{j,l} \le V(R_l)^3$ and $\log M_{j,l} \le$ const.$V(R_l)$. We complete the proof. □

4. Proof of Proposition 3

We introduce an operator D, first appeared in Littlewood-Offord [9]. Suppose that $\psi(r)$ is a real C^∞-function on an interval $[a,b]$ $(a>0)$ and m is a non-negative integer. Then we define $D(m)\psi(r)$ by

$$D(m)\psi(r) = r^{m+1}\frac{d}{dr}\frac{\psi(r)}{r^m}.$$

For a finite set of non-negative integers $E = \{m_1, m_2, \dots, m_p\}$, $D(E)$ is defined by

$$D(E) = D(m_1)D(m_2)\cdots D(m_p). \tag{4.1}$$

It is obvious that $D(m)D(n)\psi(r) = D(n)D(m)\psi(r)$, so that (4.1) is well-defined.

LEMMA 5 (LEMMA 7 in [9]). *Let $E = \{m_1, m_2, \dots, m_p\}$ be a finite set of non-negative integers. If*

$$|D(E)\psi(r)| \geq M$$

for all r in $[a, b]$, then there exist $p+2$ numbers η satisfying

$$a = \eta_0 < \eta_1 < \cdots < \eta_p < \eta_{p+1} = b$$

and

$$|\psi(r)| \geq \frac{M}{2^{p(p-1)/2}p!}b^{-p}\left(\frac{a}{b}\right)^{m_1+\cdots+m_p}\Psi(r; \eta_0, \dots, \eta_{p+1}),$$

where $\Psi(r; \eta_0, \dots, \eta_{p+1})$ is the function on $[a, b]$ defined by

$$\Psi(r; \eta_0, \dots, \eta_{p+1}) = \min\{(r-\eta_i)^p, (\eta_{i+1}-r)^p\} \quad (r \in [\eta_i, \eta_{i+1}]).$$

PROOF OF PROPOSITION 3. Let θ_k be the argument $\arg c_k$ in $[0, 2\pi)$, $n_0 = 0$ and $c_0 = 1$. Then we can write

$$f(re^{i\theta}) = \sum_{k=0}^{\infty}|c_k|e^{i\theta_k}r^{n_k}e^{in_k\theta}.$$

Taking a $\theta \in [0, 2\pi)$ to be fixed, we consider the function $\psi_l(r) = \psi_l(r, \theta)$ defined by

$$\begin{aligned}
\psi_l(r) &= \Re\left[e^{-i(\theta_{k_l}+n_{k_l}\theta)}\sum_{k=0}^{\infty}|c_k|e^{i\theta_k}r^{n_k}e^{in_k\theta}\right] \\
&= \Re\left[\sum_{k=0}^{k_l-1} + |c_{k_l}|r^{n_{k_l}} + \sum_{k=k_l+1}^{\infty}\right] \\
&= \Re\left[\sum_{k=0}^{k_l-1}\right] + |c_{k_l}|r^{n_{k_l}} + \Re\left[\sum_{k=k_l+1}^{\infty}\right].
\end{aligned}$$

It is obvious that $|\psi_l(r)| \leq |f(re^{i\theta})|$.

Let $E_l^- = \{n_0, \dots, n_{s+1}\}$ and $E_l^+ = \{n_{k_l+1}, \dots, n_{k_l+t}\}$ be the set of non-negative integers, where

$$s = \min\left\{\sigma \geq 0 : \frac{1}{q^{\sigma+1}-1} \leq \frac{1}{108}\left\{\prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right)\right\}^2\right\}$$

and

$$t = \min\{\tau \geq 1 : x^{s+\tau+3}\exp(-2x) \leq x^{-(s+1)} \quad (x \geq q^{\tau+1})\}.$$

Note that both s and t are constants depending only on f.

Now we proceed to estimate $|D(E_l^- \cup E_l^+)\psi_l(r)| \quad (r \in [r_l^1, r_l^2])$. Let l_0 be defined by

$$l_0 = \min\{l \in \mathbf{N} : (1-3/n_{k_l})^{n_{k_l}/3} \geq 1/3\}.$$

Then we obtain, for any $l \geq l_0$, by (2.1), the following inequalities:

$$\begin{aligned}
&|D(E_l^- \cup E_l^+)|c_{k_l}|r^{n_{k_l}}| \\
=&|c_{k_l}|(n_{k_l}-n_0)\cdots(n_{k_l}-n_{s+1})(n_{k_l+1}-n_{k_l})\cdots(n_{k_l+t}-n_{k_l})r^{n_{k_l}} \\
=&|c_{k_l}|n_{k_l}^{s+2}\left(1-\frac{n_0}{n_{k_l}}\right)\cdots\left(1-\frac{n_{s+1}}{n_{k_l}}\right) \\
&\times n_{k_l+1}\cdots n_{k_l+t}\left(1-\frac{n_{k_l}}{n_{k_l+1}}\right)\cdots\left(1-\frac{n_{k_l}}{n_{k_l+t}}\right)r^{n_{k_l}} \\
\geq&|c_{k_l}|n_{k_l}^{s+2}\left\{\prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right)\right\}n_{k_l+1}\cdots n_{k_l+t}\left\{\prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right)\right\}\left\{\left(1-\frac{3}{n_{k_l}}\right)^{n_{k_l}/3}\right\}^3 \\
\geq&\frac{1}{27}\left\{\prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right)\right\}^2|c_{k_l}|n_{k_l}^{s+2}n_{k_l+1}\cdots n_{k_l+t},
\end{aligned}$$

$$\begin{aligned}
&\left|D(E_l^- \cup E_l^+)\Re\left[\sum_{k=0}^{k_l-1}\right]\right| \\
\leq\ &\sum_{k=s+2}^{k_l-1}|c_k|(n_k-n_0)\cdots(n_k-n_{s+1})(n_{k_l+1}-n_k)\cdots(n_{k_l+t}-n_k) \\
\leq\ &n_{k_l+1}\cdots n_{k_l+t}\sum_{k=s+2}^{k_l-1}(|c_k|n_k)n_k^{s+1} \\
\leq\ &|c_{k_l}|n_{k_l}n_{k_l+1}\cdots n_{k_l+t}\sum_{k=s+2}^{k_l-1}n_k^{s+1}
\end{aligned}$$

$$
\begin{aligned}
&= |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} \sum_{k=s+2}^{k_l-1} \left(\frac{n_k}{n_{k_l}}\right)^{s+1} \\
&\leq |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} \sum_{k=s+2}^{k_l-1} q^{(s+1)(k-k_l)} \\
&\leq \frac{1}{q^{s+1}-1} |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} \\
&\leq \frac{1}{108} \left\{\prod_{n=1}^{\infty}\left(1-\frac{1}{q^n}\right)\right\}^2 |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} ,
\end{aligned}
$$

and

$$
\begin{aligned}
&\left| D(E_l^- \cup E_l^+) \Re \left[\sum_{k=k_l+1}^{\infty} \right] \right| \\
&\leq \sum_{k=k_l+t+1}^{\infty} |c_k| (n_k - n_0) \cdots (n_k - n_{s+1})(n_k - n_{k_l+1}) \cdots (n_k - n_{k_l+t}) r^{n_k} \\
&\leq \sum_{k=k_l+t+1}^{\infty} |c_k| n_k^{s+t+2} r^{n_k} \\
&= \sum_{k=k_l+t+1}^{\infty} |c_k| n_k^{-1} n_k^{s+t+3} r^{n_k} \\
&\leq |c_{k_l}| n_{k_l}^{-1} \sum_{k=k_l+t+1}^{\infty} n_k^{s+t+3} r^{n_k} \\
&= |c_{k_l}| n_{k_l}^{s+t+2} \sum_{k=k_l+t+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{s+t+3} r^{n_k} \\
&\leq |c_{k_l}| n_{k_l}^{s+t+2} \sum_{k=k_l+t+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{s+t+3} \left\{\left(1-\frac{2}{n_{k_l}}\right)^{n_{k_l}/2}\right\}^{\frac{2n_k}{n_{k_l}}} \\
&\leq |c_{k_l}| n_{k_l}^{s+t+2} \sum_{k=k_l+t+1}^{\infty} \left(\frac{n_k}{n_{k_l}}\right)^{s+t+3} \exp\left(-\frac{2n_k}{n_{k_l}}\right) \\
&\leq |c_{k_l}| n_{k_l}^{s+t+2} \sum_{k=k_l+t+1}^{\infty} \left(\frac{n_{k_l}}{n_k}\right)^{s+1}
\end{aligned}
$$

$$\begin{aligned}
&\le |c_{k_l}| n_{k_l}^{s+t+2} \sum_{k=k_l+t+1}^{\infty} q^{(k_l-k)(s+1)} \\
&\le \frac{1}{q^{s+1}-1} |c_{k_l}| n_{k_l}^{s+t+2} \\
&\le \frac{1}{108} \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{1}{q^n}\right) \right\}^2 |c_{k_l}| n_{k_l}^{s+t+2} .
\end{aligned}$$

These inequalities yield that

$$|D(E_l^- \cup E_l^+)\psi_l(r)| \ge \frac{1}{54} \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{1}{q^n}\right) \right\}^2 |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} ,$$

for all $r \in [r_l^1, r_l^2]$.

Therefore, by Lemma 5, there exist $(s+t+4)$-numbers

$$r_l^1 = 1 - 3/n_{k_l} = \eta_0 < \eta_1 < \cdots < \eta_{s+t+2} < \eta_{s+t+3} = 1 - 2/n_{k_l} = r_l^2$$

such that

$$\begin{aligned}
|\psi_l(r)| \ge &\frac{1}{54} \left\{ \prod_{n=1}^{\infty} \left(1 - \frac{1}{q^n}\right) \right\}^2 |c_{k_l}| n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} \\
&\times \frac{1}{2^{(s+t+2)(s+t+1)/2}(s+t+2)!} (1 - 2/n_{k_l})^{-(s+t+2)} \\
&\times \left(\frac{1 - 3/n_{k_l}}{1 - 2/n_{k_l}} \right)^{n_0 + \cdots + n_{s+1} + n_{k_l+1} + \cdots + n_{k_l+t}} \Psi(r; \eta_0, \ldots, \eta_{s+t+3}) .
\end{aligned}$$

Since $\log^+ ab \le \log^+ a + \log^+ b$ $(a, b > 0)$, we have

$$\begin{aligned}
\log^+ 1/|\psi_l(r)| \le &\log^+ 1/|c_{k_l}| \\
&+ \log^+ 1/n_{k_l}^{s+2} n_{k_l+1} \cdots n_{k_l+t} \Psi(r; \eta_0, \ldots, \eta_{s+t+3}) \\
&+ C(f) \\
\le &\log^+ 1/|c_{k_l}| \\
&+ \log^+ 1/n_{k_l}^{s+t+2} \Psi(r; \eta_0, \ldots, \eta_{s+t+3}) \\
&+ C(f) ,
\end{aligned}$$

so that we obtain

$$\begin{aligned}
&\min\{\log 1/|\psi_l(r) : \ r_l^1 \le r \le r_l^2\} \\
\le &\frac{1}{r_l^2 - r_l^1} \int_{r_l^1}^{r_l^2} \log^+ 1/|\psi_l(r)| dr
\end{aligned}$$

$$
\begin{aligned}
&\leq \log^+ 1/|c_{k_l}| \\
&\quad + \frac{1}{r_l^2 - r_l^1} \int_{r_l^1}^{r_l^2} \log^+ 1/n_{k_l}^{s+t+2} \Psi(r; \eta_0, \ldots, \eta_{s+t+3}) dr \\
&\quad + C(f) \\
&\leq \log^+ 1/|c_{k_l}| \\
&\quad + (s+t+2) \sum_{i=1}^{s+t+2} \frac{1}{r_l^2 - r_l^1} \int_{r_l^1}^{r_l^2} \log^+ 1/n_{k_l} |r - \eta_i| \, dr \\
&\quad + C(f) \\
&\leq \log^+ 1/|c_{k_l}| + C(f) .
\end{aligned}
$$

This inequality yields (2.14). We complete the proof. □

References

[1] M. BIERNACKI, Sur les équations algébriques contenant des paramètres arbitaires, Bull. Int. Acd. Polon. Sci. Lett. Sér. A: Sci. Math. (Ⅲ), (1927), 542–685.

[2] L. FEJÉR, Über die Wurzel vom kleinsten absoluten Betrage einer algebraishen Gleichung, Math. Ann. **65** (1908), 413–423.

[3] W. H. J. FUCHS, On the zeros of power series with Hadamard gaps, Nagoya Math. J. **29** (1967), 167–174.

[4] W. H. J. FUCHS, Topics in Nevanlinna theory, *Proc. NRL Conference on Classical Function Theory*, Math. Res. Center, Naval Research Laboratory, Washington, D.C., 1970, 1–32.

[5] J. HADAMARD, Essai sur l'étude des fonctions données par leur développement de Taylor, J. Math. **8** (1892), 101–186.

[6] W. K. HAYMAN, *Meromorphic Functions*, Oxford Mathematical Monographs (1964).

[7] F. JOHN and L. NIRENBERG, On functions of bounded mean oscillation, Comm. Pure Appl. Math. **14** (1961), 415–426.

[8] E. KOCHNEFF, Y. SAGHER and K. ZHOU, BMO estimates for lacunary series, Arkiv för Matematik **28** (1990), 301–310.

[9] J. E. LITTLEWOOD and A. C. OFFORD, On the distribution of zeros and a-values of a random integral function (2), Ann. of Math. **49** (1948), 885–952.

[10] T. MURAI, Sur la distribution des valeurs des séries lacunaires, J. London Math. Soc. (2) **21** (1980), 93–110.

[11] T. MURAI, The value-distribution of lacunary series and a conjecture of Paley, Ann. Inst. Fourier, Grenoble **31-1** (1981), 135–156.

[12] T. MURAI, The deficiency of entire function with Fejér gaps, Ann. Inst. Fourier, Grenoble **33-3** (1983), 39–58.

[13] V. P. PETRENKO, Growth of meromorphic functions of finite order, Math. USSR Izv. **3** (1969), 391–431.

[14] R. SALEM and A. ZYGMUND, On lacunary trigonometric series, Proc. Nat. Acad. Sci. **33** (1947), 333–338.

[15] S. SIDON, Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourier-reihen mit Lücken, Math. Ann. **97** (1927), 675–676.

Present Address:
CAE STRATEGY DEPARTMENT ENGINEERING TECHNOLOGY DIVISION,
JSOL CORPORATION,
2–5–24, HARUMI, CHUO-KU,
TOKYO 104–0053, JAPAN.
e-mail: tsuboi.narufumi@jsol.co.jp