TOKYO J. MATH.
VOL. 40, NO. 1, 2017

On the Stokes Matrices of the tt^*-Toda Equation

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(Communicated by S. Hosono)

Abstract. We derive a formula for the signature of the symmetrized Stokes matrix $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$ for the tt^*-Toda equation, reminiscent of a formula of Beukers and Heckmann for the generalized hypergeometric equation. The condition $\mathcal{S} + \mathcal{S}^{\mathrm{T}} > 0$ is prominent in the work of Cecotti and Vafa on the tt^* equation; using our formula, we show that the Stokes matrices $\mathcal{S}$ satisfying this condition are parameterized by the points of an open convex polytope.

1. Introduction

The classical Stokes phenomenon for meromorphic ODEs has begun to play an important role in geometry, notably in singularity theory, Frobenius manifolds and mirror symmetry. For a (real) Stokes matrix $\mathcal{S}$, the symmetrized matrix $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$ arises in the context of Frobenius manifolds and the tt^* equation (e.g. [6, 7, 13, 14]).

The tt^* equation is a nonlinear PDE which appeared in the work of Cecotti and Vafa [2, 3, 4] on the classification of supersymmetric field theories in physics. It is a special case of the harmonic map equation in differential geometry for maps from a surface to a noncompact symmetric space [11]. Dubrovin [5] showed that it admits an isomonodromic deformation interpretation, as well as a zero-curvature formulation. This leads to a Riemann-Hilbert correspondence between (local) solutions and monodromy data of a meromorphic ODE. Clarifying this correspondence is a subject of current research activity relating several fields of mathematics, including Hodge theory and algebraic geometry.

There are very few examples where solutions can be found. A special case of the tt^* equation, introduced by Cecotti and Vafa, and studied mathematically by Guest-Its-Lin [9, 10] and Mochizuki [15], is the tt^*-Toda equation. This is, essentially, the well-known Toda field equation (2-dimensional Toda lattice), although even in this case the existence of the solutions predicted by Cecotti and Vafa was proved only recently (in the aforementioned references).

This article was motivated by the conjectures of Cecotti and Vafa regarding the symmetrized Stokes matrix $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$, in the case of the tt^*-Toda equation. We shall give a necessary and sufficient condition for $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$ to be positive definite, and a formula for the signature of

Received October 8, 2015; revised January 18, 2016
2010 *Mathematics Subject Classification*: 81T40 (Primary), 53D45, 35J60, 34M40 (Secondary)

$\mathcal{S} + \mathcal{S}^{\mathrm{T}}$, reminiscent of a formula of Beukers and Heckmann for the generalized hypergeometric equation [1].

Let us now state the tt^*-Toda equation and explain the relevant Stokes matrix. The equations are:

$$2(w_i)_{z\bar{z}} = -e^{2(w_{i+1}-w_i)} + e^{2(w_i-w_{i-1})}, \quad w_i : \mathbb{C}^* \to \mathbb{R} \tag{1.1}$$

subject to two further conditions:

1. the "anti-symmetry" condition: $w_i + w_{n-i} = 0$; and
2. the radial condition: $w_i = w_i(|z|)$.

We use the convention that $w_i = w_{i+n+1}$ for all $i \in \mathbb{Z}$. In what follows, we write $n + 1 = 2m$ or $n + 1 = 2m + 1$.

This equation is the compatibility condition for the linear system:

$$\begin{cases} \Psi_z = (\mathrm{w}_z + \frac{1}{\lambda}\mathrm{W})\Psi \,, \\ \Psi_{\bar{z}} = (-\mathrm{w}_{\bar{z}} + \lambda \mathrm{W}^{\mathrm{T}})\Psi \,, \end{cases}$$

where:

$$\mathrm{w} = \mathrm{diag}\,(w_0, \ldots, w_n)\,, \qquad \mathrm{W} = \begin{bmatrix} & e^{w_1-w_0} & & \\ & & \ddots & \\ & & & e^{w_n-w_{n-1}} \\ e^{w_0-w_n} & & & \end{bmatrix}.$$

If we write $x = |z|$, then the radial version of (1.1) is the compatibility condition for a linear system, which may then be transformed to (see Equation (1.4) of [10]):

$$\begin{cases} \Psi_\zeta = \left(-\frac{1}{\zeta^2}\mathrm{W} - \frac{1}{\zeta}x\mathrm{w}_x + x^2\mathrm{W}^{\mathrm{T}}\right)\Psi \,, \\ \Psi_x = \left(\frac{1}{x\zeta}\mathrm{W} + x\zeta\mathrm{W}^{\mathrm{T}}\right)\Psi \,, \end{cases} \tag{1.2}$$

where $\zeta = \frac{\lambda}{z}$.

The ζ-system of (1.2) is a meromorphic linear ODE in the complex variable ζ, with poles of order two at both $\zeta = 0$ and $\zeta = \infty$. The Stokes matrices at these two poles are equivalent, so we shall only consider the Stokes matrix at $\zeta = \infty$, and denote it by $\mathcal{S}$. By the general theory of isomonodromic deformations (e.g. [8]), Stokes matrices $\mathcal{S}$ correspond to local solutions near 0 (i.e. defined on intervals of the form $(0, \varepsilon)$) of the tt^*-Toda equation. Further details and explanation may be found in [9, 10], where $\mathcal{S}$ is computed in terms of the asymptotic behaviour of the functions w_i.

It was conjectured by Cecotti and Vafa that the condition $\mathcal{S} + \mathcal{S}^{\mathrm{T}} > 0$ implies that the corresponding local solution of the tt^*-Toda equation is globally defined on $\mathbb{C}^*$ (i.e. such that $\varepsilon = \infty$). This was confirmed in [9, 10, 11, 15], and in Theorem 5.6 of [10], a stronger result (also suggested by Cecotti and Vafa) was shown: a necessary and sufficient condition for the local solution of the tt^*-Toda equation to be globally defined on $\mathbb{C}^*$ is that the eigenvalues of the monodromy $\mathcal{S}(\mathcal{S}^{-1})^{\mathrm{T}}$ are unimodular.

It is, therefore, of interest to describe the set of such Stokes matrices explicitly, and this is our first main result. For such Stokes matrices, we prove the following explicit characterization of the signature of $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$, showing that they form an open convex polytope described by simple equations, and we expect this result to be of use in future investigations of the tt^*-Toda equation:

THEOREM. $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$ *has the same signature as the diagonal matrix*:

$$\operatorname{diag}\left((-1)^{n+1}p(\pi_0),\dots,(-1)^{n+1}p(\pi_n)\right).$$

Here, π_k *are the* $n+1$ *roots of* $x^{n+1}-(-1)^{n+1}$*, and the real polynomial* $p(x)$ *is the characteristic polynomial of a certain matrix* $\mathcal{R}$ *satisfying* $(-1)^n\mathcal{R}^{n+1}=\mathcal{S}\mathcal{S}^{-\mathrm{T}}$.

COROLLARY. $\mathcal{S}+\mathcal{S}^{\mathrm{T}}>0$ iff $(-1)^{n+1}p(\pi_k)>0$ *for all* k*, and the set of such Stokes matrices is in* 1-1 *correspondence with an open, convex polytope of* $\mathbb{R}^m$.

Our second main result is a formula for the sign of $p(\pi_k)$ when the eigenvalues of $\mathcal{S}(\mathcal{S}^{-1})^{\mathrm{T}}$ are unimodular. We refer the reader to Corollary 2.13 for the precise statement of this result. This characterizes the signature of $\mathcal{S}+\mathcal{S}^{\mathrm{T}}$ in terms of the configurations of the eigenvalues of $\mathcal{R}$ with respect to the roots π_k on the unit circle.

These results are given in Section 2 for a conveniently defined, “idealized Stokes matrix” S. In Section 3, we explain the precise relation between this “idealized Stokes matrix”, and the “actual” Stokes matrices of [9, 10].

Notational remark: In this paper, $\mathbb{N}$ shall denote the natural numbers, $\mathbb{Z}$ the integers, $\mathbb{Z}_{\geq 0}$ the non-negative integers, $\mathbb{R}$ the real numbers, $\mathbb{C}$ the complex numbers, and $\mathbb{C}^*=\mathbb{C}\setminus\{0\}$ the complex plane punctured at the origin. For a matrix A, its transpose is denoted A^{T}, and $\mathrm{A}^{-\mathrm{T}}$ will denote the inverse of A^{T}.

ACKNOWLEDGMENT. The author wishes to thank Martin Guest, for explaining the tt^*-Toda equation and suggesting the problem which is solved in this article, and the anonymous referee for their comments and suggestions on improving this paper (especially § 3, and for supplying a shorter proof of Proposition 2.2).

2. Main Results

Let $\varepsilon:=(-1)^n$ and let $\mathrm{E}_{i,j}$ be the matrix $\left(\delta_{i,k}\delta_{j,\ell}\right)_{k,\ell=1}^{n}$. It will be convenient to write $\mathrm{E}_{i,j}:=0$ whenever at least one of i or $j\notin\{1,\dots,n\}$. The following two matrices will be the main focus of this section:

DEFINITION 2.1. Let $\mathrm{R} \in \mathrm{SL}_n\mathbb{R}$ be given by:

$$\mathrm{R} := -\sum_{k=1}^{n} p_{n-k}\mathrm{E}_{k,1} + \sum_{k=1}^{n} \mathrm{E}_{k,k+1} = \begin{bmatrix} -p_{n-1} & \cdot & \dots & \cdot \\ \vdots & \vdots & \mathrm{I}_{n-1} & \vdots \\ -p_1 & \cdot & \dots & \cdot \\ -p_0 & 0 & \dots & 0 \end{bmatrix}, \tag{2.1}$$

where $p_0 := \varepsilon$ and:

$$p_{n-k} = \varepsilon p_k \quad \forall\, 0 \le k \le n\,. \tag{2.2}$$

Let S be the upper-triangular Toeplitz matrix:

$$\mathrm{S} := \varepsilon \sum_{i,j=1}^{n} p_{j-i}\mathrm{E}_{i,j} = \begin{bmatrix} 1 & \varepsilon p_1 & \dots & \varepsilon p_{n-2} & \varepsilon p_{n-1} \\ 0 & 1 & \ddots & \ddots & \varepsilon p_{n-2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \varepsilon p_1 \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix}, \tag{2.3}$$

where, for notational convenience, $p_k := 0$ whenever $k < 0$ or $k > n$.

Note that the characteristic polynomial $p(x) = \sum_{k=0}^{n} p_k x^k$ of R is a *signed-palindromic* polynomial:

$$p(x) = (-x)^n p\left(\tfrac{1}{x}\right).$$

These matrices satisfy the following important relation:

PROPOSITION 2.2. $-\varepsilon\mathrm{R}^n = \mathrm{S}\mathrm{S}^{-\mathrm{T}}$.

PROOF. Using our convention for the indices of the elementary matrices $\mathrm{E}_{i,j}$ and the coefficients p_k of $p(x)$, we first observe that, due to (2.2):

$$\varepsilon\mathrm{R}\mathrm{S}^{\mathrm{T}} = -\sum_{k=1}^{n} p_k\mathrm{E}_{k,1} + \sum_{i,j=1}^{n} p_{j-i}\mathrm{E}_{j-1,i}\,.$$

We claim that for all $m \ge 1$:

$$\varepsilon\mathrm{R}^m\mathrm{S}^{\mathrm{T}} = -\sum_{\ell=1}^{m}\sum_{k=1}^{n} p_k\mathrm{E}_{k+\ell-m,\ell} + \sum_{i,j=1}^{n} p_{j-i}\mathrm{E}_{j-m,i}\,.$$

This follows by a straightforward proof by induction, with the above observation establishing the base case for $m = 1$. Hence, for $m = n$, we find that:

$$\varepsilon\mathrm{R}^n\mathrm{S}^{\mathrm{T}} = -\sum_{k,\ell=1}^{n} p_k\mathrm{E}_{k+\ell-n,\ell} + \sum_{i,j=1}^{n} p_{j-i}\mathrm{E}_{j-n,i}\,.$$

The second summation vanishes since $j < n$, and using (2.2), a simple manipulation of the indices in the first summation yields $\varepsilon \mathrm{R}^n \mathrm{S}^{\mathrm{T}} = -\mathrm{S}$, as was to be shown. □

For the next proposition, let $\omega := e^{\frac{2\pi i}{n}}$, and define Π_ε by:

$$\Pi_\varepsilon := \left[\begin{array}{c|c} 0 & \mathrm{I}_{n-1} \\ \hline \varepsilon & 0 \end{array}\right]. \tag{2.4}$$

Then the characteristic polynomial of Π_ε is $x^n - \varepsilon$, and thus, its eigenvalues are:

$$\pi_k := \begin{cases} \omega^k\ , & 0 \le k \le n-1\ ,\ n = 2m\ , \\ \omega^{k+\frac{1}{2}}, & 0 \le k \le n-1\ ,\ n = 2m+1\ . \end{cases}$$

PROPOSITION 2.3. *The eigenvalues of* $\varepsilon(\mathrm{S}+\mathrm{S}^{\mathrm{T}})$ *are* $p(\pi_0), \ldots, p(\pi_{n-1})$.

PROOF. Since $p(x)$ satisfies (2.2), it is evident that $\varepsilon(\mathrm{S}+\mathrm{S}^{\mathrm{T}}) = p(\Pi_\varepsilon)$, and hence, $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ commutes with Π_ε, so they may be diagonalized simultaneously. To each π_k, then, we let $\mathbf{v}_k$ denote the corresponding eigenvector of Π_ε:

$$\mathbf{v}_k := \frac{1}{n}\,(1, \pi_k, \pi_k^2, \ldots, \pi_k^{n-1})^{\mathrm{T}}\ , \quad 0 \le k \le n-1\ , \tag{2.5}$$

It then follows that $\varepsilon(\mathrm{S}+\mathrm{S}^{\mathrm{T}})\mathbf{v}_k = p(\pi_k)\mathbf{v}_k$ for each k. □

COROLLARY 2.4. $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ *has the same signature as the diagonal matrix* $\mathrm{diag}\,(\varepsilon p(\pi_0), \ldots, \varepsilon p(\pi_{n-1}))$. *In particular*:

- $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ *is positive definite* iff $\varepsilon p(\pi_k) > 0$ *for all* k.
- *The number of zero eigenvalues is the number of common eigenvalues of* R *and* Π_ε.

□

As before, let us write $n = 2m$ for even n, and $n = 2m+1$ for odd n. We recall that a complex number of unit norm is said to be *unimodular*.

PROPOSITION 2.5 (cf. [4], [10]).

1. *If* $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ *is positive definite, then the eigenvalues of* R *are unimodular.*
2. *The set of all* R *such that* $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ *is positive definite is in* 1-1 *correspondence with the bounded convex region of* $\mathbb{R}^m$ *defined by*:

$$\mathcal{P} := \bigcap_{k=0}^{m} \{(p_1, \ldots, p_m) \in \mathbb{R}^m \mid \varepsilon p(\pi_k) > 0\}\ .$$

PROOF. $\mathrm{SS}^{-\mathrm{T}}$ preserves the symmetric bilinear form defined by $\mathrm{S}+\mathrm{S}^{\mathrm{T}}$ [4]:

$$(\mathrm{SS}^{-\mathrm{T}})(\mathrm{S}+\mathrm{S}^{\mathrm{T}})(\mathrm{SS}^{-\mathrm{T}})^{\mathrm{T}} = \mathrm{S}+\mathrm{S}^{\mathrm{T}}\ .$$

Hence, SS^{-T} must be orthogonal, and all eigenvalues of SS^{-T} are unimodular. As a result, by Proposition 2.2, the eigenvalues of R must be unimodular, as well.

Next, we establish that the set of all R defined by (2.1), such that $S + S^T > 0$, is in 1-1 correspondence with $\mathcal{P}$:

- By Corollary 2.4, $\varepsilon p(\pi_k) > 0$ for all k, and thus, the entries $p_1, \dots, p_m$ of each such R determine a unique point $(p_1, \dots, p_m) \in \mathcal{P}$.
- Conversely, given a point $P := (p_1, \dots, p_m) \in \mathcal{P}$, if we define R and S according to (2.1), (2.2) and (2.3) via the components of P, it then follows by the definition of $\mathcal{P}$ and by Corollary 2.4 that $S + S^T$ is positive definite. (Thus, by the first assertion (1) of this proposition, all eigenvalues of R are unimodular.)

Since the entries $p_1, \dots, p_m$ of R are the elementary symmetric polynomials of the eigenvalues of R, all of which lie, by assumption, in the compact set S^1, $\mathcal{P}$ is consequently contained in the continuous image of a compact set, and hence, is bounded.

Lastly, as each inequality $\varepsilon p(\pi_k) > 0$ defines a convex region of $\mathbb{R}^m$, and the intersection of any collection of convex sets is convex, we see that $\mathcal{P}$ is convex. □

Taking $S + S^T$ non-degenerate, henceforth, we shall now consider the dependence of the signature σ of $S + S^T$ on the eigenvalues of R, when R has only unimodular eigenvalues. By Proposition 2.3 [1], we note that σ is constant with respect to any variation (within S^1) of an eigenvalue of R such that the eigenvalue does not pass through a root of $x^n - \varepsilon$. Hence, σ is a function of only the number of eigenvalues of R between each root of $x^n - \varepsilon$. When $n = 2m + 1$, the conjugate symmetry of the eigenvalues implies that σ is also a function of the number of eigenvalues in the arc $\{e^{i\theta} \mid \theta \in [0, \frac{\pi}{2m+1})\}$. We now introduce some notation to assist in discussing this:

DEFINITION 2.6. Assume R has only unimodular eigenvalues $e^{\pm i\theta_j}$, $1 \leq j \leq m$. (For $n = 2m + 1$, we do not include the guaranteed eigenvalue $z = 1$ in this list.) When $n = 2m$, the *configuration* ρ *of* R is defined to be $\rho = (\rho_1, \dots, \rho_m) \in \mathbb{Z}^m_{\geq 0}$ such that $\rho_k := \#\left\{j \,\middle|\, \theta_j \in \left(\frac{(k-1)\pi}{m}, \frac{k\pi}{m}\right)\right\}$. For $n = 2m + 1$, the *configuration* ρ *of* R is defined to be $\rho = (\rho_0, \dots, \rho_m) \in \mathbb{Z}^{m+1}_{\geq 0}$ such that:

$$\rho_k := \begin{cases} \#\left\{j \,\middle|\, \theta_j \in \left[0, \frac{\pi}{2m+1}\right)\right\}, & k = 0, \\ \#\left\{j \,\middle|\, \theta_j \in \left(\frac{(2k-1)\pi}{2m+1}, \frac{(2k+1)\pi}{2m+1}\right)\right\}, & 1 \leq k \leq m. \end{cases}$$

Necessarily, the sum of the components of ρ is always m, for all n, and we say that two matrices (for the same n) have the same configuration whenever their configuration sequences agree.

[1] This was observed in [4, p.27] in the context of the general tt^* equation.

To find σ in terms of ρ, we shall apply Descartes' Rule of Signs to a polynomial with only real roots uniquely derived from $p(x)$, and then relate this to a formula proved via an adaptation of the proof of Theorem 4.5 of [1]. The result will then be, in principle, a formula to determine σ from only the entries of S.

2.1. Recall that Descartes' Rule of Signs is the following classical result, whose proof we omit:

PROPOSITION 2.7 (cf. [12, Theorem 6.2d]). *Let $p(x) \in \mathbb{R}[x]$ have degree $n \in \mathbb{N}$, with non-zero leading coefficient a_n, and let ν denote the number of sign changes in the sequence of non-zero coefficients of $p(x)$, starting with a_n and listed in decreasing order of the power of x. If r denotes the number of real, positive roots of $p(x)$, where each root is counted according to its algebraic multiplicity, then $\nu - r$ is even and non-negative.*

To refine this for polynomials with only real roots, we first prove:

LEMMA 2.8. *Let $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathbb{R}[x]$ be non-zero, and let ν and μ be the number of sign changes in the decreasing sequence of non-zero coefficients of $p(x)$ and $p(-x)$, respectively. Then $\nu + \mu \leq n$, and equality holds* iff *$a_k \neq 0$ for all k.*

PROOF. Let $\sigma(a_k, a_{k-1})$ denote the number of sign changes in the 2-term sequence (a_k, a_{k-1}), allowing this to be 0 when at least one of a_k or a_{k-1} are 0. Then by definition of ν, $\nu = \sum_{k=1}^{n} \sigma(a_k, a_{k-1})$. Now, by inspection:

$$\chi_k := \sigma(a_k, a_{k-1}) + \sigma((-1)^k a_k, (-1)^{k-1} a_{k-1}) = \begin{cases} 1\,, & a_k \neq 0 \ \text{ and } \ a_{k-1} \neq 0\,, \\ 0\,, & a_k = 0 \ \text{ or } \ a_{k-1} = 0\,, \end{cases}$$

so summing over all k, it follows that $\sum_{k=1}^{n} \chi_k = \mu + \nu$. But the left-hand side is an n-term summation of 1s and 0s, so $\mu + \nu \leq n$, and $\mu + \nu = n$ *iff* all n terms of the sum are 1, *iff* $a_k \neq 0$ for all $0 \leq k \leq n$. □

COROLLARY 2.9. *If $p(x) \in \mathbb{R}[x]$ has only real roots, then $\nu = r$.*

PROOF. Let ν and μ be defined as in the lemma, and let r and s be the number of positive roots of $p(x)$ and $p(-x)$, respectively. (Clearly, s is the number of negative roots of $p(x)$.) Then by Descartes' Rule applied to both $p(x)$ and $p(-x)$, there are $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ such that $\nu - r = 2\alpha$ and $\mu - s = 2\beta$.

First, assume that $p(x)$ has only non-zero real roots, so that $n = r + s$. Then $n = \nu + \mu - 2(\alpha + \beta)$, so by the lemma and non-negativity of α and β, $\alpha = \beta = 0$.

Now suppose that $p(x)$ has only real roots, and t of them zero. Then $p(x)x^{-t}$ has only non-zero real roots, and the number of its positive roots is also r, so by the previous assertion, if ν' is the number of sign changes of $p(x)x^{-t}$, then $\nu' = r$. But evidently, $\nu = \nu'$, which proves the assertion. □

Now, consider the following: for any $p(x) \in \mathbb{R}[x]$ satisfying (2.2), it may be shown by induction that there is a unique monic $\tilde{p}(x) \in \mathbb{R}[x]$ such that:

$$\begin{cases} p(x) = x^m \tilde{p}\big(x + \frac{1}{x}\big)\,, & n = 2m\,, \\ p(x) = (x-1)x^m \tilde{p}\big(x + \frac{1}{x}\big)\,, & n = 2m+1\,, \end{cases} \tag{2.6}$$

(For $n = 2m + 1$, we note that $p(1) = 0$ by (2.2), so after factoring $p(x) = (x - 1)q(x)$, we see that $q(x)$ satisfies (2.2), and thus, the even case factorization applies.)

As all eigenvalues of R are unimodular, $\tilde{p}(x)$ has one root $2\cos\theta_j$ for each conjugate pair of roots $e^{\pm i\theta_j}$ of $p(x)$. This motivates the following:

PROPOSITION 2.10. *Given $\tilde{p}(x)$ as in* (2.6), *let $\tilde{p}^{[0]}(x) := \tilde{p}(x + 2)$ for all n, and*:

$$\tilde{p}^{[k]}(x) := \begin{cases} \tilde{p}\big(x + 2\cos\frac{k\pi}{m}\big)\,, & 1 \le k \le m\,,\ n = 2m\,, \\ \tilde{p}\big(x + 2\cos\frac{(2k-1)\pi}{2m+1}\big)\,, & 1 \le k \le m+1\,,\ n = 2m+1\,. \end{cases}$$

Denote by ν_k the number of sign changes in the sequence of non-zero coefficients of $\tilde{p}^{[k]}(x)$, as in Descartes' Rule. Then the configuration ρ of R *satisfies*:

$$\nu_k - \nu_{k-1} = \begin{cases} \rho_k\,, & 1 \le k \le m\,,\ n = 2m\,, \\ \rho_{k-1}\,, & 1 \le k \le m+1\,,\ n = 2m+1\,. \end{cases}$$

Conversely, ν_k *is given by*:

$$\nu_k = \begin{cases} \sum_{j=1}^{k} \rho_j\,, & 0 \le k \le m\,,\ n = 2m\,, \\ \sum_{j=0}^{k-1} \rho_j\,, & 0 \le k \le m+1\,,\ n = 2m+1\,. \end{cases}$$

PROOF. Evidently, when $n = 2m$, ρ_k is the number of roots of $\tilde{p}(x)$ in the interval $\big(2\cos\frac{k\pi}{m}, 2\cos\frac{(k-1)\pi}{m}\big)$, and when $n = 2m + 1$, ρ_k is the number of roots of $\tilde{p}(x)$ in:

$$\begin{cases} (2\cos\frac{\pi}{2m+1}, 2]\,, & k = 0\,, \\ \Big(2\cos\frac{(2k+1)\pi}{2m+1}, 2\cos\frac{(2k-1)\pi}{2m+1}\Big)\,, & 1 \le k \le m\,. \end{cases}$$

On the other hand, by construction of the $\tilde{p}^{[k]}(x)$, the number of positive roots r_k of $\tilde{p}^{[k]}(x)$ is the same as the number of roots of $\tilde{p}(x)$ strictly greater than $2\cos\frac{k\pi}{m}$ when $n = 2m$, and when $n = 2m + 1$, it is the same as the number of roots of $\tilde{p}(x)$ strictly greater than 2, for $k = 0$, or $2\cos\frac{(2k-1)\pi}{2m+1}$ for all other k. Moreover, all m roots of $\tilde{p}(x)$ are real, and thus, $\nu_k = r_k$ by Corollary 2.9. Applying Descartes' Rule to $\tilde{p}^{[k]}(x)$ and $\tilde{p}^{[k-1]}(x)$, it then follows that:

$$\nu_k - \nu_{k-1} = \begin{cases} \rho_k\,, & 1 \le k \le m\,,\ n = 2m\,, \\ \rho_{k-1}\,, & 1 \le k \le m+1\,,\ n = 2m+1\,. \end{cases}$$

Conversely, given ρ, it follows from the above that, for all k:

$$\nu_k = \begin{cases} \sum_{j=1}^{k} \rho_j \,, & 0 \le k \le m \,,\ n = 2m \,, \\ \sum_{j=0}^{k-1} \rho_j \,, & 0 \le k \le m+1 \,,\ n = 2m+1 \,. \end{cases}$$

□

For notational convenience, we shall always denote the sequence of the number of sign changes of the $\tilde{p}^{[k]}(x)$ by:

$$\nu := \begin{cases} (0, \nu_1, \dots, \nu_{m-1}, m) \,, & n = 2m \,, \\ (0, \nu_1, \dots, \nu_m, m) \,, & n = 2m+1 \,. \end{cases}$$

REMARK 2.11. For the matrix R with characteristic polynomial $p(x) = x^{2m} + 1$, $\mathrm{S} + \mathrm{S}^{\mathrm{T}} = 2\mathrm{I}_{2m}$, and the configuration is $\rho = (1, 1, \dots, 1)$, which corresponds to the sequence $\nu = (0, 1, 2, \dots, m-1, m)$. Hence, by connectedness of $\mathcal{P}$ in Proposition 2.5, for any R with only unimodular eigenvalues when $n = 2m$, $\mathrm{S} + \mathrm{S}^{\mathrm{T}} > 0$ *iff* its configuration is $(1, 1, \dots, 1)$, which is *iff* its sequence of sign-change numbers is $(0, 1, 2 \dots, m-1, m)$. Similarly, for the matrix R with characteristic polynomial $p(x) = x^{2m+1} - 1$, we have $\mathrm{S} + \mathrm{S}^{\mathrm{T}} = 2\mathrm{I}_{2m+1}$, $\rho = (0, 1, 1, \dots, 1)$, and $\nu = (0, 0, 1, 2, \dots, m-1, m)$, and this is the only configuration for which $\mathrm{S} + \mathrm{S}^{\mathrm{T}} > 0$.

This observation has the interpretation (cf. Corollary 4.7 of [1]) that $\mathrm{S} + \mathrm{S}^{\mathrm{T}} > 0$ *iff* the eigenvalues $e^{\pm i\theta_j}$ of R *interlace* with the roots π_k of ε (including the guaranteed root $z = e^{i\,0}$ when $n = 2m+1$), in the following sense:

$$\begin{cases} 0 < \theta_1 < \frac{\pi}{m} < \theta_2 < \frac{2\pi}{m} < \cdots < \frac{(m-1)\pi}{m} < \theta_m < \pi \,, & n = 2m \,, \\ \frac{-\pi}{2m+1} < 0 < \frac{\pi}{2m+1} < \theta_1 < \frac{3\pi}{2m+1} < \cdots < \frac{(2m-1)\pi}{2m+1} < \theta_m < \pi \,, & n = 2m+1 \,. \end{cases}$$

2.2. Inspired by Sections 3 and 4 of [1], we prove a formula for σ using the sequence ν of sign-change numbers of the $\tilde{p}^{[k]}(x)$. Let $\mathbf{e}_k \in \mathbb{C}^n$ be the k^{th} canonical unit vector. Before proving the formula, we note the following:

1. Since the characteristic polynomial $p(x)$ of R satisfies (2.2), it follows that $\det(x\mathrm{I}_n - \mathrm{R}^{-\mathrm{T}}) = p(x)$, as well.
2. Letting $\mathrm{D} := \mathrm{I}_n - \Pi_\varepsilon \mathrm{R}^{\mathrm{T}}$, we remark that D has rank 1, and for all $\mathbf{x} \in \mathbb{C}^n$, $\mathrm{D}\mathbf{x} = \big(\mathbf{e}_n^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})\mathbf{x}\big)\mathbf{e}_n$. Note, as well, that $(\Pi_\varepsilon \mathrm{R}^{\mathrm{T}})^2 = \mathrm{I}_n$.
3. [1] If a rank one $n \times n$ matrix M acts on $\mathbb{C}^n$ as $\mathrm{M}\mathbf{x} = w(\mathbf{x})\mathbf{u}$ for some linear form w and for some $\mathbf{u} \in \mathbb{C}^n$, then $\det(\mathrm{I}_n + \mathrm{M}) = 1 + w(\mathbf{u})$.
4. Letting $\mathbf{v}_k$ be defined as in (2.5), we note that $\sum_{k=0}^{n-1} \pi_k \mathbf{v}_k = \Pi_\varepsilon \mathbf{e}_1 = \varepsilon\, \mathbf{e}_n$.

We now prove what is essentially a special case of Theorem 4.5 of [1]:

PROPOSITION 2.12. *For* R *with only unimodular eigenvalues such that* $\mathrm{S} + \mathrm{S}^{\mathrm{T}}$ *is non-degenerate, denote the eigenvalues of* R *as* $z_k := e^{2\pi i\theta_k}$, *where* $\theta_k \in [0, 1)$, *for* $0 \le k \le n-1$.

When $n = 2m+1$, *we take* $z_m = 1$, *so* $\theta_m = 0$. *If* $n_j := \#\{k \mid \theta_k < \arg\pi_j\}$ *for each* $0 \le j \le n-1$, *then* $sgn(p(\pi_j)) = (-1)^{n_j - j}$.

PROOF. We adapt the proof of Theorem 4.5 of [1] as follows: Expanding $p(x)$ and $x^n - \varepsilon$ into their complex linear factors, we use the above remarks to obtain:

$$\begin{aligned}\prod_{k=0}^{n-1}(z_k - x)(\pi_k - x)^{-1} &= \det(\mathrm{R}^{-\mathrm{T}} - x\mathrm{I}_n)\det(\Pi_\varepsilon - x\mathrm{I}_n)^{-1}\\ &= \det\big((\mathrm{R}^{-\mathrm{T}}\Pi_\varepsilon^{-1} - x\Pi_\varepsilon^{-1})(\mathrm{I}_n - x\Pi_\varepsilon^{-1})^{-1}\big)\\ &= \det\big((-\mathrm{D} + \mathrm{I}_n - x\Pi_\varepsilon^{-1})(\mathrm{I}_n - x\Pi_\varepsilon^{-1})^{-1}\big)\\ &= \det\big(\mathrm{I}_n - \mathrm{D}(\mathrm{I}_n - x\Pi_\varepsilon^{-1})^{-1}\big)\\ &= 1 - \mathbf{e}_n^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})(\mathrm{I}_n - x\Pi_\varepsilon^{-1})^{-1}\mathbf{e}_n\\ &= 1 - \mathbf{e}_n^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})(\Pi_\varepsilon - x\mathrm{I}_n)^{-1}\Pi_\varepsilon\mathbf{e}_n\,.\end{aligned}$$

Using the expression for $\mathbf{e}_n$ in terms of the $\mathbf{v}_k$ then yields:

$$\prod_{k=0}^{n-1}(z_k - x)(\pi_k - x)^{-1} = 1 - \sum_{k,j=0}^{n-1}\frac{\pi_j^2\bar{\pi}_k}{\pi_j - x}\mathbf{v}_k^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})\mathbf{v}_j = 1 - \sum_{j=0}^{n-1}\frac{\pi_j}{\pi_j - x}\mathbf{v}_j^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})\mathbf{v}_j\,.$$

Taking the residue at $x = \pi_j$, and inserting[2] $\varepsilon = i\prod_{k=0}^{n-1}\pi_k^{\frac{1}{2}}z_k^{-\frac{1}{2}}$, we find that:

$$\begin{aligned}\varepsilon\,\mathbf{v}_j^*(\mathrm{S} + \mathrm{S}^{\mathrm{T}})\mathbf{v}_j &= -\varepsilon(z_j - \pi_j)\pi_j^{-1}\prod_{k\neq j}(z_k - \pi_j)(\pi_k - \pi_j)^{-1}\\ &= -\left(i\prod_{k=0}^{n-1}\pi_k^{\frac{1}{2}}z_k^{-\frac{1}{2}}\right)(z_k - \pi_j)\pi_j^{-1}\prod_{k\neq j}(z_j - \pi_j)(\pi_k - \pi_j)^{-1}\\ &= -i\left(\frac{z_j^{\frac{1}{2}}}{\pi_j^{\frac{1}{2}}} - \frac{\pi_j^{\frac{1}{2}}}{z_j^{\frac{1}{2}}}\right)\prod_{k\neq j}\left(\frac{z_k^{\frac{1}{2}}}{\pi_j^{\frac{1}{2}}} - \frac{\pi_j^{\frac{1}{2}}}{z_k^{\frac{1}{2}}}\right)\left(\frac{\pi_k^{\frac{1}{2}}}{\pi_j^{\frac{1}{2}}} - \frac{\pi_j^{\frac{1}{2}}}{\pi_k^{\frac{1}{2}}}\right)^{-1}\\ &= 2\sin\pi(\theta_j - \arg\pi_j)\prod_{k\neq j}\frac{\sin\pi(\theta_k - \arg\pi_j)}{\sin(\arg\pi_k - \arg\pi_j)}\,.\end{aligned}$$

The sign of the denominator is $(-1)^j$, by inspection, and the sign of the numerator is $(-1)^{n_j}$, by definition of n_j. Thus, $sgn(p(\pi_j)) = (-1)^{n_j - j}$, by Proposition 2.3. □

COROLLARY 2.13. *For* R *as in Proposition* 2.12, *let* ν *be the sequence of sign-change numbers of the* $\tilde{p}^{[k]}(x)$, *and let* $\mathrm{S} + \mathrm{S}^{\mathrm{T}}$ *have signature* $\sigma = (n_+, n_-)$, *where* n_+ *and* n_- *are*

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[2] Note that $z_k^{\frac{1}{2}}z_{n-k}^{\frac{1}{2}} = -1$ for all k, since $\theta_k \in [0, 1)$ and the z_k come in conjugate pairs, except for $z_m = 1$ when $n = 2m+1$. A similar statement holds for the π_k.

the number of positive and negative eigenvalues, respectively. Then for $n = 2m$:

$$\begin{cases} n_j = \nu_j\,, & 0 \le j \le m\,, \\ n_{2m-j} = 2m - \nu_j\,, & 1 \le j \le m-1\,, \end{cases}$$

and for $n = 2m+1$:

$$\begin{cases} n_j - 1 = \nu_{j+1}\,, & 0 \le j \le m\,, \\ n_{2m-j} - 1 = 2m - \nu_{j+1}\,, & 0 \le j \le m-1\,. \end{cases}$$

Consequently, for all j:

$$sgn(p(\pi_j)) = \begin{cases} (-1)^{\nu_j - j}\,, & n = 2m\,, \\ (-1)^{\nu_{j+1}-(j+1)}\,, & n = 2m+1\,, \end{cases}$$

and thus, for all n:

$$n_+ - m - 1 = \begin{cases} \sum_{j=1}^{m-1} (-1)^{\nu_j - j}\,, & n = 2m\,, \\ -\sum_{j=1}^{m} (-1)^{\nu_j - j}\,, & n = 2m+1\,. \end{cases}$$

PROOF. The relations between n_j and ν_j follow from the definition of n_j and from Proposition 2.10. (Note that, when $n = 2m+1$, $n_j \ge 1$ for all j due to the guaranteed root $z_m = 1$ of $p(x)$.) By Proposition 2.12, the formula for $sgn(p(\pi_j))$ then follows immediately. Since $n_+ + n_- = n$ whenever $\mathrm{S} + \mathrm{S}^{\mathrm{T}}$ is non-degenerate, it follows by the above formula for $sgn(p(\pi_j))$ that:

$$\varepsilon(2n_+ - n) = \varepsilon(n_+ - n_-) = \begin{cases} (-1)^{\nu_0} + (-1)^{\nu_m - m} + 2\sum_{j=1}^{m-1} (-1)^{\nu_j - j}\,, & n = 2m\,, \\ (-1)^{\nu_{m+1}-(m+1)} + 2\sum_{j=1}^{m} (-1)^{\nu_j - j}\,, & n = 2m+1\,. \end{cases}$$

When $n = 2m$, $\nu_j = j$ for $j = 0$ and $j = m$, and hence:

$$n_+ = m + 1 + \sum_{j=1}^{m-1} (-1)^{\nu_j - j}\,.$$

When $n = 2m+1$, $\nu_{m+1} = m$ for $j = m$, and hence:

$$n_+ = m + 1 - \sum_{j=1}^{m} (-1)^{\nu_j - j}\,.$$

□

REMARK 2.14. We now provide some sample calculations. When $n = 4$ (cf. [10]), the relation between $\sigma = (n_+, n_-)$ and $\nu = (0, \nu_1, 2)$ is:

ν_1	0	1	2
σ	$(2,2)$	$(4,0)$	$(2,2)$

When $n = 6$, the signatures may be tabulated as follows:

$\nu_1 \diagdown \nu_2$	0	1	2	3
0	$(4,2)$	$(2,4)$	$(4,2)$	$(2,4)$
1	$\times$	$(4,2)$	$(6,0)$	$(4,2)$
2	$\times$	$\times$	$(4,2)$	$(2,4)$
3	$\times$	$\times$	$\times$	$(4,2)$

3. Application to the tt^*-Toda equation

Now we shall apply the results of the previous section to the symmetrized Stokes matrices of the tt^*-Toda equation, which were calculated in [9, 10, 15]. Let us consider the pole at infinity (of order 2) of Equation (1.2), for arbitrary $n+1$. In Section 4 of [9], the case $n+1 = 4$ was treated in detail. The same method applies for any $n \geq 3$; for the sake of exposition, we explain only the case $n+1 = 2m$ in detail, and point out major differences from the $n+1 = 2m+1$ case in footnotes.

Recall that we wish to determine the Stokes data at $\zeta = \infty$ for the ODE:

$$\Psi_\zeta = \left(-\frac{1}{\zeta^2}\mathrm{W} - \frac{1}{\zeta}x\mathrm{w}_x + x^2\mathrm{W}^{\mathrm{T}}\right)\Psi\,, \tag{3.1}$$

where w and W are defined before (1.2). If $\eta := \zeta^{-1}$, then we may re-write this as:

$$\Psi_\eta = \left(-\frac{1}{\eta^2}x^2\mathrm{W}^{\mathrm{T}} + O\left(\frac{1}{\eta}\right)\right)\Psi\,.$$

Letting $\omega := e^{\frac{2\pi i}{n+1}}$, $\mathrm{d}_{n+1} := \mathrm{diag}\,(1, \omega, \ldots, \omega^n)$, and $\Omega := \left(\omega^{ij}\right)_{i,j=0}^n$, we may use the matrix $\mathrm{P}_\infty := \mathrm{diag}\,(e^{w_0}, \ldots, e^{w_n})\,\Omega^{-1}$ to diagonalize W^{T} as:

$$\mathrm{W}^{\mathrm{T}} = \mathrm{P}_\infty \mathrm{d}_{n+1} \mathrm{P}_\infty^{-1}\,.$$

Then by Proposition 1.1 of [8], and reverting back to ζ, we see that there exists a unique formal solution Ψ_f^∞ of (3.1) of the form:

$$\Psi_f^\infty = \mathrm{P}_\infty\left(\mathrm{I}_{n+1} + \sum_{j\geq 1}\psi_j^\infty \zeta^{-j}\right)e^{\Lambda_0 \log\eta + x^2\zeta\mathrm{d}_{n+1}}\,.$$

It may then be verified, by direct substitution into (3.1), that $\Lambda_0 = 0$. By Theorem 1.4 of [8], there is then a unique holomorphic solution Ψ to (3.1), with asymptotic expansion Ψ_f^∞, on any Stokes sector based at $\zeta = \infty$.

There are $2(n+1)$ Stokes rays, given by all $\zeta \in \mathbb{C}^*$ satisfying:

$$\cos\left(\arg\zeta + \arg(\omega^j - \omega^k)\right) = 0\,, \tag{3.2}$$

where $\omega := e^{\frac{2\pi i}{n+1}}$. As fundamental Stokes sectors, we take:

$$\Omega_1^\infty = \left\{\zeta \in \mathbb{C}^* \,\middle|\, -\frac{\pi}{2} < \arg\zeta < \frac{\pi}{2} + \frac{\pi}{n+1}\right\},$$
$$\Omega_2^\infty = \left\{\zeta \in \mathbb{C}^* \,\middle|\, \frac{\pi}{2} < \arg\zeta < \frac{3\pi}{2} + \frac{\pi}{n+1}\right\}.$$

The Stokes matrix S_1^∞ is defined by $\Psi_2^\infty = \Psi_1^\infty \mathrm{S}_1^\infty$, where Ψ_j^∞ is the unique holomorphic solution asymptotic to Ψ_f^∞ on Ω_j^∞ (for $j = 1, 2$), and where the analytic continuation of Ψ_1^∞ to Ω_2^∞ is taken in the positive direction.

Letting $\Pi := \left(\begin{smallmatrix} 0 & \mathrm{I}_n \\ 1 & 0 \end{smallmatrix}\right)$, and using the symmetries of (1.2), as in Section 4 of [9], we find that:

$$\mathrm{S}_1^\infty = \left(\mathrm{Q}_1^\infty \mathrm{Q}_{1\frac{1}{n+1}}^\infty \Pi\right)^m \Pi^{-m}\,, \quad \mathrm{S}_2^\infty = (\mathrm{S}_1^\infty)^{-\mathrm{T}}\,,$$

and the inverse of the monodromy of Ψ_1^∞ is:

$$\mathrm{S}_1^\infty \mathrm{S}_2^\infty = \left(\mathrm{Q}_1^\infty \mathrm{Q}_{1\frac{1}{n+1}}^\infty \Pi\right)^{n+1}.$$

Here, the matrices Q_k^∞ are the "Stokes factors" of S_1^∞ and S_2^∞, defined with respect to the Stokes sectors $\Omega_{k+1}^\infty = e^{k\pi i}\Omega_1^\infty$ for all $k \in \frac{1}{n+1}\mathbb{Z}$ (i.e. $\Psi_{k+\frac{1}{n+1}}^\infty = \Psi_k^\infty \mathrm{Q}_k^\infty$).

As in Section 5 of [10], we may convert to real matrices $\tilde{\mathrm{S}}_k^\infty$ and $\tilde{\mathrm{Q}}_k^\infty$ by using the matrix $\tilde{P}_\infty := \mathrm{P}_\infty \mathrm{d}_\infty$, for $\mathrm{d}_\infty^{-1} := \mathrm{diag}\left(1, \omega^{\frac{1}{2}}, \omega, \ldots, \omega^{\frac{n}{2}}\right)^r$, $r = \frac{1}{2}$, in the diagonalization of W^T. We then obtain solutions $\tilde{\Psi}_k := \Psi_k \mathrm{d}_\infty$ asymptotic to $\tilde{\Psi}_f^\infty := \Psi_f^\infty \mathrm{d}_\infty$ on their respective Stokes sectors with corresponding real Stokes factors and real Stokes matrices:

$$\tilde{\mathrm{Q}}_k^\infty := \mathrm{d}_\infty^{-1} \mathrm{Q}_k^\infty \mathrm{d}_\infty\,, \quad \tilde{\mathrm{S}}_k^\infty := \mathrm{d}_\infty^{-1} \mathrm{S}_k^\infty \mathrm{d}_\infty\,.$$

Letting:

$$\hat{\Pi} := \begin{bmatrix} 0 & \mathrm{I}_n \\ -1 & 0 \end{bmatrix}, \quad \mathcal{R} := \tilde{\mathrm{Q}}_1^\infty \tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty \hat{\Pi}\,, \tag{3.3}$$

$$\mathcal{J} := \hat{\Pi}^m = \begin{pmatrix} 0 & \mathrm{I}_m \\ -\mathrm{I}_m & 0 \end{pmatrix}, \tag{3.4}$$

we then obtain[3]:

$$\mathcal{S} := \tilde{\mathrm{S}}_1^\infty = \mathcal{R}^m \mathcal{J}^{-1}, \quad \mathcal{S}\mathcal{S}^{-\mathrm{T}} = \tilde{\mathrm{S}}_1^\infty \tilde{\mathrm{S}}_2^\infty = -\mathcal{R}^{n+1}. \tag{3.5}$$

We are now almost in the situation of Section 2 of this article, but to achieve exactly those matrices, a further transformation is required. To find the correct transformation, we need explicit expressions for the matrices $\tilde{\mathrm{Q}}_1^\infty$ and $\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty$. Using (3.2) for the Stokes rays, it can be deduced that:

LEMMA 3.1. *The diagonal entries of* $\tilde{\mathrm{Q}}_1^\infty$ *and* $\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty$ *are* 1, *and the other entries satisfy the rule* (*for* $1 \le i \ne j \le n+1$):

$$\begin{cases} \arg(\omega^{i-1} - \omega^{j-1}) \ne \frac{n\pi}{n+1} \mod 2\pi \quad \Rightarrow \quad (\tilde{\mathrm{Q}}_1^\infty)_{i,j} = 0, \\ \arg(\omega^{i-1} - \omega^{j-1}) \ne \frac{(n-1)\pi}{n+1} \mod 2\pi \quad \Rightarrow \quad (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty)_{i,j} = 0. \end{cases}$$

PROOF. We follow the proof of Lemma 4.4 of [9]. For the complex Stokes factors Q_k^∞, $k \in \frac{1}{n+1}\mathbb{Z}$, we have:

$$\mathrm{Q}_k^\infty = \lim_{\zeta\to\infty} (\Psi_k^\infty)^{-1} \Psi_{k+\frac{1}{n+1}}^\infty = \lim_{\zeta\to\infty} e^{-\zeta x^2 \mathrm{d}_{n+1}} \big(\mathrm{I}_{n+1} + O(\tfrac{1}{\zeta})\big) e^{\zeta x^2 \mathrm{d}_{n+1}},$$

and hence, $(\mathrm{Q}_k^\infty)_{ii} = 1$ for all i. On the other hand, for (i, j) such that $1 \le i \ne j \le n+1$, the entry $(\mathrm{Q}_k^\infty)_{ij} = 0$ so long as there is a path $\zeta_t \to \infty$ in $\Omega_k \cap \Omega_{k+\frac{1}{n+1}}$ such that $\operatorname{Re} \zeta_t(\omega^{i-1} - \omega^{j-1}) < 0$. Since $\Omega_k \cap \Omega_{k+\frac{1}{n+1}}$ is a sector of angle π, it follows that $(\mathrm{Q}_k^\infty)_{ij}$ is necessarily zero only if $(\omega^{i-1} - \omega^{j-1})\, \Omega_k \cap \Omega_{k+\frac{1}{n+1}}$ is equal to the closed half-plane $\{\operatorname{Re}\zeta \le 0\}$. This, in turn, occurs *iff* $\arg(\omega^{i-1} - \omega^{j-1}) \ne \frac{(2n+1-(n+1)k)\pi}{n+1} \mod \pi$.

It then follows by the definition of the $\tilde{\mathrm{Q}}_k^{(\infty)}$ that their entries satisfy the same conditions, and by substituting $k = 1$ and $k = \frac{1}{n+1}$, the assertion of the lemma follows. □

Consequently, the potentially non-zero entries are:

$$(\tilde{\mathrm{Q}}_1^\infty)_{m-k,1+k}, \quad (\tilde{\mathrm{Q}}_1^\infty)_{m+1+k,n+1-k}, \quad (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty)_{m-1-k,1+k}, \quad (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty)_{m+k,n+1-k},$$

for $0 \le k \le \ell$, $\ell := \lfloor \frac{m}{2} \rfloor$. Taking into account all of the symmetry conditions (see Section 5 of [10]), it is straightforward to deduce, using the above lemma, that:

[3] For $n+1 = 2m+1$, (3.5) holds for $r = m+1$, $\mathcal{R} := \tilde{\mathrm{Q}}_1^\infty \tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^\infty \Pi$ and $\mathcal{J}^{-1} := \tilde{\mathrm{Q}}_1^\infty \Pi^{-m}$.

PROPOSITION 3.2. *For $0 \leq k \leq \ell$, the entries of $\tilde{\mathrm{Q}}_1^{\infty}$ and $\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty}$ satisfy*[4]:

$$\begin{cases} (\tilde{\mathrm{Q}}_1^{\infty})_{m-k,1+k} + (\tilde{\mathrm{Q}}_1^{\infty})_{m+1+k,n+1-k} = 0\,, \\ (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m-1-k,1+k} + (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m+1+k,n+1-1-k} = 0\,. \end{cases} \qquad \square$$

Hence, $\tilde{\mathrm{Q}}_1^{\infty}$ and $\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty}$ are block-matrices of the form:

$$\tilde{\mathrm{Q}}_1^{\infty} = \left[\begin{array}{c|c} \mathrm{L}_1{}^{\infty} & 0 \\ \hline 0 & \left(\mathrm{L}_1{}^{\infty}\right)^{\mathrm{T}} \end{array}\right], \quad \tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty} = \left[\begin{array}{c|c} \mathrm{L}_{1\frac{1}{n+1}}^{\infty} & p_m \mathrm{E}_{m,m} \\ \hline 0 & \left(\mathrm{L}_{1\frac{1}{n+1}}^{\infty}\right)^{-\mathrm{T}} \end{array}\right],$$

where L_1^{∞}, $\mathrm{L}_{1\frac{1}{n+1}}^{\infty}$ are lower-triangular with only 1s on the diagonal, and where, for notational convenience, we define $p_m := (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m,n+1}$ and:

$$-p_{m-2k} := (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m-1-k,1+k}\,, \quad -p_{m-2k-1} := (\tilde{\mathrm{Q}}_1^{\infty})_{m-k,1+k}\,. \tag{3.6}$$

To facilitate the next few propositions, we introduce the permutation matrix $\Delta := \sum_{k=0}^{n} \mathrm{E}_{n+1-k,1+k}$ and the block-matrix[5]:

$$\mathrm{F} := \left[\begin{array}{c|c} \mathrm{L}^{[m]} & 0 \\ \hline 0 & \mathrm{U}^{[m]} \end{array}\right], \tag{3.7}$$

where $\mathrm{L}^{[m]}$ and $\mathrm{U}^{[m]}$ are defined as follows:

$$\mathrm{L}^{[m]} := \mathrm{I}_m + \sum_{k=0}^{\ell-1} p_{m-2k-1} \sum_{j=0}^{k} \mathrm{E}_{m-2k+j,j+1} + \sum_{k=0}^{\ell-1} p_{m-2k-2} \sum_{j=0}^{k} \mathrm{E}_{m-2k-1+j,j+1}\,,$$

$$\mathrm{U}^{[m]} := \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \Delta \mathrm{L}^{[m-1]} \Delta \end{array}\right].$$

[4]For $n+1 = 2m+1$, the row indices are increased by 1, and we instead have:

$$(\tilde{\mathrm{Q}}_1^{\infty})_{m+1-k,1+k} + (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m+1+k,n+1-k} = 0 \quad \text{and} \quad (\tilde{\mathrm{Q}}_{1\frac{1}{n+1}}^{\infty})_{m-k,1+k} + (\tilde{\mathrm{Q}}_1^{\infty})_{m+2+k,n+1-k} = 0\,.$$

[5]For $n+1 = 2m+1$, the upper-left block of F is $\mathrm{L}^{[m+1]}$, and it is convenient to instead define $p_{m-2k} := (-1)^m (\tilde{Q}_1^{\infty})_{m+1-k,1+k}$ and $p_{2m-k-1} := (-1)^{m-1} (\tilde{Q}_{1\frac{1}{n+1}}^{\infty})_{m-k,1+k}$.

For example, when $m = 4$:

$$\mathrm{L}^{[4]} = \begin{bmatrix} 1 & & & \\ p_1 & 1 & & \\ p_2 & p_1 & 1 & \\ p_3 & 0 & 0 & 1 \end{bmatrix}, \quad \mathrm{U}^{[4]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & p_2 \\ & & 1 & p_1 \\ & & & 1 \end{bmatrix}.$$

LEMMA 3.3. *The following block-matrix identities hold for all $m \geq 1$:*

$$\mathrm{F}\tilde{\mathrm{Q}}_1^\infty = \left[\begin{array}{c|c} \mathrm{L}^{[m-1]} & 0 \\ \hline 0 & \mathrm{U}^{[m+1]} \end{array}\right], \tag{3.8}$$

$$\mathrm{F}\tilde{\mathrm{Q}}_1^\infty \tilde{\mathrm{Q}}^\infty_{1\frac{1}{n+1}} = \left[\begin{array}{c|c} \mathrm{L}^{[m-2]} & 0 \\ \hline 0 & \mathrm{U}^{[m+2]} \end{array}\right]. \tag{3.9}$$

PROOF. These follow directly from (3.6) and (3.7). □

To make the connection with Section 2, we now define R as in (2.1) using the entries $p_1, \ldots, p_m$, defined above, and take $p_{n+1-k} := p_k$. Then:

PROPOSITION 3.4. $\mathrm{R} = \mathrm{F}\mathcal{R}\mathrm{F}^{-1}$. *Hence, the characteristic polynomial of $\mathcal{R}$ is*:

$$p(x) = x^{n+1} + \sum_{k=1}^{n} p_k x^k + 1 .$$

PROOF. Using (3.9), we obtain $\mathrm{RF}\hat{\Pi}^{-1} = \mathrm{F}\tilde{\mathrm{Q}}_1^\infty \tilde{\mathrm{Q}}^\infty_{1\frac{1}{n+1}}$. □

PROPOSITION 3.5. *Let* S *be defined as in* (2.3) *via the entries of* $\mathrm{R} = \mathrm{F}\mathcal{R}\mathrm{F}^{-1}$. *Then* $\mathrm{S} = \mathrm{F}\mathcal{S}\mathrm{F}^{\mathrm{T}}$.

PROOF. Letting $\mathrm{J}_p^{-1} := \mathrm{F}\mathcal{J}^{-1}\mathrm{F}^{\mathrm{T}}$, we find, by definition (3.5) of $\mathcal{S}$ and Lemma 3.4, that $\mathrm{F}\mathcal{S}\mathrm{F}^{\mathrm{T}} = \mathrm{R}^m\mathrm{J}_p^{-1}$, so we shall prove the proposition by determining the entries of $\mathrm{R}^m\mathrm{J}_p^{-1}$. Inspecting J_p^{-1}, first, we see that[6]:

$$\mathrm{J}_p^{-1} = \left[\begin{array}{c|c} 0 & -\mathrm{L}^{[m]}\mathrm{U}^{[m]\,\mathrm{T}} \\ \hline \mathrm{U}^{[m]}\mathrm{L}^{[m]\,\mathrm{T}} & 0 \end{array}\right].$$

Since $\mathrm{R}\mathbf{e}_k = \mathbf{e}_{k-1}$ for all $1 \leq k \leq n$, we then find that:

$$\mathrm{R}^m\mathrm{J}_p^{-1}\mathbf{e}_k = \begin{cases} \mathbf{e}_1 , & k = 1 , \\ p_{k-1}\mathbf{e}_1 + \cdots + p_1\mathbf{e}_{k-1} + \mathbf{e}_k , & 2 \leq k \leq m , \end{cases}$$

[6]When $n + 1 = 2m + 1$, this follows by (3.8), and the bottom-left block is $\mathrm{U}^{[m+1]}\mathrm{L}^{[m+1]\,\mathrm{T}}$.

and the first m columns of $\mathrm{F}\mathcal{S}\mathrm{F}^{\mathrm{T}}$ and S agree. We now determine the last m columns of $\mathrm{R}^m\mathrm{J}_p^{-1}$ by applying it to the flag:

$$\mathcal{F} \;:\; \mathbf{0} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_n \cong \mathbb{R}^{n+1}\,, \quad \mathcal{F}_k := < \mathbf{e}_{n+1}, \ldots, \mathbf{e}_{n+1-k} >_{\mathbb{R}}\,.$$

From the form of $\mathrm{L}^{[m]}\mathrm{U}^{[m]\,\mathrm{T}}$, we observe that $-\mathrm{J}_p^{-1}\mathbf{e}_{n+1} = \mathbf{e}_m$, and that:

$$-\mathrm{J}_p^{-1}\mathbf{e}_{n+1-k} - \mathbf{e}_{m-k} \in < \mathbf{e}_m, \ldots, \mathbf{e}_{m-k+1} >_{\mathbb{R}}\,, \quad \forall\, 1 \le k \le m-1\,.$$

As a result, $-\mathrm{J}_p^{-1}\mathcal{F}_k = \mathrm{R}^m\mathcal{F}_k$ for all $0 \le k \le m-1$, and hence, $\mathrm{R}^m\mathrm{J}_p^{-1}\mathcal{F}_k = \mathrm{S}\mathrm{S}^{-\mathrm{T}}\mathcal{F}_k$ by Proposition 2.2. But $\mathcal{F}_k$ is fixed by $\mathrm{S}^{-\mathrm{T}}$ since it is lower-triangular with 1s on its diagonal, so $\mathrm{R}^m\mathrm{J}_p^{-1}\mathcal{F}_k = \mathrm{S}\mathcal{F}_k$ for $0 \le k \le m-1$. Therefore, the last m columns of $\mathrm{F}\mathcal{S}\mathrm{F}^{\mathrm{T}}$ are the last m columns of S, and this concludes the proof. □

We remind the reader that, bearing in mind the points in the footnotes, all of the above results hold for all $n+1 \ge 4$. Thus, we arrive at the following:

THEOREM 3.6. *$\mathcal{S} + \mathcal{S}^{\mathrm{T}}$ has the same signature as the diagonal matrix* $\mathrm{diag}\,((-1)^{n+1}p(\pi_0), \ldots, (-1)^{n+1}p(\pi_n))$.

PROOF. By Proposition 3.5, $\mathrm{S} + \mathrm{S}^{\mathrm{T}}$ and $\mathcal{S} + \mathcal{S}^{\mathrm{T}}$ are congruent via F^{T}. Since real symmetric matrices are diagonalizable, congruence implies that they have equal rank and signature, and thus, the theorem follows by Propositions 3.4 and 2.3. □

To conclude, we state what this means in terms of solutions to the tt^*-Toda equation (1.1). It was shown in [9, 10, 11, 15] that solutions $w_i : \mathbb{C}^* \to \mathbb{R}$ are in one-to-one correspondence with real numbers γ_i satisfying $\gamma_i - \gamma_{i-1} \ge -2$ for all i, where $2w_i(z) \sim \gamma_i \log|z|$ as $|z| \to 0$. When $n+1 = 2m$, the corresponding eigenvalues of R are $exp\big(\pm\frac{i\pi}{n+1}(\gamma_j + 2j + 1)\big)$, $0 \le j \le m-1$, with

$$0 \le \tfrac{\pi}{2m}(\gamma_0 + 1) \le \tfrac{\pi}{2m}(\gamma_1 + 3) \le \cdots \le \tfrac{\pi}{2m}(\gamma_{m-1} + 2m - 1) \le \pi\,.$$

The condition $\mathcal{S} + \mathcal{S}^{\mathrm{T}} > 0$ means that these points must interlace with the $(n+1)^{\mathrm{th}}$ roots of unity, implying that:

$$0 < \gamma_0 + 1 < 2 < \gamma_1 + 3 < 4 < \cdots < 2m - 2 < \gamma_{m-1} + 2m - 1 < 2m\,,$$

and this means that $|\gamma_j| < 1$ for $j = 0, \ldots, m-1$.

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