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A Note on the Shuffle Variant of Jeśmanowicz' Conjecture

Dedicated to Kálmán Györy on the occasion of his 75th birthday.

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Abstract. Let (a, b, c) be a primitive Pythagorean triple. In 1956, Jeśmanowicz conjectured that the equation $a^x + b^y = c^z$ has the unique solution $(x, y, z) = (2, 2, 2)$ in positive integers. In 2010 Miyazaki proposed a similar problem. He conjectured that if (a, b, c) is again a primitive Pythagorean triple with b even, then the equation $c^x + b^y = a^z$ with x, y and z positive integers has the unique solution $(x, y, z) = (1, 1, 2)$ if $c = b + 1$ and no solutions if $c > b + 1$. He also proved that his conjecture is true if $c \equiv 1 \pmod{b}$. We extend Miyazaki's result to the case $c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}$.

1. Introduction

Suppose that a, b and c are known positive integers, and consider the exponential diophantine equation

$$a^x + b^y = c^z\,, \tag{1}$$

with indeterminates x, y and $z \in \mathbf{Z}_{>0}$. The application of Baker's theorem about effective lower bounds on linear forms of logarithms led to many exciting results concerning such equations (see for example [ST86]). The triple of positive integers (a, b, c) is called a Pythagorean triple, if

$$a^2 + b^2 = c^2.$$

Also, (a, b, c) is called a *primitive Pythagorean triple*, if a, b and c are co-prime.

The study of equation (1) with Pythagorean triple (a, b, c) has a long history. In 1955, Sierpiński proved that for the smallest and most famous Pythagorean triple $(a, b, c) = (3, 4, 5)$, the corresponding equation (1) has the unique solution $(x, y, z) = (2, 2, 2)$ (see [Sie56]). Similar results were given by Jeśmanowicz in 1956. He showed that if

$$(a, b, c) \in \{(5, 12, 13), (7, 24, 25), (9, 40, 41), (11, 60, 61)\}\,,$$

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then the only solution of (1) is again $(x, y, z) = (2, 2, 2)$. Based on his results he proposed the following conjecture (also known as *Jeśmanowicz's conjecture*).

CONJECTURE 1. *Let* (a, b, c) *be a primitive Pythagorean triple such that* $a^2+b^2 = c^2$. *Then the only solution of* (1) *is* $(x, y, z) = (2, 2, 2)$.

Conjecture 1 and its generalizations have received a great deal of attention over the years, however the problem in its general form is still open. It is well known that for any primitive Pythagorean triple (a, b, c), we can write

$$a = m^2 - n^2 , \quad b = 2mn , \quad c = m^2 + n^2 , \tag{2}$$

where m and n are positive co-prime integers of different parities with $m > n$. In 1959, Lu [Lu59], and in 1965 Dem'janenko [Dem65] proved Conjecture 1 for

$$n = 1 ; \quad (a, b, c) = (m^2 - 1,\ 2m,\ m^2 + 1)$$

and

$$n = m - 1 ; \quad (a, b, c) = (2m - 1,\ 2m(m - 1),\ 2m^2 - 2m + 1)$$

respectively. Since 1990 a lot of progress has been made towards the proof of Conjecture 1. In 1993, Takakuwa and Asaeda, and Takakuwa (See [TA93a], [TA93b], [Tak93]) proved Conjecture 1 for various infinite families of triples (a, b, c). In several papers between 1995 and 2009 Le ([Le95], [Le96], [Le09]) applied the theory of linear forms in logarithms to give quantitative results, and prove Conjecture 1 for many triples. In 1994, Terai [Ter94] introduced a generalization of Conjecture 1 (known as Terai's conjecture). In the following years he proved it for several special cases (see for example [Ter95], [Ter96], [TT97]). In the last few years, Miyazaki made many important contributions to this field. He proved both Conjecture 1 and Terai's conjecture for various infinite families of triples (see for example [Miy09], [Miy11b]). A comprehensive collection of classical and recent results on Jeśmanovicz' conjecture, and its generalizations can be found in [Miy12].

For any positive integer N, denote by $\mathrm{rad}(N)$ the radical of N (i.e. the product of the distinct prime divisors of N), and $\mathrm{ord}_2(N)$ the 2-order of N (i.e. the largest non-negative integer k, such that $2^k | N$). In their recent papers, Miyazaki [Miy13] and Miyazaki, Yuan and Wu [MYW14] prove (among others) the following theorems.

THEOREM A. *If* $c \equiv 1 \pmod{b}$, *then Conjecture* 1 *is true.*

THEOREM B. *Let* b_0 *be a divisor of* b, *such that* b_0 *is divisible by* $\mathrm{rad}(b)$. *Suppose that Conjecture* 1 *is true for*

$$c \equiv 1 \pmod{b_0} .$$

Then Conjecture 1 *is true for all* $c \equiv 1 \pmod{b_0/2}$.

THEOREM C. *If* $c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}}$, *then Conjecture* 1 *is true.*

Note that here b is always even thus Theorem C is an improvement of Theorem A. It was noted by Miyazaki in [Miy11a] that, if (a, b, c) is a primitive Pythagorean triple and $c = b + 1$, then

$$c + b = a^2 .$$

From this, he proposed the following problem. Let (a, b, c) be a given primitive Pythagorean triple such that $a^2 + b^2 = c^2$, and consider the equation

$$c^x + b^y = a^z \tag{3}$$

with indeterminates x, y and $z \in \mathbf{Z}_{>0}$.

CONJECTURE 2. *With the above conditions, equation* (3) *has the only solution* $(x, y, z) = (1, 1, 2)$ *if* $c = b + 1$. *If* $c > b + 1$ *then* (3) *has no solution.*

This is referred to as the *shuffle* variant of Jeśmanovicz' problem. In [Miy11a], Miyazaki proved that Conjecture 2 is true if $c \equiv 1 \pmod b$. This result is stated as the following lemma.

LEMMA 1. *If* $c \equiv 1 \pmod b$, *then Conjecture* 2 *is true.*

In June 2014, during a visit to Hungary, Miyazaki proposed the following problem. Is it possible to give a generalization of Lemma 1, similar to the way Theorem C generalizes Theorem A? The current paper gives a positive answer to this question. Our main results are the following.

THEOREM 1. *Let* b_0 *be a divisor of* b, *such that* b_0 *is divisible by* $\mathrm{rad}(b)$. *Suppose that Conjecture* 2 *is true for all Pythagorean triples* (a, b, c) *with*

$$c \equiv 1 \pmod{b_0} . \tag{4}$$

Then Conjecture 2 *is true for all Pythagorean triples* (a, b, c) *with*

$$c \equiv 1 \pmod{b_0/2} . \tag{5}$$

THEOREM 2. *Conjecture* 2 *is true for all Pythagorean triples* (a, b, c) *with*

$$c \equiv 1 \pmod{b/2^{\mathrm{ord}_2(b)}} .$$

Combining Lemma 1 and Theorem 1, it is easy to verify Theorem 2. We will give a proof of Theorem 1 in sections 2 and 3. In the last section, we will report numerical results concerning (3) about cases, that are not covered by Lemma 1 and Theorem 2, giving some further evidence for Conjecture 2.

2. Preliminaries and auxiliary results

By (2), we can rewrite (3) into the form

$$(m^2 + n^2)^x + (2mn)^y = (m^2 - n^2)^z , \tag{6}$$

where m and n are given co-prime positive integers of different parities with $m > n$, and x, y and z are unknown positive integers.

Our proof of Theorem 1 will closely follow the work of Miyazaki, Yuan and Wu in [MYW14]. We start with several auxiliary results and general observations. In the proof, the parities of the exponents x, y and z will play a crucial role. Thus first we give some preliminary remarks about the exponents. The following notation was previously established by Miyazaki in [Miy13]. By Lemma 1, we may suppose that in (3) $c \neq b+1$ and $n \neq 1$. Define integers α, β and e with $\alpha \geq 1$, $\beta \geq 2$ and $e = \pm 1$ and odd positive integers i and j as follows:

$$\begin{array}{lll} m = 2^{\alpha} i\,, & n = 2^{\beta} j + e & \text{if } m \text{ is even}\,, \\ m = 2^{\beta} j + e\,, & n = 2^{\alpha} i & \text{if } m \text{ is odd}\,. \end{array} \tag{7}$$

Now, assume that Conjecture 2 holds with (4), and suppose that it does not hold for (5) (or in other words (3) has a solution with (5)). We will show that this will result in a contradiction. Again it is clear that both b and b_0 are even. By (5), we have $c \equiv 1 \pmod{b_0/2}$ that is $c = 1 + t \cdot b_0/2$ for some positive integer t. Since $b_0/2$ is a divisor of $b/2 = mn$, we can write

$$b_0/2 = m_0 n_0\,,$$

where $\gcd(m_0, n_0) = 1$, $m_0|m$ and $n_0|n$. Moreover, m_0 and n_0 are uniquely determined. Thus we have

$$m^2 + n^2 = 1 + m_0 n_0 t\,. \tag{8}$$

If $2||b_0$, then $b_0/2$ is odd. However, since $c = m^2 + n^2$ is odd, we have that t is even. Thus we have $c = 1 + (t/2)b_0$, which means that $c \equiv 1 \pmod{b_0}$, for which Conjecture 2 is true by assumption. Thus, in what follows, we can assume that $4|b_0$. We may also assume that t is odd, else we have again $c = 1 + (t/2)b_0$, for which Conjecture 2 is true. Then (8) implies that m_0 or n_0 is even, and

$$\operatorname{rad}(m_0) = \operatorname{rad}(m)\,, \quad \operatorname{rad}(n_0) = \operatorname{rad}(n)\,.$$

From (8), we have that

$$m^2 \equiv 1 \pmod{n_0}\,, \quad n^2 \equiv 1 \pmod{m_0}\,. \tag{9}$$

Next, we present some lemmas, which will be used in the proof.

LEMMA 2. *With the above notation, we have*

$$c - 1 \equiv 0 \pmod{2^{\min(2\alpha, \beta+1)}} \tag{10}$$

and

$$\begin{array}{lll} a - 1 \equiv 0 \pmod{2^{\min(2\alpha, \beta+1)}} & \textit{if } m \textit{ is odd}\,, \\ a + 1 \equiv 0 \pmod{2^{\min(2\alpha, \beta+1)}} & \textit{if } m \textit{ is even}\,. \end{array} \tag{11}$$

PROOF. This lemma can be proven similarly to Lemma 4 in [MYW14], by simply substituting (7) into (6). □

LEMMA 3. *With the above notations, we have* $2\alpha \neq \beta + 1$. *Moreover, we have* $\alpha \geq \beta + 1$.

PROOF. By Lemma 2, and (8), we have

$$\min(2\alpha, \beta + 1) \leq \mathrm{ord}_2(c-1) = \mathrm{ord}_2(m_0 n_0 t) \leq \mathrm{ord}_2(mn) = \alpha .$$

This implies our lemma. □

LEMMA 4. *Let* $d > 1$ *and let* u, v *be non-zero co-prime integers. Let* p *be a prime factor of* $u - v$. *If* p *is odd, or* $p = 2$ *and* 4 *divides* $u - v$, *then*

$$\mathrm{ord}_p(u^d - v^d) = \mathrm{ord}_p(u - v) + \mathrm{ord}_p(d) .$$

PROOF. See for example on p. 11 in [Rib94]. □

The next lemma is similar to Lemma 3.1 in [Miy11a]. However, we prove it in detail, because we want to emphasize a somewhat different conclusion. We will use this alternate statement to avoid Baker's method during the proof of Theorem 1.

LEMMA 5. *Assume that* $\alpha > 1$, $\alpha \neq \beta$ *and* $2\alpha \neq \beta + 1$. *Let* (x, y, z) *be a solution of* (6). *Then both* x *and* z *are even.*

PROOF. Set

$$M = \begin{cases} 4, & \text{if } m \text{ is even}, \\ m_0, & \text{if } m \text{ is odd}. \end{cases} \tag{12}$$

It is clear that $M \geq 3$. Taking (6) modulo M and using (9), we see that

$$1 \equiv (-1)^z \pmod{M} .$$

Since $M \geq 3$, we conclude that z is even. Now, assume that x is odd and m is even. Then from (6) we have

$$(2mn)^y \equiv -m^2(zn^{2z-2} + xn^{2x-2}) + n^{2z} - n^{2x} \pmod{2^{2\alpha+1}} .$$

Write

$$A = -m^2(zn^{2z-2} + xn^{2x-2}), \quad B = n^{2z} - n^{2x} .$$

Since x is odd, $zn^{2z-2} + xn^{2x-2}$ is odd, thus by Lemma 4

$$\mathrm{ord}_2(A) = \mathrm{ord}_2(m^2) = 2\alpha ,$$

$$\mathrm{ord}_2(B) = \mathrm{ord}_2(n^{2|x-z|} - 1) = \mathrm{ord}_2(n^2 - 1) = \beta + 1 .$$

Since $\mathrm{ord}_2((2mn)^y) = (\alpha+1)y$, and $2\alpha \neq \beta + 1$, we have

$$(\alpha+1)y = \begin{cases} 2\alpha & \text{if } \ 2\alpha < \beta + 1 \\ \beta + 1 & \text{if } \ 2\alpha > \beta + 1 \end{cases}$$

which means that either $\alpha = 1$ and $y = 1$ or $\alpha = \beta$ and $y = 1$ holds. The case, where m is odd can be treated similarly. □

LEMMA 6. *Assume that* $2\alpha \neq \beta + 1$. *Let* (x, y, z) *be a solution of* (6). *If* $y > 1$ *and* x *and* z *are even, then* $X \equiv Z \pmod 2$, *where* $x = 2X$ *and* $z = 2Z$ *for some* $X, Z \geq 1$.

PROOF. See Lemma 3.1 and Lemma 3.2 in [Miy11a]. □

3. Proof of Theorem 1

We are now ready to prove Theorem 1. It follows from Lemmas 3 and 5 that both x and z are even. So, we can write $x = 2X$, $z = 2Z$ with integers $X, Z > 1$, and

$$(2mn)^y = D \cdot E$$

with

$$D = (m^2 - n^2)^Z + (m^2 + n^2)^X, \quad E = (m^2 - n^2)^Z - (m^2 + n^2)^X .$$

Now, if $y = 1$, then

$$(m-n)^2 = m^2 + n^2 - 2mn \leq (m^2 + n^2)^X - 2mn = \frac{D-E}{2} - DE \leq 0,$$

which is a contradiction, since $m \neq n$. Thus, in what follows, we can assume that $y > 1$ holds.

By Lemma 6, we have

$$X \equiv Z \pmod 2 .$$

Suppose that X and Z are both even. Then the congruences

$$D \equiv 2 \pmod 4, \quad D \equiv 2 \pmod{m_0}, \quad D \equiv 2 \pmod{n_0}$$

are obtained by (9). These imply that $D/2$ is odd, and co-prime to $m_0 n_0$, thus to mn. Therefore we get $D = 2$ which is impossible. Hence both X and Z are odd. Then we compute

$$(D, E) \equiv \begin{cases} (0, 2) \pmod 4 & \text{if } \ m \text{ is even}, \\ (2, 0) \pmod 4 & \text{if } \ m \text{ is odd}, \end{cases}$$

and

$$D \equiv 2 \pmod{n_0}, \quad E \equiv -2 \pmod{n_0}$$

which yield the equality

$$(D, E) = \begin{cases} (2^{y-1}m^y, 2n^y) & \text{if } m \text{ is even}, \\ (2m^y, 2^{y-1}n^y) & \text{if } m \text{ is odd}. \end{cases}$$

Now, we discuss the two cases separately.

The case that m is even; If m is even, then we have

$$\frac{D-E}{2} = 2^{y-2}m^y - n^y = (1 + m_0n_0t)^X .$$

Reducing both sides modulo m_0, we get

$$n^y \equiv -1 \pmod{m_0} .$$

If y is even, then

$$-1 \equiv n^y \equiv \left(n^2\right)^{y/2} \equiv 1 \pmod{m_0} ,$$

which is a contradiction, if $m_0 \geq 3$. Thus, either y is odd, or $m_0 = 2$. In both cases we have $n \equiv -1 \pmod{m_0}$. However, using this we get

$$\begin{aligned} \mathrm{ord}_2(m_0) &\leq \mathrm{ord}_2(n+1) < \mathrm{ord}_2(n^2-1) \\ &= \mathrm{ord}_2(-m^2 + m_0n_0t) = \mathrm{ord}_2(m_0) + \mathrm{ord}_2(-m^2/m_0 + n_0t) = \mathrm{ord}_2(m_0) , \end{aligned}$$

which is a contradiction. Thus, neither of the above cases are possible.

The case that m is odd; Proceeding in a similar way, we get

$$m^y - 2^{y-2}n^y = (1 + m_0n_0t)^X ,$$

which yields

$$m^y \equiv 1 \pmod{n_0} .$$

Suppose now that y is odd. Then $m \equiv 1 \pmod{n_0}$. This yields a contradiction as in the previous case by estimating $\mathrm{ord}_2(n_0)$. Thus, we now have that m is odd, and $y = 2Y$, with some integer Y. We complete the proof of Theorem 1 by proving the following proposition.

PROPOSITION 1. *Let m and n be co-prime positive integers with n even, m odd and $m > n$. Then the system of equations*

$$\begin{cases} (m^2-n^2)^Z + (m^2+n^2)^X = 2m^{2Y} , \\ (m^2-n^2)^Z - (m^2+n^2)^X = 2^{2Y-1}n^{2Y} \end{cases} \tag{13}$$

has no solution in positive integers X, Y and Z.

PROOF. Note that the equations are equivalent to

$$\begin{cases}(m^2-n^2)^Z = m^{2Y} + 2^{2Y-2}n^{2Y}\,, \\ (m^2+n^2)^X = m^{2Y} - 2^{2Y-2}n^{2Y}\,, \end{cases} \tag{14}$$

simultaneously. Assume that there are positive integer solutions X, Y and Z. First we shall show

$$1 < X < Y\,.$$

Indeed, the inequality $X < Y$ is obtained by

$$m^{2X} < (m^2+n^2)^X = m^{2Y} - 2^{2Y-2}n^{2Y} < m^{2Y}\,.$$

Further, if $X = 1$, then $Y \geq 2$ and

$$m^2+n^2 = m^{2Y} - 2^{2Y-2}n^{2Y} \geq m^Y + 2^{Y-1}n^Y \geq m^2 + 2n^2$$

that is impossible. Next we claim that

$$n \equiv 0 \pmod 4\,.$$

If not, then we have $\pm n^2 \equiv 4 \pmod 8$ and

$$5^X \equiv 5^Z \equiv 1 + 2^{2Y-2}4^Y = 1 + 4^{2Y-1} \equiv 1 \pmod 8\,.$$

Therefore both X and Z are even. Multiplying the left and right hand sides of (13) respectively, we get a solution of the equation $S^4 - T^4 = U^2$. But it is well-known that this has no non-trivial solutions, and the congruence $n \equiv 0 \pmod 4$ has been shown. Now, from the second equation of (14), we get

$$(m^2+n^2)^X = m^{2Y} - 2^{2Y-2}n^{2Y} = (m^Y + 2^{Y-1}n^Y)(m^Y - 2^{Y-1}n^Y)\,.$$

Since $\gcd(m^Y + 2^{Y-1}n^Y, m^Y - 2^{Y-1}n^Y) = 1$, there are co-prime positive integers s, t satisfying

$$st = m^2+n^2\,, \quad s^X = m^Y + 2^{Y-1}n^Y\,, \quad t^X = m^Y - 2^{Y-1}n^Y\,.$$

Note that $X > 1$ and $s - t \equiv 0 \pmod 4$. Thus we can apply Lemma 4 so that

$$\mathrm{ord}_2(s-t) + \mathrm{ord}_2(X) = \mathrm{ord}_2((2n)^Y) = (1 + \mathrm{ord}_2(n))Y \geq 3Y\,,$$

by $n \equiv 0 \pmod 4$, while we can confirm that $\mathrm{ord}_2(X) < Y$, using $X < Y < 2^Y$. Then we get $\mathrm{ord}_2(s-t) > 2Y$, in particular,

$$2^{2Y} \leq s - t < st = m^2 + n^2\,.$$

On the other hand, since $n^2 \equiv -m^2 \pmod{m^2+n^2}$, we have from (14) again,

$$0 \equiv m^{2Y} - 2^{2Y-2}n^{2Y} \equiv (1 \pm 2^{2Y-2})m^{2Y} \pmod{m^2+n^2}\,.$$

Then it follows from $\gcd(m, m^2+n^2) = 1$ that $2^{2Y-2} \pm 1$ is divisible by m^2+n^2. Note that $2^{2Y-2} - 1 > 0$, since $Y > X \geq 2$. Hence

$$m^2 + n^2 \leq 2^{2Y-2} \pm 1 < 2^{2Y},$$

which is inconsistent with the inequality shown above. This completes the proof of Proposition 1, and thus the proof of Theorem 1. □

4. Examples

In this section we show how to utilize Lemma 1 and Theorem 2 to prove Conjecture 2 for a finite set of triples. For this purpose we will consider all primitive Pythagorean triples (a, b, c) for which

$$a^2 + b^2 = c^2 \tag{15}$$

and

$$5 \leq c \leq 100, \tag{16}$$

and prove the following proposition.

PROPOSITION 2. *If (a, b, c) is a primitive Pythagorean triple with $a^2 + b^2 = c^2$ and $5 \leq c \leq 100$, then Conjecture 2 is true.*

PROOF. Altogether there are sixteen triples with (15) and (16), ten of these are covered by either Lemma 1 or Theorem 2. The remaining six cases are

$$(a, b, c) \in \{(21, 20, 29), (45, 28, 53), (33, 56, 65), (39, 80, 89), (77, 36, 85), (65, 72, 97)\}.$$

Since the bases are thus fixed in (3), it is possible to use the classical theory of S-unit equations. However we will apply here a more recent approach based on a paper of Bertók and Hajdu [BH15]. In this paper the authors use basic search for small solutions and modulo arithmetic to give very good upper bounds for the size of the solutions, and also provide a program code written in SAGE to do the calculations. Consider first the triple $(a, b, c) = (21, 20, 29)$. This gives us the equation

$$29^x + 20^y = 21^z, \tag{17}$$

where x, y and z are positive unknown integers. Since $(x, y, z) = (0, 1, 1)$ is a solution of (17), it is impossible to find a suitable integer M, such that the congruence

$$29^x + 20^y \equiv 21^z \pmod{M}$$

is not solvable. However using the program of Bertók and Hajdu we get that if we choose

$$M = 3^2 \cdot 7^2 \cdot 13 \cdot 17 \cdot 19 \cdot 37 \cdot 73 \cdot 97 \cdot 109 \cdot 163 \cdot 193 \cdot 257 \cdot 433 \cdot 487 \cdot 577 \cdot 769,$$

then the congruence

$$29^x + 20^y \equiv 21^2 \cdot 21^{z_0} \pmod{M}$$

is not solvable for any non-negative integers x, y and z_0. Thus in (17) we have that $z \le 1$, that is

$$29^x + 20^y = 21\,,$$

which has no solutions in positive integers (and the obvious solution $(x, y, z) = (0, 1, 1)$ in non-negative integers). The remaining five cases do not possess trivial solution, and can be dealt with similarly. We omit the details, and only list the results in the following table.

(a, b, c)	Modulus	Result
$(45, 28, 53)$	$13 \cdot 19 \cdot 37 \cdot 73 \cdot 109$	No solutions
$(33, 56, 65)$	$17 \cdot 19 \cdot 37 \cdot 73$	No solutions
$(39, 80, 89)$	$3^2 \cdot 7 \cdot 13^2$	No solutions
$(77, 36, 85)$	$13 \cdot 19 \cdot 37 \cdot 73$	No solutions
$(65, 72, 97)$	$17 \cdot 19 \cdot 37 \cdot 73 \cdot 577$	No solutions

Thus we covered all the six cases, proving Proposition 2. □

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