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On Zeta Elements and Refined Abelian Stark Conjectures

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Abstract. We apply recent methods of Burns, Kurihara and Sano in [5] to study connections between the values at $s = 0$ of the higher derivatives of abelian L-functions of number fields and the higher Fitting ideals of the canonical Selmer groups of $\mathbf{G}_m$. Whereas Burns, Kurihara and Sano apply these methods to the setting of the 'Rubin-Stark conjecture', we study the 'evaluators' defined in a more general setting by Emmons and Popescu in [7] and by Vallieres in [14].

This allows us to conjecture that the ideals formed from the images of the evaluators can be described precisely in terms of the higher Fitting ideals of the canonical Selmer groups of $\mathbf{G}_m$. Moreover, we are able to prove that this conjecture follows from the equivariant Tamagawa number conjecture.

1. Introduction

In the 1970s and 80s Stark wrote a seminal series of four papers [11] in which he formulated conjectures concerning the values at $s = 0$ of the Artin L-functions attached to finite Galois extensions of number fields. Stark's work was reinterpreted and extended by Tate in [12] and a large number of related conjectures were subsequently formulated and studied by many different authors.

Of particular interest to us is the so-called 'Rubin-Stark conjecture', formulated by Rubin in [10, Conj. B']. This is an 'abelian rank r' Stark's conjecture in that one considers S-truncated L-functions for characters that factor through some abelian extension of number fields F/E, where S is a finite set of places of E containing r places which split completely in F so that the truncated L-functions each vanish to order at least r at $s = 0$. The conjecture then asserts the existence of a canonical 'Rubin-Stark element' that acts as an 'evaluator' for the values at $s = 0$ of the r-th derivatives of the truncated L-functions.

In later work Emmons and Popescu [7] considered the more general situation in which one assumes that the truncated L-functions vanish to order at least r at $s = 0$ but not that S contains a prescribed subset of splitting places. In this setting they defined a natural generalisation of the Rubin-Stark 'evaluator' and conjectured that it satisfies precise integrality conditions.

A little earlier Burns [1] had showed that a special case of the very general 'equivariant Tamagawa number conjecture' (from [3]) implied the validity of the Rubin-Stark conjecture,

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as well as that of a host of related conjectures due to Gross, to Tate and to Popescu among others. The approach used in [1] was then adapted by Vallières in [14] to show that the same case of the equivariant Tamagawa number conjecture also implied the validity of a refined version of the conjecture of Emmons and Popescu.

In some very recent work Burns, Kurihara and Sano have significantly improved upon the proof in [1] and provided new methods for studying Stark type conjectures (see [5]).

In this paper we will apply these new methods to study the evaluators defined by Popescu, Emmons and Vallières and, in so doing, we will show that the proof of the main result of Vallières in [14] can be both significantly simplified and improved (for details see §3). This allows us to formulate a stronger conjecture.

In particular, while the conjecture of Emmons and Popescu asserts that the images under certain homomorphisms of their evaluator form an ideal of $\mathbf{Z}[\mathrm{Gal}(F/E)]$, in this paper we are led to conjecture that this ideal can be described precisely in terms of the r-th Fitting ideal over $\mathbf{Z}[\mathrm{Gal}(F/E)]$ of the canonical Selmer groups that one can associate to the multiplicative group $\mathbf{G}_m$ over F. This is formulated in Conjecture 1.8.

In §4 we prove the main result of this paper, Theorem 1.9, which states that our conjectural description of the ideal follows from the validity of the equivariant Tamagawa number conjecture and thereby deduce that it is unconditionally true in some important cases.

1.1. Notations and definitions

1.1.1. Galois modules. We let K/k be a finite abelian extension of number fields with Galois group G. We fix a finite set S of places of k such that S contains all infinite places and all places that ramify in K/k. For any place v of k and place w of K lying above v, write G_v for the decomposition subgroup of w in G. (Note that this is independent of our choice of w lying above v since G is abelian.) Let $\mathcal{I}_{G_v}$ be the augmentation ideal of G_v.

We write S_K for the set of places of K lying above those in S and we let $Y_{K,S}$ be the free abelian group on the set S_K. We let $X_{K,S}$ be the kernel of the homomorphism from $Y_{K,S}$ to $\mathbf{Z}$ that sends each place in S_K to one. We write $\mathcal{O}_{K,S}$ for the ring of S_K-integers and $\mathcal{O}_{K,S}^{\times}$ for the group of S_K-units.

We fix a non-empty finite auxiliary set of places T with $S \bigcap T = \emptyset$ and such that $\mathcal{O}_{K,S,T}^{\times}$ is $\mathbf{Z}$-torsion-free. The (S_K, T_K)-units are defined by

$$\mathcal{O}_{K,S,T}^{\times} = \left\{ x \in \mathcal{O}_{K,S}^{\times} : x \equiv 1 \mod w \text{ for all } w \in T_K \right\}.$$

We let $\mathrm{Cl}_S(K)$ denote the S_K-class group and $\mathrm{Cl}_{S,T}(K)$ denote the (S_K, T_K)-class group, which is defined by

$$\mathrm{Cl}_{S,T}(K) := \frac{\left\{\text{fractional ideals of } \mathcal{O}_{K,S} \text{ prime to } T_K\right\}}{\left\{x \cdot \mathcal{O}_{K,S} : x \equiv 1 \mod w \text{ for all } w \in T_K\right\}}$$

All of these groups are stable under the action of G, so form $\mathbf{Z}[G]$-modules.

If M is a $\mathbf{Z}[G]$-module, we write $\mathbf{R}M$ for the $\mathbf{R}[G]$-module $\mathbf{R} \otimes_{\mathbf{Z}} M$, and $\mathbf{C}M$ for the $\mathbf{C}[G]$-module $\mathbf{C} \otimes_{\mathbf{Z}} M$. We let $M^{\vee} := \mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$ denote the Pontryagin dual of M and we let $M^* := \mathrm{Hom}_{\mathbf{Z}[G]}(M, \mathbf{Z}[G])$.

If A is a set let $|A|$ denote the cardinality of A and $\wp_r(A)$ denote the set of subsets of A of cardinality r. Let $\wp_r(m)$ denote the set of r-tuples $(n_1, \ldots, n_r)$ of integers between 1 and m that satisfy $n_1 < \cdots < n_r$.

1.1.2. Characters and L-functions. As G is abelian, all irreducible representations of G are one dimensional. Hence the set of characters $\hat{G} := \mathrm{Hom}(G, \mathbf{C}^\times)$ forms a group under multiplication. We let χ_1 represent the trivial character. For each $\chi \in \hat{G}$ define

$$e_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g \, .$$

For any $\mathbf{Z}[G]$-module M and character $\chi \in \hat{G}$, we write M_χ for the χ component of $\mathbf{C}M$. This is the submodule of $\mathbf{C}M$ defined by $M_\chi := e_\chi \mathbf{C}M$.

For each character χ of G we have the S-truncated Artin L-function attached to χ given by the product formula

$$L_{K/k,S}(\chi, s) = \prod_{\wp \notin S} \left(1 - \chi(\sigma_\wp)\mathbf{N}\wp^{-s}\right)^{-1} \in \mathbf{C}[G]$$

where $\sigma_\wp$ is the Frobenius automorphism of $\wp$ in G, $s \in \mathbf{C}$ and $\mathbf{N}\wp$ is the size of the residue field, (recall that we are excluding factors for infinite and ramifying primes).

We also modify this L-function to take into account our auxiliary set of places T. Let

$$\delta_T(s) = \prod_{\wp \in T} \left(1 - \sigma_\wp^{-1}\mathbf{N}(\wp)^{1-s}\right)$$

Then the S-truncated T-modified L-function is defined as

$$L_{K/k,S,T}(\chi, s) = \bar{\chi}(\delta_T(s)) \cdot L_{K/k,S}(\chi, s) \, .$$

In the situation that we study all of these L-functions will vanish at $s = 0$. Let $L^*_{K/k,S,T}(\chi, 0)$ denote the first non-vanishing Taylor coefficient of $L_{K/k,S,T}(\chi, s)$ at $s = 0$.

We can gather the L-functions for different characters together to obtain the S-truncated T-modified equivariant L-function given by

$$\theta_{K/k,S,T}(s) := \sum_{\chi \in \hat{G}} L_{K/k,S,T}(\chi, s) \cdot e_{\bar{\chi}} \, .$$

We also define

$$\theta^*_{K/k,S,T} := \sum_{\chi \in \hat{G}} L^*_{K/k,S,T}(\chi, 0) \cdot e_{\bar{\chi}} \, .$$

1.1.3. Regulator maps. The Dirichlet logarithm induces an isomorphism of $\mathbf{C}[G]$ modules

$$\lambda_{K,S} : \mathbf{C}\mathcal{O}_{K,S,T}^{\times} \longrightarrow \mathbf{C}X_{K,S}$$

such that at each element $u \in \mathcal{O}_{K,S,T}^{\times}$ we have

$$\lambda_{K,S}(u) = -\sum_{w \in S_K} \log|u|_w \cdot w\,,$$

where $|\cdot|_w$ is the normalised absolute value at w.

The map $\lambda_{K,S}$ also induces an isomorphism on the t-th exterior power, for any positive integer t, which we will also denote by $\lambda_{K,S}$

$$\lambda_{K,S} : \bigwedge_{\mathbf{C}[G]}^{t} \mathbf{C}\mathcal{O}_{K,S,T}^{\times} \xrightarrow{\sim} \bigwedge_{\mathbf{C}[G]}^{t} \mathbf{C}X_{K,S}\,.$$

When K and S are clear from the context we will just write λ.

1.2. Order of vanishing of L-functions. The order of vanishing of S-truncated L-functions is well understood and given by the following lemma. (See, for example [13, Chap. I, Prop. 3.4] for proof.)

LEMMA 1.1. *Let K/k be an abelian extension of number fields and S a finite set of primes of k containing all infinite primes. We let $r_S(\chi) := \mathrm{ord}_{s=0} L_{K/k,S}(\chi, s)$ Then*

$$r_S(\chi) = \dim_{\mathbf{C}}\big(\mathbf{C}\mathcal{O}_{K,S}^{\times} \cdot e_\chi\big) = \begin{cases} |S| - 1, & \text{if}\chi = \chi_1 \\ \big|\{v \in S : G_v \subseteq \mathrm{Ker}(\chi)\}\big| & \text{if}\chi \neq \chi_1\,. \end{cases}$$

Let $r := \min\{r_S(\chi) : \chi \in \hat{G}\}$ so that r is the minimal order of vanishing at $s = 0$ of the S-truncated L-function for any character χ of G. Then

$$\hat{G}_{r,S} := \{\chi \in \hat{G} : r_S(\chi) = r\}$$

is the set of characters whose S-truncated L-functions vanish at $s = 0$ to this minimal order. It follows from Lemma 1.1 that

$$e_{r,S} := \sum_{\chi \in \hat{G}_{r,S}} e_\chi$$

is a rational idempotent.

If $\chi_1 \in \hat{G}_{r,S}$ it follows that $|S| = r + 1$. In this case Emmons and Popescu have shown in [7] that S must contain at least r places that split completely in K. Thus we may pick r

such places $v_1, \ldots, v_r$. Then for any $\chi \in \hat{G}_{r,S}$ we define the set V_χ as follows:

$$V_\chi := \begin{cases} \{v \in S : v \text{ splits completely in } K^{\mathrm{Ker}(\chi)}/k\} & \text{if} \chi \neq \chi_1 \\ \{v_1, \ldots, v_r\} & \text{if} \chi = \chi_1 . \end{cases}$$

Then we let

$$V_S = \bigcup_{\chi \in \hat{G}_{r,S}} V_\chi \subset S .$$

For each $I \in \wp_r(V_S)$ define

$$\hat{G}_{r,S,I} := \{\chi \in \hat{G}_{r,S} : V_\chi = I\} .$$

So for $|S| \neq r + 1$ we have

$$\hat{G}_{r,S,I} = \{\chi \in \hat{G}_{r,S} : G_v \subseteq \mathrm{Ker}(\chi), \text{ for all } v \in I\}$$

and

$$\hat{G}_{r,S} = \bigcup_{I \in \wp_r(V_S)} \hat{G}_{r,S,I} .$$

We may think of the sets I as defining equivalence classes for the set of characters in $\hat{G}_{r,S}$. We define idempotents corresponding to these sets. For each $I \in \wp_r(V_S)$ let

$$e_I = e_{r,S,I} := \sum_{\chi \in \hat{G}_{r,S,I}} e_\chi .$$

Again, e_I is a rational idempotent.

1.3. Statements of main results. Assume that $S \neq V_S$. We fix an ordering of $S = \{v_0, v_1, \ldots, v_n\}$ such that $V_S = \{v_1, \ldots, v_m\}$. This means that for each $I \in \wp_r(V_S)$ and non-trivial character $\chi \in \hat{G}_{r,S,I}$ we have $G_{v_0} \not\subseteq \mathrm{Ker}(\chi)$. Fix places $w_0, w_1, \ldots, w_n$ lying above $\{v_0, v_1, \ldots, v_n\}$ in K. We let $\bar{I} = \{i_1, i_2, \ldots, i_r\}$ with $i_1 < i_2 < \cdots < i_r$ and so that $I = \{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$.

Following Vallières and Emmons and Popescu (see [14] and [7]) we define the following elements:

DEFINITION 1.2. Let

$$\eta^I_{K/k,S,T} \in \mathbf{C} \bigwedge_{\mathbf{Z}[G]}^{r} \mathcal{O}^\times_{K,S,T} \cdot e_I$$

be the unique element such that

$$\lambda(\eta^I_{K/k,S,T}) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I ,$$

where $w_I = (w_{i_1} - w_0) \wedge (w_{i_2} - w_0) \wedge \cdots \wedge (w_{i_r} - w_0)$.

We sum over all I to obtain

$$\eta_{K/k,S,T} = \sum_{I \in \wp_r(V_S)} \eta^I_{K/k,S,T} \in \mathbf{C} \bigwedge_{\mathbf{Z}[G]}^{r} \mathcal{O}^{\times}_{K,S,T} \cdot e_{r,S}\,.$$

Emmons and Popescu refer to these elements $\eta^I_{K/k,S,T}$ and $\eta_{K/k,S,T}$ as evaluators.

In order to recall the previous conjecture of Emmons and Popescu we define the Rubin Lattice.

DEFINITION 1.3. The Rubin Lattice is defined by

$$\Lambda_{K/k,S,T} := \left\{ u \in \mathbf{Q} \bigwedge_{\mathbf{Z}[G]}^{r} \mathcal{O}^{\times}_{K,S,T} : \Phi(u) \in \mathbf{Z}[G] \text{ for all } \Phi \in \bigwedge_{\mathbf{Z}[G]}^{r} (\mathcal{O}^{\times}_{K,S,T})^{*} \right\},$$

where Φ is regarded as an element of $(\bigwedge^r_{\mathbf{Z}[G]} \mathcal{O}^{\times}_{K,S,T})^*$ as described in §2.1.

The element $\eta_{K/k,S,T}$ is the analogue of the Rubin-Stark element. This leads Emmons and Popescu to make the following conjecture (see [7, Conj. 3.8]).

CONJECTURE 1.4 (Emmons-Popescu). *Let K/k be a finite abelian extension of number fields. Let S be a finite set of places of k containing all infinite places and places that ramify in K/k. Let T be another finite set of places of k such that $S \cap T = \emptyset$ and such that $\mathcal{O}^{\times}_{K,S,T}$ is $\mathbf{Z}$-torsion-free. Assume $S \neq V_S$. Let $\eta_{K/k,S,T} \in \mathbf{C} \bigwedge^r_{\mathbf{Z}[G]} \mathcal{O}^{\times}_{K,S,T} \cdot e_{r,S}$ be defined as above. Then*

$$\eta_{K/k,S,T} \in \Lambda_{K/k,S,T}\,.$$

REMARK 1.5. Suppose S contains precisely r places that split completely in K/k so that the conditions of the Rubin-Stark conjecture are satisfied. Then V_S consists of these r places and therefore there is only one $I \in \wp_r(V_S)$. Then $\eta^I_{K/k,S,T} = \eta_{K/k,S,T}$ is a Rubin-Stark element for K/k and Conjecture 1.4 reduces to the Rubin-Stark conjecture.

REMARK 1.6. The condition $S \neq V_S$ is not trivial. In the classical case, where S contains r places that split completely, this guarantees the existence of a distinguished place v_0 that does not contribute to the vanishing of the L-function that vanishes to order precisely r. In our more general case the existence of such a place is not guaranteed. This place is used to show that the denominators of the rational idempotents e_I do not lead us to lose integrality properties. Without this hypothesis integrality results may not always hold.

In [14] Vallières considers the case where $|S| > r + 1$ and conjectures that

$$\Phi\big(\eta^I_{K/k,S,T}\big) \in \mathbf{Z}[G] \text{ for all } \Phi \in \bigwedge_{\mathbf{Z}[G]}^{r} \big(\mathcal{O}^{\times}_{K,S,T}\big)^{*}. \tag{1}$$

Vallières uses the method developed by Burns in [1] to show that (1) (and therefore also Conjecture 1.4) follows from a special case of the equivariant Tamagawa number conjecture called the leading term conjecture ($LTC(K/k)$).

In §3 we use recent work by Burns, Kurihara and Sano (see [5]) to significantly simplify and improve upon Vallières' proof. The key idea is to express the element $\eta^I_{K/k,S,T}$ as the 'canonical projection' of the zeta element $z_{K/k,S,T}$ arising in a particular formulation of the leading term conjecture. The zeta element interpolates the leading terms at $s = 0$ of the S-truncated T-modified L-functions, $L_{K/k,S,T}(\chi, s)$. (See §2.3 for details.)

In their recent paper, Burns, Kurihara and Sano work under the assumptions of the Rubin-Stark conjecture, so $\eta_{K/k,S,T}$ is a Rubin-Stark element. Their description of $\eta_{K/k,S,T}$ in terms of the zeta element $z_{K/k,S,T}$ allows them to calculate the ideal of $\mathbf{Z}[G]$ generated by $\Phi(\eta_{K/k,S,T})$ as Φ runs over $\bigwedge^r_{\mathbf{Z}[G]}(\mathcal{O}^\times_{K,S,T})^*$ in terms of the r-th Fitting ideal of $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})$, the 'transpose' of a canonical 'S-relative T-trivialised integral dual Selmer group' for the multiplicative group over K. We denote this r-th Fitting ideal by $\mathrm{Fitt}^r_{\mathbf{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))$. (See §2.3 for full details.)

In §4 we investigate the ideal of $\mathbf{Z}[G]$ generated by the elements $\Phi(\eta^I_{K/k,S,T})$ in the more general case, when we don't assume that S contains r split places. We prove the following theorem, which tells us more about the structure of our Fitting ideal.

THEOREM 1.7. *Let K/k be a finite abelian extension of number fields. Let S be a finite set of places of k containing all infinite places and places that ramify in K/k. Let T be another finite set of places of k such that $S \cap T = \emptyset$ and such that $\mathcal{O}^\times_{K,S,T}$ is $\mathbf{Z}$-torsion-free. Assume $S \neq V_S$. Then for each $I \in \wp_r(V_S)$*

$$e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) \subseteq \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big).$$

Here D_I denotes the group generated by the decomposition groups of the places in I and the idempotent e_{D_I} is defined in §4.

This result leads us to make the following conjecture which improves upon (1).

CONJECTURE 1.8. *Assume the conditions of Theorem* 1.7 *hold. Let $\eta^I_{K/k,S,T} \in \mathbf{C}\bigwedge^r_{\mathbf{Z}[G]}\mathcal{O}^\times_{K,S,T} \cdot e_I$ be defined as above. Then we have*

$$\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) = \bigoplus_{I \in \wp_r(V_S)} \left\{\Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]}\big(\mathcal{O}^\times_{K,S,T}\big)^*\right\}.$$

The main evidence that we can give for this conjecture is the following result.

THEOREM 1.9. *Assume that K/k and S satisfy the hypotheses of Conjecture* 1.4. *Then the validity of LTC(K/k) implies the validity of Conjecture* 1.8.

We recall that Burns and Greither have proved in [4] that LTC(K/k) holds away from the prime 2 when K is an abelian extension of $\mathbf{Q}$ (and k is any intermediate field of $K/\mathbf{Q}$) and that Flach proved the same result at the prime 2 in [8].

Given this, the following result is an immediate consequence of Theorem 1.9.

COROLLARY 1.10. *If K is an abelian extension of $\mathbf{Q}$, then Conjecture* 1.8 *is valid for any intermediate field k of $K/\mathbf{Q}$.*

2. Preliminaries

2.1. Exterior powers. When working with exterior powers, we will need to deal with changes of sign arising when we use ordered bases. We will use the following notations (for further details see [14, §6.1]).

Let $\bar{I} = (i_1, \dots, i_{r_1}) \in \wp_{r_1}(m)$ and $\bar{J} = (j_1, \dots, j_{r_2}) \in \wp_{r_2}(m)$. Define

$$(-1)^{\bar{I}+\bar{J}} := (-1)^{i_1+\cdots+i_{r_1}+j_1+\cdots+j_{r_2}} .$$

If in addition we have that $\bar{J} \subseteq \bar{I}$ then let $\tau_{\bar{I},\bar{J}}$ denote the permutation $\bar{I} \mapsto \bar{J} \cdot (\bar{I} \setminus \bar{J})$, where $\cdot$ means concatenation. If t is a positive integer let

$$[t] = (1, 2, \dots, t) .$$

Then

$$\operatorname{sgn}\big(\tau_{[m],\bar{I}}\big) = (-1)^{\bar{I}+[r_1]} .$$

We will also need some constructions for exterior powers of homomorphisms. Suppose M is a $\mathbf{Z}[G]$-module and $f \in M^*$. Then for every positive integer t we can define a $\mathbf{Z}[G]$-module homomorphism

$$\bigwedge_{\mathbf{Z}[G]}^{t} M \longrightarrow \bigwedge_{\mathbf{Z}[G]}^{t-1} M$$

by

$$m_1 \wedge \cdots \wedge m_t \mapsto \sum_{i=1}^{t} (-1)^{i-1} f(m_i) m_1 \wedge \cdots \wedge m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_t .$$

We will still denote this morphism by f.

Furthermore this construction allows us to regard elements of $\bigwedge_{\mathbf{Z}[G]}^{s} M^*$ as elements of $\operatorname{Hom}_{\mathbf{Z}[G]}(\bigwedge_{\mathbf{Z}[G]}^{t} M, \bigwedge_{\mathbf{Z}[G]}^{t-s} M)$ for non negative integers t and s with $t \geq s$. We do this by defining a homomorphism

$$\bigwedge_{\mathbf{Z}[G]}^{s} M^* \longrightarrow \operatorname{Hom}_{\mathbf{Z}[G]}\Big(\bigwedge_{\mathbf{Z}[G]}^{t} M, \bigwedge_{\mathbf{Z}[G]}^{t-s} M\Big)$$

by

$$f_1 \wedge \cdots \wedge f_s \mapsto \big(m \mapsto f_s \circ \cdots \circ f_1(m)\big)\,.$$

When $s = t$ this is the map

$$(f_1 \wedge \cdots \wedge f_t)(m_1 \wedge \cdots \wedge m_t) = \det\big(f_i(m_j)\big)\,.$$

Finally we will use the following results from [5] which we state without proof.

PROPOSITION 2.1. *Let Q be a $\mathbf{Z}[G]$-algebra. Let $m_1, \ldots, m_t \in M$ and $f_1, \ldots, f_s \in \operatorname{Hom}_{\mathbf{Z}[G]}(M, Q)$. Then we have*

$$(f_1 \wedge \cdots \wedge f_s)(m_1 \wedge \cdots \wedge m_t) = \sum_{\sigma \in \mathfrak{S}_{t,s}} \operatorname{sgn}(\sigma) m_{\sigma(s+1)} \wedge \cdots \wedge m_{\sigma(t)} \otimes \det\big(f_i(m_{\sigma(j)})\big)_{1 \le i,j \le s}\,,$$

where

$$\mathfrak{S}_{t,s} := \big\{\sigma \in \mathfrak{S}_t : \sigma(1) < \cdots < \sigma(s) \text{and} \sigma(s+1) < \cdots < \sigma(t)\big\}\,.$$

PROOF. See [5, Prop. 4.1]. □

LEMMA 2.2. *Let D be a field, and A an n-dimensional D-vector space. If we have a D-linear map*

$$\Psi : A \longrightarrow D^{\oplus m}\,,$$

where $\Psi = \bigoplus_{i=1}^m \psi_i$ with $\psi_1, \ldots, \psi_m \in \operatorname{Hom}_D(A, D)$ $(m \le n)$, then we have

$$\operatorname{Im}\Big(\bigwedge_{1 \le i \le m} \psi_i : \bigwedge_D^n A \longrightarrow \bigwedge_D^{n-m} A\Big) = \begin{cases} \bigwedge_D^{n-m} \operatorname{Ker}(\Psi) & \text{if } \Psi \text{ is surjective,} \\ 0 & \text{if } \Psi \text{ is not surjective.} \end{cases}$$

PROOF. See [5, Lem. 4.2]. □

LEMMA 2.3. *Let P be a finitely generated projective $\mathbf{Z}[G]$-module and $j : \mathcal{O}_{K,S,T}^\times \hookrightarrow P$ be an injection whose cokernel is $\mathbf{Z}$-torsion-free. If we regard $\mathcal{O}_{K,S,T}^\times$ as a submodule of P via j, then we have*

$$\Lambda_{K/k,S,T} = \Big(\mathbf{Q} \bigwedge_{\mathbf{Z}[G]}^r \mathcal{O}_{K,S,T}^\times\Big) \cap \bigwedge_{\mathbf{Z}[G]}^r P\,.$$

PROOF. See [5, Lem. 4.7(ii)]. □

2.2. Restriction and corestriction maps. Here we recall the definitions of the restriction and corestriction maps. We follow the notation in [14].

Let $k \subseteq L \subseteq K$ be a tower of number fields, where K/k is abelian. As before let G be the Galois group of K/k and then let H be the Galois group of K/L and $\Gamma = G/H$ be the Galois group of L/k.

The restriction map

$$\mathrm{res}_{K/L} : \mathbf{C}[G] \longrightarrow \mathbf{C}[\Gamma]$$

is the $\mathbf{C}[G]$-algebra morphism defined by

$$\sigma \mapsto \sigma|_L$$

for $\sigma \in G$.

The corestriction map

$$\mathrm{cor}_{K/L} : \mathbf{C}[\Gamma] \longrightarrow \mathbf{C}[G]$$

is the $\mathbf{C}[G]$-module morphism defined by

$$\gamma \mapsto \sum_{\substack{\sigma \in G \\ \sigma|_L = \gamma}} \sigma = \tilde{\gamma} \cdot N_H ,$$

where $\tilde{\gamma}$ is any extension of γ and $N_H = \sum_{h \in H} h$.

REMARK 2.4. The restriction and corestriction map satisfy the following properties

(i) For $\lambda_1, \lambda_2 \in \mathbf{C}[\Gamma]$ we have

$$\mathrm{cor}_{K/L}(\lambda_1 \cdot \lambda_2) = \frac{1}{|H|} \mathrm{cor}_{K/L}(\lambda_1) \cdot \mathrm{cor}_{K/L}(\lambda_2) .$$

(ii) For $\sigma \in \mathbf{C}[G]$ we have

$$\mathrm{cor}_{K/L} \circ \mathrm{res}_{K/L}(\sigma) = N_H \cdot \sigma .$$

(iii) For $\gamma \in \mathbf{C}[\Gamma]$ we have

$$\mathrm{res}_{K/L} \circ \mathrm{cor}_{K/L}(\gamma) = |H| \cdot \gamma .$$

We will also need the following result.

LEMMA 2.5. *We have the following isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathbf{Z}[G]}\big(\mathcal{O}_{L,S,T}^{\times}, \mathbf{Z}[G]\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}[\Gamma]}\big(\mathcal{O}_{L,S,T}^{\times}, \mathbf{Z}[\Gamma]\big) ,$$

given by

$$\phi \mapsto \frac{1}{|H|} \cdot \mathrm{res}_{K/L} \circ \phi ,$$

for $\phi \in \mathrm{Hom}_{\mathbf{Z}[G]}(\mathcal{O}_{L,S,T}^{\times}, \mathbf{Z}[G])$. *The inverse map is given by*

$$\phi \mapsto \mathrm{cor}_{K/L} \circ \phi$$

for $\phi \in \mathrm{Hom}_{\mathbf{Z}[\Gamma]}(\mathcal{O}_{L,S,T}^{\times}, \mathbf{Z}[\Gamma])$.

PROOF. See [14, Lem. 4.20]. □

2.3. The leading term conjecture

2.3.1. The canonical complex. Burns has constructed a canonical complex which allows $\mathrm{LTC}(K/k)$ to be expressed in terms of a zeta element and the determinant module of this complex. Burns first uses this complex in [2], but it is developed further in [5] where it is expressed in terms of an 'integral dual Selmer group for $\mathbf{G}_m$'. Here we give the necessary definitions and properties of the complex. For full details see [5, §2].

DEFINITION 2.6. The 'S-relative T-trivialised integral dual Selmer group for $\mathbf{G}_m$' is defined by

$$\mathcal{S}_{S,T}(\mathbf{G}_{m/K}) := \mathrm{coker}\Big(\prod_{w \notin S_K \cup T_K} \mathbf{Z} \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(K_T^{\times}, \mathbf{Z}\big)\Big),$$

where $K_T^{\times}$ is the subgroup of $K^{\times}$ defined by

$$K_T^{\times} := \left\{ a \in K^{\times} : \mathrm{ord}_w(a-1) > 0 \text{ for all } w \in T_K \right\},$$

and the homomorphism in the right hand side is defined by

$$(x_w)_w \mapsto \Big(a \mapsto \sum_{w \notin S_K \cup T_K} \mathrm{ord}_w(a) x_w \Big).$$

$\mathcal{S}_{S,T}(\mathbf{G}_{m/K})$ can be better understood, conjecturally at least, as a cohomology group of a canonical complex of G-modules using 'Weil-ètale cohomology'.

PROPOSITION 2.7. *There exists a perfect complex* $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z})$ *such that*

(i) $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z})$ *is acyclic outside degrees one, two and three.*

(ii) *There are canonical isomorphisms*

$$H^i\big(R\Gamma_{c,T}\big((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z}\big)\big) \simeq \begin{cases} Y_{K,S}/\Delta_S(\mathbf{Z}) & \text{if } i = 1 \\ \mathcal{S}_{S,T}(\mathbf{G}_{m/K}) & \text{if } i = 2 \\ \big(K_{T,\mathrm{tors}}^{\times}\big)^{\vee} & \text{if } i = 3 \end{cases}$$

where Δ_S *is the natural diagonal map and* $(K_{T,\mathrm{tors}}^{\times})^{\vee}$ *is the Pontryagin dual of the torsion subgroup of* $K_T^{\times}$.

(iii) *There is an exact sequence*

$$0 \longrightarrow \mathrm{Cl}_{S,T}(K)^{\vee} \longrightarrow \mathcal{S}_{S,T}(\mathbf{G}_{m/K}) \longrightarrow \mathrm{Hom}_{\mathbf{Z}}\big(\mathcal{O}_{K,S,T}^{\times}, \mathbf{Z}\big) \longrightarrow 0.$$

PROOF. See [5, Prop. 2.2 and Prop. 2.4]. □

We define a ‘dual’ of the complex $R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z})$ Let

$$R\Gamma_T\big((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m\big) := R\mathrm{Hom}_{\mathbf{Z}}\big(R\Gamma_{c,T}\big((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z}\big), \mathbf{Z}\big)[-2]\,.$$

DEFINITION 2.8. We define the ‘transpose’ of $\mathcal{S}_{S,T}(\mathbf{G}_{m/K})$ by

$$\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}) := H^1_T(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m)) = H^{-1}(R\mathrm{Hom}_{\mathbf{Z}}(R\Gamma_{c,T}((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{Z}), \mathbf{Z}))\,.$$

PROPOSITION 2.9. *The complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m)$ is acyclic outside degrees zero and one. There exist canonical isomorphisms*

$$H^i\big(R\Gamma_T\big((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m\big)\big) \simeq \begin{cases} \mathcal{O}^{\times}_{K,S,T} & \text{if } i = 0 \\ \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}) & \text{if } i = 1 \end{cases}$$

and there is a canonical exact sequence

$$0 \longrightarrow \mathrm{Cl}_{S,T}(K) \longrightarrow \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}) \longrightarrow X_{K,S} \longrightarrow 0\,.$$

PROOF. See [5, Rem. 2.7]. □

2.3.2. The ‘zeta element’. If $D^{\bullet}_{K,S,T} : D^0 \longrightarrow D^1$ is a representative of the complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m)$, we define the regulator isomorphism

$$\rho_{K,S} : \mathbf{C}\mathrm{det}_{\mathbf{Z}[G]}\big(D^{\bullet}_{K,S,T}\big) \xrightarrow{\sim} \mathbf{C}[G]$$

as follows:

$$\begin{aligned} \mathbf{C}\mathrm{det}_{\mathbf{Z}[G]}\big(D^{\bullet}_{K,S,T}\big) &\xrightarrow{\sim} \mathrm{det}_{\mathbf{C}[G]}\big(\mathbf{C}D^0_{K,S,T}\big) \otimes_{\mathbf{C}[G]} \mathbf{C}\mathrm{det}^{-1}_{\mathbf{C}[G]}\big(\mathbf{C}D^1_{K,S,T}\big) \\ &\xrightarrow{\sim} \mathrm{det}_{\mathbf{C}[G]}\big(\mathbf{C}\mathcal{O}^{\times}_{K,S,T}\big) \otimes_{\mathbf{C}[G]} \mathrm{det}_{\mathbf{C}[G]}\big(\mathbf{C}\mathrm{Im}(d^0)\big) \\ &\qquad \otimes_{\mathbf{C}[G]} \mathrm{det}^{-1}_{\mathbf{C}[G]}\big(\mathbf{C}\mathrm{Im}(d^0)\big) \otimes_{\mathbf{C}[G]} \mathrm{det}^{-1}_{\mathbf{C}[G]}\big(\mathbf{C}X_{K,S}\big) \\ &\xrightarrow{\sim} \mathrm{det}_{\mathbf{C}[G]}\big(\mathbf{C}\mathcal{O}^{\times}_{K,S,T}\big) \otimes_{\mathbf{C}[G]} \mathrm{det}^{-1}_{\mathbf{C}[G]}\big(\mathbf{C}X_{K,S}\big) \\ &\xrightarrow{\sim} \mathrm{det}_{\mathbf{C}[G]}\big(\mathbf{C}X_{K,S}\big) \otimes_{\mathbf{C}[G]} \mathrm{det}^{-1}_{\mathbf{C}[G]}\big(\mathbf{C}X_{K,S}\big) \\ &\xrightarrow{\sim} \mathbf{C}[G]\,. \end{aligned}$$

The first isomorphism is the definition of $\mathrm{det}_{\mathbf{Z}[G]}(D^{\bullet}_{K,S,T})$. The second follows from the properties of determinants and Proposition 2.9. The third isomorphism follows from the evaluation map and the fourth by applying the Dirichlet logarithm λ. The final isomorphism is again the evaluation map.

DEFINITION 2.10. The *zeta element* $z_{K/k,S,T} \in \mathbf{C}\mathrm{det}_{\mathbf{Z}[G]}(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m))$ is the unique element such that

$$\rho_{K,S}(z_{K/k,S,T}) = \theta^*_{K/k,S,T}\,.$$

CONJECTURE 2.11 (Leading Term Conjecture - LTC(K/k)). *In* $\mathrm{Cdet}_{\mathbf{Z}[G]}(R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m))$ *one has*

$$\mathbf{Z}[G]z_{K/k,S,T} = \mathrm{det}_{\mathbf{Z}[G]}\big(R\Gamma_T\big((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m\big)\big)\,.$$

2.4. A representative for $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m)$. Before we show that the leading term conjecture implies the extended abelian Stark conjectures due to Vallières, Emmons and Popescu, we construct the representative $D^\bullet_{K,S,T}$ of the complex $R\Gamma_T((\mathcal{O}_{K,S})_{\mathcal{W}}, \mathbf{G}_m)$.

In [1] Burns constructs a Yoneda two extension of $\mathcal{O}^\times_{K,S,T}$ by $X_{K,S}$ using a free $\mathbf{Z}[G]$-module F and a homomorphism π which maps F onto $X_{K,S}$. Vallières follows this method in [14]. This method first works under the assumption that $\mathrm{Cl}_{S,T}(K)$ is trivial, and then shows how this may be removed.

In [5] Burns et al. adapt this method to create a representative for $D^\bullet_{K,S,T}$ by constructing a Yoneda two-extension of $\mathcal{O}^\times_{K,S,T}$ by $H^1(C^\bullet_{K,S,T})$ and making the necessary adaptions to F and π. In the case that $\mathrm{Cl}_{S,T}(K)$ is trivial, this method recovers the method used by Burns in [1] and Vallières in [14]. We follow this second method.

2.4.1. The free $\mathbf{Z}[G]$-module F. Here we recall the construction of F and π by Burns et al. ([5, §5.4]).

Let $d \in \mathbf{Z}$ be sufficiently large. Let F be a free $\mathbf{Z}[G]$-module with basis $\{b_i\}_{1\le i\le d}$. We will construct a surjective $\mathbf{Z}[G]$-homomorphism

$$\pi : F \longrightarrow H^1\big(D^\bullet_{K,S,T}\big)$$

in two parts. Firstly write $F = F_1 \oplus F_2$, where F_1 is the free $\mathbf{Z}[G]$-module generated by $\{b_i\}_{1\le i\le n}$ and F_2 is the free $\mathbf{Z}[G]$-module generated by $\{b_i\}_{n<i\le d}$.

Since F_1 is free we may choose a homomorphism $\pi_1 : F_1 \longrightarrow H^1(C^\bullet_{K,S,T})$ such that the composition

$$F_1 \xrightarrow{\pi_1} H^1\big(D^\bullet_{K,S,T}\big) \longrightarrow X_{K,S}$$

sends b_i to $w_i - w_0$.

Now let A be the kernel of the composition

$$H^1\big(D^\bullet_{K,S,T}\big) \longrightarrow X_{K,S} \longrightarrow Y_{K,S\setminus\{v_0\}}\,,$$

where the last map sends places above v_0 to 0. We chose d to be sufficiently large so that we can now choose a surjection

$$\pi_2 : F_2 \longrightarrow A\,.$$

This then gives us a surjective map π defined by:

$$\pi := \pi_1 \oplus \pi_2 : F = F_1 \oplus F_2 \longrightarrow H^1\big(D^\bullet_{K,S,T}\big)\,.$$

2.4.2. Constructing the representative. We use F and the map π to construct a representative for the Yoneda extension class of $D^\bullet_{K,S,T}$ in $\mathrm{Ext}^2_{\mathbf{Z}[G]}(H^1(D^\bullet_{K,S,T}), \mathcal{O}^\times_{K,S,T})$. Since $D^\bullet_{K,S,T}$ is perfect we can represent it by an exact sequence

$$0 \longrightarrow \mathcal{O}^\times_{K,S,T} \longrightarrow P \xrightarrow{\psi} F \xrightarrow{\pi} H^1\big(D^\bullet_{K,S,T}\big) \longrightarrow 0$$

where P is a cohomologically trivial $\mathbf{Z}[G]$-module. Since $\mathcal{O}^\times_{K,S,T}$ is torsion-free, P is torsion-free and hence projective.

LEMMA 2.12. *For every prime p there is a natural isomorphism of $\mathbf{Z}_{(p)}[G]$-modules*

$$P_{(p)} \simeq F_{(p)}\,.$$

PROOF. We have that $\mathbf{C}\mathcal{O}^\times_{K,S,T} \simeq \mathbf{C}H^1(D^\bullet_{K,S,T})$. Therefore for every prime p

$$\mathbf{Q}_{(p)}\mathcal{O}^\times_{K,S,T\,(p)} \simeq \mathbf{Q}_{(p)}H^1\big(D^\bullet_{K,S,T}\big)_{(p)}\,.$$

Since $\mathbf{Q}_{(p)}[G]$ is semisimple we get

$$\mathbf{Q}_{(p)}P_{(p)} \simeq \mathbf{Q}_{(p)}F_{(p)}\,.$$

Finally our required result follows from Swan's Theorem since P and F are both projective and torsion-free. (See [6] for details of Swan's Theorem.) □

REMARK 2.13. Since

$$\det_{\mathbf{Z}[G]}\big(D^\bullet_{K,S,T}\big) \simeq \det_{\mathbf{Z}[G]}(P) \otimes_{\mathbf{Z}[G]} \det^{-1}_{\mathbf{Z}[G]}(F)\,,$$

if LTC(K/k) is valid then we have that the $\mathbf{Z}[G]$-module P is isomorphic to F.

3. The leading term conjecture implies the extended abelian Stark conjectures

In this section we prove the following result:

THEOREM 3.1 (Vallières). *Let K/k be a finite abelian extension of number fields. Let S be a finite set of places of k containing all infinite places and places that ramify in K/k. Let T be another finite set of places of k such that $S \cap T = \emptyset$ and such that $\mathcal{O}^\times_{K,S,T}$ is $\mathbf{Z}$-torsion-free. Assume $S \neq V_S$. Let $\eta^I_{K/k,S,T} \in \mathbf{C}\bigwedge^r_{\mathbf{Z}[G]}\mathcal{O}^\times_{K,S,T} \cdot e_I$ be defined as before.*

Then LTC(K/k) implies that $\eta^I_{K/k,S,T} \in \Lambda_{K/k,S,T}$.

REMARK 3.2. Theorem 3.1 is originally proved by Vallières in [14, Thm. 6.12] but here we use methods from [5] to improve and simplify the proof. Since r is defined to be the minimal order of vanishing of the S-truncated L-functions, we do not need to assume that S is an 'r-cover'. (This condition essentially asserts that all orders of vanishing are at least r.)

Crucially this proof allows us to express $\eta^I_{K/k,S,T}$ in terms of the zeta element $z_{K/k,S,T}$. This new proof provides us with the construction we need to to prove our results involving Fitting ideals in §4.

PROOF. Assume LTC(K/k) holds. If $|S| = r+1$ then S contains at least r places that split completely in K. Then the statement $\eta^I_{K/k,S,T} \in \Lambda_{K/k,S,T}$ is just the Rubin-Stark conjecture, and the theorem has been proved by Burns in [1].

Assume $|S| > r+1$ and therefore that $\chi_1 \notin \hat{G}_{r,S}$. We adapt the method from the proof of [5, Theorem 5.11].

It follows from Remark 2.13 that P is free of rank d. It also follows from LTC(K/k) that we can define $z_b \in \bigwedge^d_{\mathbf{Z}[G]} P$ to be the element that corresponds to the zeta element $z_{K/k,S,T} \in \det_{\mathbf{Z}[G]}(D^\bullet_{K,S,T})$ via the isomorphism

$$\kappa : \bigwedge^d_{\mathbf{Z}[G]} P \xrightarrow{\sim} \bigwedge^d_{\mathbf{Z}[G]} P \otimes \bigwedge^d_{\mathbf{Z}[G]} F^* \simeq \det_{\mathbf{Z}[G]}\big(D^\bullet_{K,S,T}\big),$$

where the first isomorphism is given by

$$a \mapsto a \otimes \bigwedge_{1 \le i \le d} b_i^*,$$

where $b_i^* \in F^*$ is the dual basis of $b_i \in F$, and the second isomorphism is given by

$$\det_{\mathbf{Z}[G]}\big(D^\bullet_{K,S,T}\big) \simeq \det_{\mathbf{Z}[G]}(P) \otimes_{\mathbf{Z}[G]} \det^{-1}_{\mathbf{Z}[G]}(F).$$

For each $1 \le i \le d$, we define

$$\psi_i := b_i^* \circ \psi \in P^*.$$

Then the theorem follows from the next proposition. □

PROPOSITION 3.3. *Regard $\mathcal{O}^\times_{K,S,T} \subset P$. Then we have*

1. $e_I \cdot (\bigwedge_{i \notin \bar{I}} \psi_i)(z_b) \in \Lambda_{K/k,S,T} (\subset \bigwedge^r_{\mathbf{Z}[G]} P)$
2. $\eta^I_{K/k,S,T} = e_I \cdot (-1)^{\bar{I}+[r]+r(d-r)} (\bigwedge_{i \notin \bar{I}} \psi_i)(z_b)$.

PROOF. Let $\chi \in \hat{G}_{r,S,I}$. Define a map

$$\Psi := \bigoplus_{i \notin \bar{I}} \psi_i : e_\chi \cdot \mathbf{C}P \to e_\chi \cdot \mathbf{C}[G]^{\oplus(d-r)}.$$

We will show that Ψ is surjective and then apply Lemma 2.2. We have that $e_\chi \cdot \mathbf{C}\,\mathrm{Im}(\psi) = \bigoplus_{i \notin \bar{I}} \mathbf{C}[G] \cdot b_i \cdot e_\chi$, (see [14, Lem. 6.5],) which implies surjectivity. Therefore by Lemma 2.2 we have $e_\chi \cdot (\bigwedge_{i \notin \bar{I}} \psi_i)(z_b) \in e_\chi \cdot \mathbf{C} \bigwedge^r_{\mathbf{Z}[G]} \mathcal{O}^\times_{K,S,T}$. Summing over all $\chi \in \hat{G}_{r,S,I}$ we get

$$e_I \cdot \Big(\bigwedge_{i \notin \bar{I}} \psi_i\Big)(z_b) \in e_I \cdot \mathbf{C} \bigwedge^r_{\mathbf{Z}[G]} \mathcal{O}^\times_{K,S,T}.$$

Since e_I is a rational character it follows from Lemma 2.3 and Lemma 3.4 below that

$$e_I \cdot \Big(\bigwedge_{i\notin\bar I} \psi_i\Big)(z_b) \in \Big(\mathbf{Q} \bigwedge_{\mathbf{Z}[G]}^{r} \mathcal{O}_{K,S,T}^{\times}\Big) \cap \bigwedge_{\mathbf{Z}[G]}^{r} P = \Lambda_{K/k,S,T}\,.$$

By Lemma 3.5 below we have that

$$\lambda\Big(e_I \cdot (-1)^{\bar I+[r]+r(d-r)}\Big(\bigwedge_{i\notin\bar I}\psi_i\Big)(z_b)\Big) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I\,.$$

Since λ is an isomorphism it follows that

$$\eta^I_{K/k,S,T} = e_I \cdot (-1)^{\bar I+[r]+r(d-r)}\Big(\bigwedge_{i\notin\bar I}\psi_i\Big)(z_b)\,.$$

□

LEMMA 3.4. *Assume LTC(K/k). Then*

$$e_I \cdot \Big(\bigwedge_{i\notin\bar I}\psi_i\Big)(z_b) = \Big(\bigwedge_{i\notin\bar I}\psi_i\Big)(z_b)\,.$$

PROOF. This proof uses methods developed in [1], [5] and [14, Lem. 6.8].

By Proposition 2.1 it suffices to prove that for every $\sigma \in \mathfrak{S}_{d,r}$ we have that

$$e_I \cdot \det\big(\psi_i\big(b_{\sigma(j)}\big)\big)_{i\notin\bar I, r<j\le d} = \det\big(\psi_i\big(b_{\sigma(j)}\big)\big)_{i\notin\bar I, r<j\le d}\,.$$

We prove this prime by prime after localisation.

Fix a prime p. By Lemma 2.12 the $\mathbf{Z}_{(p)}[G]$-modules $P_{(p)}$ and $F_{(p)}$ are free of the same rank d. Thus we may assume that $P_{(p)} = F_{(p)}$. The basis $\{b_1, \dots, b_d\}$ of F thus gives a basis for both $P_{(p)}$ and $F_{(p)}$, for which we will use the same notation.

Fix $\sigma \in \mathfrak{S}_{d,r}$. Let $\psi_{I,\sigma,p}$ denote the composite map

$$\psi_{I,\sigma,p} : \bigoplus_{r<j\le d} \mathbf{Z}_{(p)}[G]\cdot b_{\sigma(j)} \xrightarrow{\psi_{res}} F_{(p)} \xrightarrow{q} \bigoplus_{i\notin\bar I}\mathbf{Z}_{(p)}[G]\cdot b_i$$

where q is the natural projection. The matrix $(\psi_i(b_{\sigma(j)}))_{i\notin\bar I, r<j\le d}$ corresponds to the morphism $\psi_{I,\sigma,p}$. We will show that if $\chi \notin \hat G_{r,S,I}$ then $\psi^{\chi}_{I,\sigma,p} := e_\chi \cdot (\mathbf{C} \otimes_{\mathbf{Z}_{(p)}} \psi_{I,\sigma,p})$ is singular (and so has determinant equal to zero). This will suffice since $e_I + \sum_{\chi\notin\hat G_{r,S,I}} e_\chi = 1$. We will examine two separate cases.

Fix $\chi \notin \hat G_{r,S}$, then $\dim_{\mathbf{C}}(\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi) > r$. Suppose $\psi^{\chi}_{I,\sigma,p}$ is not singular. Then

$$\big(\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi\big) \cap \Big(\bigoplus_{r<j\le d}\mathbf{C}[G]\cdot b_{\sigma(j)}\cdot e_\chi\Big) = 0\,.$$

It follows that $\dim_{\mathbf{C}}(\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi+\bigoplus_{r<j\le d}\mathbf{C}[G]\cdot b_{\sigma(j)}\cdot e_\chi)=\dim_{\mathbf{C}}(\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi)+d-r>r+d-r=d$, which is a contradiction.

Now let $\chi\in\hat{G}_{r,S}\backslash\hat{G}_{r,S,I}$, then $\dim_{\mathbf{C}}(\mathrm{Im}(\psi)\cdot e_\chi)=d-r$. Again suppose $\psi^\chi_{I,\sigma,p}$ is not singular. Then

$$\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi\cap\Bigg(\bigoplus_{r<j\le d}\mathbf{C}[G]\cdot b_{\sigma(j)}\Bigg)\cdot e_\chi=0\,,$$

so by counting dimensions

$$\mathbf{C}\mathrm{Ker}(\psi)\cdot e_\chi+\Bigg(\bigoplus_{r<j\le d}\mathbf{C}[G]\cdot b_{\sigma(j)}\cdot e_\chi\Bigg)=\mathbf{C}F\cdot e_\chi\,.$$

We have $\chi\notin\hat{G}_{r,S,I}$. Therefore $\exists i_0\in\bar{I}$ such that $G_{v_{i_0}}\not\subseteq\mathrm{Ker}(\chi)$. We also have that $\pi(b_{i_0})=w_{i_0}-w_0$ in $\mathbf{C}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})=\mathbf{C}X_{K,S}$. Therefore $\pi(e_\chi\cdot b_{i_0})=e_\chi\cdot(w_{i_0}-w_0)=0$ as $G_{v_0},G_{v_{i_0}}\not\subseteq\mathrm{Ker}(\chi)$. This gives $e_\chi\cdot b_{i_0}\in\mathrm{Im}(\psi^\chi)$.

Therefore we can find x in $\mathbf{C}F\cdot e_\chi$ such that $\psi^\chi(x)=e_\chi\cdot b_{i_0}$ and write $x=y'+y$ for some $y'\in\mathrm{Ker}(\psi^\chi)$ and $y\in(\bigoplus_{r<j\le d}\mathbf{C}[G]\cdot b_{\sigma(j)})\cdot e_\chi$.

Then $\psi^\chi(x)=\psi^\chi(y)$ and since $q(e_\chi\cdot b_{i_0})=0$, we get

$$\psi^\chi_{I,\sigma,p}(y)=0\,.$$

But $y\neq 0$ since $e_\chi\cdot b_{i_0}\neq 0$, therefore $\psi^\chi_{I,\sigma,p}$ is singular. □

LEMMA 3.5. *By definition of z_b we have*

$$e_I\cdot\lambda\Bigg((-1)^{\bar{I}+[r]+r(d-r)}\Bigg(\bigwedge_{i\notin\bar{I}}\psi_i\Bigg)(z_b)\Bigg)=e_I\cdot\theta^*_{K/k,S,T}\cdot w_I\,.$$

PROOF. Let $\chi\in\hat{G}_{r,S,I}$.

We have

$$0\longrightarrow\mathcal{O}^\times_{K,S,T}\longrightarrow P\xrightarrow{\psi}F\xrightarrow{\pi}H^1\big(D^\bullet_{K,S,T}\big)\longrightarrow 0\,.$$

This breaks up into two short exact sequences, which after tensoring with $\mathbf{C}$ and taking χ-components give

$$0\longrightarrow\big(\mathcal{O}^\times_{K,S,T}\big)_\chi\longrightarrow P_\chi\xrightarrow{\psi}\mathrm{Im}(\psi)_\chi\longrightarrow 0\tag{2}$$

and

$$0\longrightarrow\mathrm{Im}(\psi)_\chi\longrightarrow F_\chi\xrightarrow{\pi}\big(X_{K,S}\big)_\chi\longrightarrow 0\,.\tag{3}$$

We now fix ordered $\mathbf{C}$-bases for the modules above. (Note that we identify $e_\chi \cdot \mathbf{C}[G]$ with $\mathbf{C}$ by letting e_χ correspond to 1.) Let $\bar{J} := \{1, \dots, d\} \setminus \bar{I} = \{j_1, j_2, \dots, j_{d-r}\}$ with $j_1 < j_2 < \dots < j_{d-r}$.

By the definition of π we have that $\pi(e_\chi \cdot b_i) = e_\chi \cdot (w_i - w_0)$ for $1 \le i \le n$. We also have that $e_\chi \cdot w_i = 0$ for $i \notin \bar{I}$. Therefore we write

$$x_{i,\chi} := e_\chi \cdot (w_i - w_0),$$

and we fix an ordered $\mathbf{C}$-basis for $(\mathbf{C}X_{K,S})_\chi$ as $\{x_{i_1,\chi}, x_{i_2,\chi}, \dots, x_{i_r,\chi}\}$.

We recall that $\mathbf{C}[G]$ is semisimple. Choose a section

$$\iota_2 : \mathbf{C}X_{K,S} \longrightarrow \mathbf{C}F$$

such that $\iota_2(x_{i,\chi}) = b_{i,\chi}$ for $i \in \bar{I}$, where $b_{i,\chi} := e_\chi \cdot b_i$.

An ordered $\mathbf{C}$-basis for F_χ is $\{b_{1,\chi}, b_{2,\chi}, \dots, b_{d,\chi}\}$. Then $\mathrm{Im}(\psi)_\chi$ has $\mathbf{C}$-rank $d-r$ and we fix $\{b_{j_1,\chi}, b_{j_2,\chi}, \dots, b_{j_{d-r},\chi}\}$ as an ordered basis. We also have that P_χ has rank d and $(\mathcal{O}^\times_{K,S,T})_\chi$ has rank r. Fix a basis $\{u_{i_1,\chi}, u_{i_2,\chi}, \dots, u_{i_r,\chi}\}$ of $(\mathcal{O}^\times_{K,S,T})_\chi$. We can then choose a section

$$\iota_1 : \mathbf{C}\mathrm{Im}(\psi) \longrightarrow \mathbf{C}P$$

with $\{u_{i_1,\chi}, \dots, u_{i_r,\chi}, \iota_1(b_{j_1,\chi}), \dots, \iota_1(b_{j_{d-r},\chi})\}$ forming a basis for P_χ.

We can now use these two sections to define an isomorphism f of $\mathbf{C}$-modules from $\mathbf{C}P$ to $\mathbf{C}F$.

$$\begin{tikzcd}
\mathbf{C}P \arrow[r, "\sim"] \arrow[d, "f"] & \mathbf{C}\mathcal{O}^\times_{K,S,T} \arrow[d, "\iota_2\circ\lambda"] & \oplus & \iota_1\big(\mathbf{C}\mathrm{Im}(\psi)\big) \arrow[d, "\psi"] \\
\mathbf{C}F \arrow[r, "\sim"] & \iota_2\big(\mathbf{C}X_{K,S}\big) & \oplus & \mathbf{C}\mathrm{Im}(\psi)
\end{tikzcd}$$

Then f also defines an isomorphism from P_χ to F_χ and we can define $f_i := b_i^* \circ f \in \mathrm{Hom}_{\mathbf{C}}(P_\chi, \mathbf{C})$.

We will show that diagram (4) below commutes. This will allow us to obtain the equality given in the proposition by mapping the element z_b across both the top and bottom arrows.

$$\begin{tikzcd}
\bigwedge^d_{\mathbf{C}} P_\chi \arrow[r, "(-1)^{r(d-r)}\cdot\rho_{K,S}\circ\kappa"'] \arrow[d, equal] & \mathbf{C} \arrow[d, equal] \\
\bigwedge^d_{\mathbf{C}} P_\chi \arrow[r, "\wedge_{i=1}^{i=d} f_i"'] & \mathbf{C}
\end{tikzcd} \tag{4}$$

We need only show that it commutes for a basis of $\bigwedge^d_{\mathbf{C}} P_\chi$. Since

$\{u_{i_1,\chi},\ldots,u_{i_r,\chi},\iota_1(b_{j_1,\chi}),\ldots,\iota_1(b_{j_{d-r},\chi})\}$ is a $\mathbf{C}$-basis for P_χ, we have that the element $u_{i_1,\chi}\wedge\cdots\wedge u_{i_r,\chi}\wedge\iota_1(b_{j_1,\chi})\wedge\cdots\wedge\iota_1(b_{j_{d-r},\chi})$ is a $\mathbf{C}$-basis for $\bigwedge_{\mathbf{C}}^d P_\chi$.

Let us first apply the top line. Unless otherwise stated all individual wedge products are taken in order of increasing index.

We start with the basis element

$$\bigwedge_{i\in\bar{I}} u_{i,\chi}\wedge\bigwedge_{j\in\bar{J}}\iota_1(b_{j,\chi})\in\bigwedge_{\mathbf{C}}^d P_\chi\,.$$

Recall that the map κ was defined via two isomorphisms. We first apply the isomorphism given by $a\mapsto a\otimes\bigwedge_{1\le i\le d} b_i^*$. This gives

$$\bigwedge_{i\in\bar{I}} u_{i,\chi}\wedge\bigwedge_{j\in\bar{J}}\iota_1(b_{j,\chi})\otimes\bigwedge_{1\le k\le d} b_{k,\chi}^*\in\bigwedge_{\mathbf{C}}^d P_\chi\otimes_{\mathbf{C}[G]}\bigwedge_{\mathbf{C}}^d(F^*)_\chi\,.$$

The second part of κ is given by the isomorphism

$$\det_{\mathbf{Z}[G]}(P)\otimes_{\mathbf{Z}[G]}\det^{-1}_{\mathbf{Z}[G]}(F)\xrightarrow{\sim}\det_{\mathbf{Z}[G]}\big(D^\bullet_{K,S,T}\big)\,.$$

However if we recall the definition of the regulator isomorphism $\rho_{K,S}$ in §2.3.2, the first step was given by the inverse of this map. Therefore our next step is to apply the second part of $\rho_{K,S}$, which is given by the isomorphisms coming from the split short exact sequences (2) and (3). This maps $\bigwedge_{i\in\bar{I}} u_{i,\chi}\wedge\bigwedge_{j\in\bar{J}}\iota_1(b_{j,\chi})\otimes\bigwedge_{1\le k\le d} b_{k,\chi}^*$ to the element

$$(-1)^{\bar{J}+[d-r]}\bigwedge_{i\in\bar{I}} u_{i,\chi}\otimes\bigwedge_{j\in\bar{J}} b_{j,\chi}\otimes\bigwedge_{j\in\bar{J}} b_{j,\chi}^*\otimes\bigwedge_{i\in\bar{I}} x_{i,\chi}^*$$

$$\in\bigwedge_{\mathbf{C}}^r\big(\mathcal{O}_{K,S,T}^\times\big)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^{d-r}\mathrm{Im}(\psi)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^{d-r}\big(\mathrm{Im}(\psi)^*\big)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^r\big(X_{K,S}^*\big)_\chi\,.$$

We then apply the evaluation map to obtain

$$(-1)^{\bar{J}+[d-r]}\bigwedge_{i\in\bar{I}} u_{i,\chi}\otimes\bigwedge_{i\in\bar{I}} x_{i,\chi}^*\in\bigwedge_{\mathbf{C}}^r\big(\mathcal{O}_{K,S,T}^\times\big)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^r\big(X_{K,S}^*\big)_\chi\,.$$

Applying the Dirichlet logarithm gives

$$(-1)^{\bar{J}+[d-r]}\bigwedge_{i\in\bar{I}}\lambda(u_{i,\chi})\otimes\bigwedge_{i\in\bar{I}} x_{i,\chi}^*\in\bigwedge_{\mathbf{C}}^r\big(X_{K,S}\big)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^r\big(X_{K,S}^*\big)_\chi\,.$$

This is equal to

$$(-1)^{\bar{J}+[d-r]}\det(C_\chi)\bigwedge_{i\in\bar{I}}(x_{i,\chi})\otimes\bigwedge_{i\in\bar{I}} x_{i,\chi}^*\in\bigwedge_{\mathbf{C}}^r\big(X_{K,S}\big)_\chi\otimes_{\mathbf{C}}\bigwedge_{\mathbf{C}}^r\big(X_{K,S}^*\big)_\chi$$

where $C_\chi = (c_{ik,\chi})_{i,k\in\bar{I}}$ is the matrix defined by $\lambda(u_{i,\chi}) = \sum_{k\in\bar{I}} c_{ik,\chi}x_{k,\chi}$.

Finally applying the evaluation map and multiplying by $(-1)^{r(d-r)}$ gives

$$(-1)^{\bar{I}+[r]}\det\bigl(C_\chi\bigr) \in \mathbf{C}\,,$$

where we note that $(-1)^{[d-r]+[r]} = (-1)^{[d]+r(d-r)}$ and $(-1)^{[d]} = (-1)^{\bar{I}+\bar{J}}$.

Let us now turn our attention to the bottom row of diagram (4). We have $\wedge_{i=1}^{i=d} f_i = \wedge_{i=1}^{i=d} b_{i,\chi}^* \circ \wedge_{i=1}^{i=d} f$. In order to apply $\wedge_{i=1}^{i=d} f$ we calculate $\det(f)$ with respect to our chosen ordered bases.

It is easier to do this by first considering the matrix of f with respect to a different ordering of the basis of F_χ. We consider the matrix of f with respect to the ordered bases $\{u_{i_1,\chi}, \dots, u_{i_r,\chi}, \iota_1(b_{j_1,\chi}), \dots, \iota_1(b_{j_{d-r},\chi})\}$ of P_χ and $\{b_{i_1,\chi}, \dots, b_{i_r,\chi}, b_{j_1,\chi}, \dots, b_{j_{d-r},\chi}\}$ of F_χ.

We have for $i \in \bar{I}$ that

$$\begin{aligned} f(u_{i,\chi}) &= \iota_2\Bigl(\sum_{k\in\bar{I}} c_{ik,\chi}x_{k,\chi}\Bigr) \\ &= \sum_{k\in\bar{I}} c_{ik,\chi}b_{k,\chi} \end{aligned}$$

where the first equality follows since $u_{i,\chi} \in (\mathcal{O}_{K,S,T}^\times)_\chi$ and the second by the definition of ι_2. For $j \in \bar{J}$ we have

$$f\bigl(\iota_1(b_{j,\chi})\bigr) = b_{j,\chi}\,.$$

Thus the matrix of f with respect to this ordering is:

$$\begin{array}{c} \\ u_{i_1,\chi} \\ \vdots \\ u_{i_r r,\chi} \\ \iota_1\bigl(b_{j_1,\chi}\bigr) \\ \vdots \\ \iota_1\bigl(b_{j_{d-r},\chi}\bigr) \end{array} \begin{array}{c} \begin{array}{cccccc} b_{i_1,\chi} & \dots & b_{i_r,\chi} & b_{j_1,\chi} & \dots & b_{j_{d-r},\chi} \end{array} \\ \left(\begin{array}{c|c} C_\chi & 0 \\ \hline 0 & \mathrm{Id} \end{array}\right) \end{array}.$$

The determinant of this matrix is equal to $\det(C_\chi)$. In order to obtain the matrix of f with respect to our original ordered basis $\{b_{1,\chi}, \dots, b_{d,\chi}\}$ of F_χ we have to permute the columns and so we obtain that

$$\det(f) = (-1)^{\bar{I}+[r]}\det\bigl(C_\chi\bigr)\,.$$

We can now apply the bottom arrow to our basis element. Again we start with the element

$$\bigwedge_{i\in\bar{I}} u_{i,\chi} \wedge \bigwedge_{j\in\bar{J}} \iota_1(b_{j,\chi}) \in \bigwedge_{\mathbf{C}}^{d} P_\chi \,.$$

We then apply $\wedge_{i=1}^{i=d} f$ to get

$$\bigwedge_{i\in\bar{I}} f\big(u_{i,\chi}\big) \wedge \bigwedge_{j\in\bar{J}} f(\iota_1(b_{j,\chi})) = \big(\det(f)\big) \bigwedge_{i=1}^{i=d} b_{i,\chi} \in \bigwedge_{\mathbf{C}}^{d} F_\chi \,.$$

Applying $\wedge_{i=1}^{i=d} b_{i,\chi}^*$ gives

$$\det(f) = (-1)^{\bar{I}+[r]}\det\big(C_\chi\big) \in \mathbf{C}\,,$$

thus proving the commutativity of the diagram.

We can use this result to map the element $e_\chi \cdot z_b \in \bigwedge_{\mathbf{C}}^{d} P_\chi$ in both directions around diagram (4).

By definition of z_b under the top line we have

$$e_\chi \cdot z_b \mapsto e_\chi \cdot z_{K/k,S,T} \mapsto (-1)^{r(d-r)} e_\chi \cdot \theta^*_{K/k,S,T} \,.$$

Under the bottom line we have

$$e_\chi \cdot z_b \mapsto \big(\wedge_{i=1}^{i=d} f_i\big)(e_\chi \cdot z_b)\,.$$

Therefore

$$\begin{aligned}
& \left(\bigwedge_{i=1}^{i=d} f_i\right)(e_\chi \cdot z_b) = (-1)^{r(d-r)} e_\chi \cdot \theta^*_{K/k,S,T} \\
\Rightarrow \quad & (-1)^{\bar{I}+[r]+r(d-r)}\left(\bigwedge_{i\in\bar{I}} f_i\right)\left(\left(\bigwedge_{i\notin\bar{I}} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T} \\
\Rightarrow \quad & (-1)^{\bar{I}+[r]+r(d-r)}(\iota_2 \circ \lambda)\left(\left(\bigwedge_{i\notin\bar{I}} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T} \\
\Rightarrow \quad & (-1)^{\bar{I}+[r]+r(d-r)}\lambda\left(\left(\bigwedge_{i\notin\bar{I}} f_i\right)(e_\chi \cdot z_b)\right) = e_\chi \cdot \theta^*_{K/k,S,T} \bigwedge_{i\in\bar{I}}(w_i - w_0) \\
\Rightarrow \quad & (-1)^{\bar{I}+[r]+r(d-r)}\lambda\left(e_I \cdot \left(\bigwedge_{i\notin\bar{I}} f_i\right)(z_b)\right) = e_I \cdot \theta^*_{K/k,S,T} \cdot w_I
\end{aligned}$$

where third line follows since $(\bigwedge_{i\notin I} f_i)(z_b) \in \bigwedge^r \mathrm{Ker}(\psi)$.

Finally let us consider $e_\chi \cdot (\bigwedge_{i\notin I} f_i)(z_b)$ for $\chi \in \hat{G}_{r,S,I}$. Recall that f is $\iota_2 \circ \lambda$ on $\mathbf{C}\mathcal{O}^\times_{K,S,T} = \mathbf{C}\mathrm{Ker}(\psi)$ and ψ on $\iota_2(\mathbf{C}\mathrm{Im}(\psi))$. We claim that $(b_i^* \circ \iota_2 \circ \lambda)^\chi = 0$ for $i \notin \bar{I}$. Considering just the χ component we have that λ maps to $(X_{K,S})_\chi$ which has basis $\{x_{i,\chi} : i \in \bar{I}\}$. These basis elements are mapped to $\{b_{i,\chi} : i \in \bar{I}\}$ by ι_2. However these elements are not seen by b_i^* for $i \notin \bar{I}$, which proves our claim. Therefore

$$e_I \cdot \left(\bigwedge_{i\notin \bar{I}} f_i\right)(z_b) = e_I \cdot \left(\bigwedge_{i\notin \bar{I}} \psi_i\right)(z_b)$$

thus proving the proposition.

□

Then the following corollary follows immediately from the above theorem.

COROLLARY 3.6. *LTC(K/k) implies Conjecture* 1.4.

4. Evaluators and higher Fitting ideals

In this section we investigate the ideal generated by $\Phi(\eta^I_{K/k,S,T})$ as Φ runs over $\bigwedge^r_{\mathbf{Z}[G]}(\mathcal{O}^\times_{K,S,T})^*$. We will apply and adapt the techniques developed by Burns et al. in [5] in the following way. We begin by defining, for every $I \in \wp_r(V_S)$, a subextension of K/k in which the hypotheses of the Rubin-Stark hold. Emmons, Popescu and Vallières have shown (see [7] and [14]) the relationship between the Rubin-Stark elements of the subextensions, and the elements $\eta^I_{K/k,S,T}$. This allows us to apply the results in [5].

Applying these methods introduces denominators that remove some of the integrality properties. However, through Theorem 1.7 we are able to regain some integrality results.

4.1. Subextensions of K/k. For each $I \in \wp_r(V_s)$ we define a subextension of K/k in which there are r places that split completely. Let

- $D_I = \langle G_v : v \in I\rangle$
- $L_I = K^{D_I}$
- $\Gamma_I = G/D_I$
- $e'_{r,S} = \sum_{\chi'\in(\hat{\Gamma}_I)_{r,S}} e_{\chi'}$
- $e_{D_I} = \frac{1}{|D_I|}\sum_{d\in D_I} d$

REMARK 4.1. The definition of e_{D_I} implies that it can be decomposed as

$$e_{D_I} = \sum_{\substack{\chi\in\hat{G}\\ \chi(D_I)=1}} e_\chi = \sum_{\substack{\chi\in\hat{G}\\ \chi(G_v)=1,\forall v\in I}} e_\chi\,.$$

By comparing this with the definition of the idempotent e_I, it is easy to see that

$$e_I \cdot e_{D_I} = e_I\,.$$

For all $i \in \{0, \dots, n\}$ fix $w_{I,i}$ as the unique place of L_I lying between v_i and w_i.
Let

$$\epsilon^I_{L_I/k,S,T} \in \mathbf{C} \bigwedge^r_{\mathbf{Z}[\Gamma_I]} \mathcal{O}^\times_{L_I,S,T} \cdot e'_{r,S}$$

be the unique element such that

$$\lambda_{L_I/k}\big(\epsilon^I_{L_I/k,S,T}\big) = \theta^*_{L_I/k,S,T}(w_{I,i_1} - w_{I,0}) \wedge \cdots \wedge (w_{I,i_r} - w_{I,0}) \cdot e'_{r,S}\,.$$

Since the set of places $\{v_i : i \in \bar{I}\}$ split completely in L_I/k, $\epsilon^I_{L_I/k,S,T}$ is a Rubin-Stark element for L_I/k.

We can therefore apply the theorem by Burns et al. to the Rubin-Stark element $\epsilon^I_{L_I/k,S,T}$.

THEOREM 4.2. *Assume LTC(L_I/k). Then*

$$\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big) = \left\{\Psi\big(\epsilon^I_{L_I/k,S,T}\big) : \Psi \in \bigwedge^r_{\mathbf{Z}[\Gamma_I]} \mathrm{Hom}_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{O}^\times_{L_I,S,T}, \mathbf{Z}[\Gamma_I]\big)\right\}.$$

PROOF. See [5, Thm. 7.3]. $\square$

In order to apply this result to our element $\eta^I_{K/k,S,T}$ we use the following result linking the Rubin-Stark of element each subextension to the corresponding element $\eta^I_{K/k,S,T}$ of the original extension.

LEMMA 4.3. *In $\mathbf{C}\bigwedge^r_{\mathbf{Z}[G]} \mathcal{O}^\times_{K,S,T}$ we have the following equality*:

$$\eta^I_{K/k,S,T} = \frac{1}{|D_I|^r}\epsilon^I_{L_I/k,S,T}\,.$$

PROOF. See [14, Prop. 4.18]. $\square$

REMARK 4.4. The Rubin-Stark conjecture asserts that $\epsilon^I_{L_I/k,S,T} \in \Lambda_{L_I/k,S,T}$. However this relationship between elements allows Vallières to make the stronger conjecture that $\epsilon^I_{L_I/k,S,T}$ is a $|D_I|$th power in $\Lambda_{L_I/k,S,T}$ and to prove that this conjecture follows from LTC(K/k).

This relationship allows us to apply results already proven for the Rubin-Stark element $\epsilon^I_{L_I/k,S,T}$, and via the restriction and corestriction maps apply them to the element $\eta_{K/k,S,T}$.

4.2. Proof of Theorem 1.7. In this section we prove Theorem 1.7.

We prove this prime by prime after localisation, i.e. we prove that for every prime p we have

$$e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}_{(p)}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})_{(p)}\big) \subseteq \mathrm{Fitt}^r_{\mathbf{Z}_{(p)}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})_{(p)}\big)\,.$$

We have the following presentation for $\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})$

$$P \xrightarrow{\psi} F \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}) \longrightarrow 0\,.$$

After localisation, we assume, as we may by Lemma 2.12, that $F_{(p)} = P_{(p)}$. Thus we obtain the presentation

$$F_{(p)} \xrightarrow{\psi} F_{(p)} \xrightarrow{\pi} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})_{(p)} \longrightarrow 0\,.$$

The $\mathbf{Z}[G]$-basis $\{b_1, \dots, b_d\}$ for F also gives a $\mathbf{Z}_{(p)}[G]$-basis for $F_{(p)}$, for which we use the same notation. Let $A_{(p)}$ be the matrix corresponding to this presentation.

For any $\sigma \in \mathfrak{S}_{d,r}$, we have that $\det(\psi_i(b_{\sigma(j)}))_{i\notin \bar{I}, r<j\le d}$ is a $(d-r)\times(d-r)$ minor of $A_{(p)}$. Furthermore the proof of Lemma 3.4 gives us that

$$e_I \cdot \det\bigl(\psi_i\bigl(b_{\sigma(j)}\bigr)\bigr)_{i\notin \bar{I}, r<j\le d} = \det\bigl(\psi_i\bigl(b_{\sigma(j)}\bigr)\bigr)_{i\notin \bar{I}, r<j\le d}\,.$$

Since we also have that $e_I \cdot e_{D_I} = e_I$ (see Remark 4.1) it's not hard to see that

$$e_{D_I} \cdot \det\bigl(\psi_i\bigl(b_{\sigma(j)}\bigr)\bigr)_{i\notin \bar{I}, r<j\le d} = \det\bigl(\psi_i\bigl(b_{\sigma(j)}\bigr)\bigr)_{i\notin \bar{I}, r<j\le d}\,.$$

The other $(d-r)\times(d-r)$ minors of $A_{(p)}$ are of the form $\det(\psi_i(b_{\sigma(j)}))_{i\in \bar{J}, r<j\le d}$ for some $\sigma \in \mathfrak{S}_{d,r}$ and some $J \in \wp_{d-r}(V_S)$ such that $I \cap J \neq \emptyset$. In this case we have that $e_{D_I} \cdot \det(\psi_i(b_{\sigma(j)}))_{i\in \bar{J}, r<j\le d} = 0$.

To see this we fix $i \in \bar{I} \cap \bar{J}$. Then $\mathrm{Im}(\psi_i) \subseteq \mathcal{I}_{G_{v_i}}$ (see [5, Lem. 5.19] for proof). Let $x \in \mathcal{I}_{G_{v_i}}$. Then $x = \sum_{\delta\in G_{v_i}} (\delta - 1)\cdot x_\delta$ for some $x_\delta \in \mathbf{Z}_{(p)}[G]$. Then

$$\begin{aligned}
e_{D_I} \cdot x &= \sum_{\delta\in G_{v_i}} e_{D_I} \cdot (\delta - 1) \cdot x_\delta \\
&= \sum_{\delta\in G_{v_i}} \sum_{\substack{\chi\in\hat{G} \\ \chi(G_v)=1, \forall v\in I}} e_\chi \cdot (\delta - 1)\cdot x_\delta \\
&= \sum_{\delta\in G_{v_i}} \sum_{\substack{\chi\in\hat{G} \\ \chi(G_v)=1, \forall v\in I}} e_\chi \cdot \bigl(\chi(\delta) - 1\bigr)\cdot x_\delta \\
&= 0\,,
\end{aligned}$$

where the final line follows since $\chi(\delta) = 1$.

We then obtain our final result since

$$\begin{aligned}
&e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}_{(p)}[G]}\bigl(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})_{(p)}\bigr) \\
&= \sum_{\substack{B\in\{(d-r)\times \\ (d-r)\text{minors of } A_{(p)}\}}} e_{D_I} \cdot \det(B) \cdot \mathbf{Z}_{(p)}
\end{aligned}$$

$$= \sum_{\sigma \in \mathfrak{S}_{d,r}} e_{D_I} \cdot \det\big(\psi_i\big(b_{\sigma(j)}\big)\big)_{i \notin \bar{I}, r<j\le d} \cdot \mathbf{Z}_{(p)}[G]$$

$$+ \sum_{\substack{J \in \wp_{d-r}(V_S) \\ I \cap J \neq \emptyset \\ \sigma \in \mathfrak{S}_{d,r}}} e_{D_I} \cdot \det\big(\psi_i\big(b_{\sigma(j)}\big)\big)_{i \in \bar{J}, r<j\le d} \cdot \mathbf{Z}_{(p)}[G]$$

$$= \sum_{\sigma \in \mathfrak{S}_{d,r}} \det\big(\psi_i\big(b_{\sigma(j)}\big)\big)_{i \notin \bar{I}, r<j\le d} \cdot \mathbf{Z}_{(p)}[G] + 0$$

$$\subseteq \mathrm{Fitt}^r_{\mathbf{Z}_{(p)}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})_{(p)}\big).$$

4.3. The proof of Theorem 1.9. As a first step we prove the following result.

THEOREM 4.5. *With the same set up as before LTC(K/k) implies that*:

$$e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) = \left\{ \Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} (\mathcal{O}^\times_{K,S,T})^* \right\}$$
$$\subseteq \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big).$$

PROOF. Assume $LTC(K/k)$.
We need only prove that

$$e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) = \left\{ \Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^* \right\}$$

since the containment follows from Theorem 1.7, proved above.

Suppose $|S| = r + 1$. Then S must contain at least r places that split completely. In this case the stronger result $\{\Phi(\eta^I_{K/k,S,T}) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]}(\mathcal{O}^\times_{K,S,T})^*\} = \mathrm{Fitt}^r_{\mathbf{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))$ has been proved by Burns et al in [5] so we are done. Therefore we assume that $|S| > r + 1$.

In order to prove the statement we apply the result from Burns et al. to each element $\epsilon^I_{L_I/k,S,T}$ for $I \in \wp_r(V_S)$. We use Lemma 4.3 and the properties of restriction and corestriction maps to apply the result to the elements $\eta^I_{K/k,S,T}$. This will allow us to show that

$$\left\{ \Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^* \right\} = \frac{1}{|D_I|} \mathrm{cor}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\big).$$

First we show that

$$\frac{1}{|D_I|} \mathrm{cor}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\big) \subseteq \left\{ \Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^* \right\}.$$

By Theorem 4.2 we may write any element of $\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I}))$ as $\Psi(\epsilon^I_{L_I/k,S,T})$ for some $\Psi \in \bigwedge^r_{\mathbf{Z}[\Gamma_I]} \mathrm{Hom}_{\mathbf{Z}[\Gamma_I]}(\mathcal{O}^\times_{L_I,S,T}, \mathbf{Z}[\Gamma_I])$. Therefore it is enough to show the inclusion above holds for the element

$$\psi_1 \wedge \cdots \wedge \psi_r\big(\epsilon^I_{L_I/k,S,T}\big) \in \mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)$$

for any $\psi_1, \dots, \psi_r \in \mathrm{Hom}_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{O}^\times_{L_I,S,T}, \mathbf{Z}[\Gamma_I]\big)$.

We apply the corestriction map and then use the properties of the restriction and corestriction maps given in Remark 2.4 and the relation $\eta^I_{K/k,S,T} = \frac{1}{|D_I|^r}\epsilon^I_{L_I/k,S,T}$ to get

$$\begin{aligned}
&\mathrm{cor}_{K/L_I}\big(\psi_1 \wedge \cdots \wedge \psi_r\big(\epsilon^I_{L_I/k,S,T}\big)\big)\\
&= \frac{1}{|D_I|^{r-1}}(\mathrm{cor}_{K/L_I} \circ \psi_1) \wedge \cdots \wedge (\mathrm{cor}_{K/L_I} \circ \psi_r)\big(\epsilon^I_{L_I/k,S,T}\big)\\
&= \frac{1}{|D_I|^{r-1}}(\mathrm{cor}_{K/L_I} \circ \psi_1) \wedge \cdots \wedge (\mathrm{cor}_{K/L_I} \circ \psi_r)\big(|D_I|^r \eta^I_{K/k,S,T}\big)\\
&= |D_I|(\mathrm{cor}_{K/L_I} \circ \psi_1) \wedge \cdots \wedge (\mathrm{cor}_{K/L_I} \circ \psi_r)\big(\eta^I_{K/k,S,T}\big)\\
&= |D_I|(\widetilde{\mathrm{cor}_{K/L_I} \circ} \psi_1) \wedge \cdots \wedge (\widetilde{\mathrm{cor}_{K/L_I} \circ} \psi_r)\big(\eta^I_{K/k,S,T}\big)\\
&\in |D_I| \cdot \left\{\Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^*\right\}
\end{aligned}$$

where $\mathrm{cor}_{K/L_I} \circ \psi_i \in \mathrm{Hom}_{\mathbf{Z}[G]}(\mathcal{O}^\times_{L_I,S,T}, \mathbf{Z}[G])$ for $i = 1, \dots, r$ by Lemma 2.5, and $\widetilde{\mathrm{cor}_{K/L_I} \circ} \psi_i$ is any lift to $(\mathcal{O}^\times_{K,S,T})^*$.

Therefore

$$\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\big) \subseteq \left\{\Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^*\right\}$$

as required.

We now show the inclusion in the other direction. Again it is enough to show that $\phi_1 \wedge \cdots \wedge \phi_r(\eta^I_{K/k,S,T}) \in \frac{1}{|D_I|}\mathrm{cor}_{K/L_I}(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})))$ for arbitrary $\phi_1, \dots, \phi_r \in (\mathcal{O}^\times_{K,S,T})^*$.

Fix $\phi_1, \dots, \phi_r \in (\mathcal{O}^\times_{K,S,T})^*$ and for each i denote the restriction of ϕ_i to $\mathcal{O}^\times_{L_I,S,T}$ by the same symbol ϕ_i. Then Lemma 2.5 gives that

$$\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_i \in \mathrm{Hom}_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{O}^\times_{L_I,S,T}, \mathbf{Z}[\Gamma]\big)$$

for $i = 1, \dots, r$. Therefore by Theorem 4.2

$$\Big(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_1\Big) \wedge \cdots \wedge \Big(\frac{1}{|D_I|}\mathrm{res}_{K/L_I} \circ \phi_r\Big)\big(\epsilon^I_{L_I/k,S,T}\big) \in \mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big).$$

Applying the corestriction map gives

$$\begin{aligned}
&\mathrm{cor}_{K/L_I}\Big(\Big(\frac{1}{|D_I|}\mathrm{res}_{K/L_I}\circ\phi_1\Big)\wedge\cdots\wedge\Big(\frac{1}{|D_I|}\mathrm{res}_{K/L_I}\circ\phi_r\Big)\big(\epsilon^I_{L_I/k,S,T}\big)\Big)\\
&=\frac{1}{|D_I|^{r-1}}\Big(\Big(\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\circ\mathrm{res}_{K/L_I}\circ\phi_1\Big)\wedge\ldots\wedge\Big(\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\circ\mathrm{res}_{K/L_I}\circ\phi_r\Big)\big(\epsilon^I_{L_I/k,S,T}\big)\Big)\\
&=\frac{1}{|D_I|^{r-1}}\Big(\Big(\frac{N_{D_I}}{|D_I|}\cdot\phi_1\Big)\wedge\cdots\wedge\Big(\frac{N_{D_I}}{|D_I|}\cdot\phi_r\Big)\big(\epsilon^I_{L_I/k,S,T}\big)\Big)\\
&=\frac{N_{D_I}}{|D_I|^{r}}\cdot\phi_1\wedge\cdots\wedge\phi_r\big(\epsilon^I_{L_I/k,S,T}\big)\\
&=\frac{1}{|D_I|^{r-1}}\cdot\phi_1\wedge\cdots\wedge\phi_r\big(\epsilon^I_{L_I/k,S,T}\big)\\
&=|D_I|\cdot\phi_1\wedge\cdots\wedge\phi_r\big(\eta^I_{K/k,S,T}\big)\in\mathrm{cor}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\big)\,,
\end{aligned}$$

which gives the required inclusion.

The final requirement that

$$\frac{1}{|D_I|}\mathrm{cor}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\big)=e_{D_I}\cdot\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)$$

follows immediately from Remark 2.4(ii) if we note that

$$\mathrm{res}_{K/L_I}\big(\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\big)=\mathrm{Fitt}^r_{\mathbf{Z}[\Gamma_I]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/L_I})\big)\,.$$

□

LEMMA 4.6.

$$\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)=e_{r,S}\cdot\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\,.$$

PROOF. We prove that for all $x\in\mathrm{Fitt}^r_{\mathbf{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))$ and all χ such that $r_S(\chi)>r$ we have that $xe_\chi=0$. If this holds then

$$x=x\cdot 1=xe_{r,S}+\sum_{\substack{\chi\in\hat{G}\\ r_S(\chi)>r}}xe_\chi=xe_{r,S}\,,$$

so that $\mathrm{Fitt}^r_{\mathbf{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))=e_{r,S}\cdot\mathrm{Fitt}^r_{\mathbf{Z}[G]}(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))$ as required.

Let χ be such that $r_S(\chi)>r$ and consider the surjective ring homomorphism

$$\mathbf{Z}[G]\longrightarrow R_\chi:=e_\chi\cdot\mathbf{Z}[\chi][G]$$

that sends x to $e_\chi\cdot x$.

Under this map the standard properties of Fitting ideals give us that

$$\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\longrightarrow\mathrm{Fitt}^r_{R_\chi}\big(R_\chi\otimes_{\mathbf{Z}[G]}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\,.$$

We show that $\mathrm{Fitt}^r_{R_\chi}(R_\chi \otimes_{\mathbf{Z}[G]} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})) = 0$.

By a result of Northcott (see [9, Ex. 3, p. 61]), and the fact that a Fitting ideal is determined by its localisations, it is enough to show that $R_\chi \otimes_{\mathbf{Z}[G]} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})$ surjects onto a projective R_χ module of rank greater than r. We show this by computing the dimensions over $\mathbf{C}$.

Note that Lemma 1.1 gives

$$\dim_{\mathbf{C}}\big(\mathbf{C}\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}) \cdot e_\chi\big) = \dim_{\mathbf{C}}\big(\mathbf{C}X_{K,S} \cdot e_\chi\big) = r_S(\chi) > r\,.$$

It follows that $R_\chi \otimes_{\mathbf{Z}[G]} \mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K}))$ spans a $\mathbf{C} \simeq \mathbf{C}[G] \cdot e_\chi$-space of dimension greater than r, proving the required result. □

REMARK 4.7. For each $I \in \wp_r(V_S)$ we have that

$$e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) = e_I \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\,.$$

To see this note that $e_{D_I} = e_I + (e_{D_I} - e_I)$ and $e_{D_I} - e_I$ is the sum of idempotents e_χ such that $r_S(\chi) > r$ and $\chi(D_I) = 1$ and follow the argument at the end of the proof of Lemma 4.6 above.

To deduce Theorem 1.9 we now need only make the following computation

$$\begin{aligned}
\mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big) &= e_{r,S} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\\
&= \Bigg(\sum_{I \in \wp_r(V_S)} e_I\Bigg) \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\\
&= \bigoplus_{I \in \wp_r(V_S)} \big(e_I \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\big)\\
&= \bigoplus_{I \in \wp_r(V_S)} \big(e_I \cdot e_{D_I} \cdot \mathrm{Fitt}^r_{\mathbf{Z}[G]}\big(\mathcal{S}^{\mathrm{tr}}_{S,T}(\mathbf{G}_{m/K})\big)\big)\\
&= \bigoplus_{I \in \wp_r(V_S)} \Bigg(e_I \cdot \bigg\{\Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^*\bigg\}\Bigg)\\
&= \bigoplus_{I \in \wp_r(V_S)} \bigg\{\Phi\big(\eta^I_{K/k,S,T}\big) : \Phi \in \bigwedge^r_{\mathbf{Z}[G]} \big(\mathcal{O}^\times_{K,S,T}\big)^*\bigg\}\,.
\end{aligned}$$

Here the first equality is Lemma 4.6 and the second is clear from the definition of the idempotents $e_{r,S}$ and e_I. The third equality follows from the mutual orthogonality of the idempotents e_I, from the equality in Remark 4.7 and from the inclusion in Theorem 1.7. The fourth equality is by Remark 4.1, the fifth by Theorem 4.5 and the final equality since $\eta^I_{K/k,S,T} = e_I \cdot \eta^I_{K/k,S,T}$.

This completes the proof of Theorem 1.9.

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