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Equivariant Bauer-Furuta invariants on Some Connected Sums of 4-manifolds

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Abstract. On some connected sums of 4-manifolds with natural actions of finite groups, we use equivariant Bauer-Furuta invariant to deduce the existence of solutions of Seiberg-Witten equations invariant under the group actions.

For example, for any integer $k \geq 2$ we show that the connected sum of k copies of a 4-manifold M with nontrivial Bauer-Furuta invariant has a nontrivial $\mathbf{Z}_k$-equivariant Bauer-Furuta invariant for the obviously glued Spin^c structure, where the $\mathbf{Z}_k$-action cyclically permutes k summands of M. This contrasts with the fact that ordinary Bauer-Furuta invariants of such connected sums are all trivial for any sufficiently large k, when $b_1(M) = 0$.

1. Introduction

Let M be a smooth closed oriented Riemannian manifold of dimension 4 with a smooth orientation-preserving isometric action of a finite group G. A second cohomology class of M is called a *G-monopole class* if it arises as the first Chern class of a G-equivariant Spin^c structure $\mathfrak{s}$ for which the Seiberg-Witten equations

$$\begin{cases} D_A\Phi = 0 \\ F_A^+ = \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2}\mathrm{Id} \end{cases}$$

admit a G-invariant solution (A, Φ) for every G-invariant Riemannian metric of M.

To detect a G-monopole class, there are two methods developed so far. The first one is a G-monopole invariant obtained by the intersection theory on the G-monopole moduli space, i.e., the space of G-invariant solutions of Seiberg-Witten equations modulo gauge transformations. The second one is G-equivariant Bauer-Furuta invariant, which is basically the $(G \times S^1)$-equivariant stable cohomotopy class of the monopole map between appropriate Hilbert manifolds given by Seiberg-Witten equations, just as the ordinary Bauer-Furuta invariant is the S^1-equivariant stable cohomotopy class of the monopole map.

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While it is difficult to compute those invariants, unless the action is free, we were able to exactly compute G-monopole invariants on some special types of connected sums in [8], and we will compute their G-equivariant Bauer-Furuta invariants in this paper as a sequel. In some cases, those invariants turn out to be nontrivial, although ordinary Bauer-Furuta invariant vanishes.

The existence of a G-monopole class can be applied to Riemannian geometry such as G-invariant Einstein metrics and G-Yamabe invariant of 4-manifolds, and some standard applications are dealt with in [7].

2. Equivariant Bauer-Furuta invariant

Let M be a smooth closed oriented 4-manifold. Suppose that a finite group G acts on M smoothly preserving the orientation, and this action lifts to an action on a Spin^c structure $\mathfrak{s}$ of M. Once there is a lifting, any other lifting differs from it by an element of $Map(G \times M, S^1)$. We fix a lifting and put a G-invariant Riemannian metric on M.

The corresponding spinor bundles $W_\pm$ are also G-equivariant, and we let $\Gamma(W_\pm)^G$ be the set of its G-invariant sections. When we put G as a superscript on the right shoulder of a set, we always mean the subset consisting of its G-invariant elements. Thus $\mathcal{A}(W_+)^G$ is the space of G-invariant connections on $\det(W_+)$, which is identified as the space $\Gamma(\Lambda^1(M))^G$ of G-invariant 1-forms, and $\mathcal{G}_o^G = Map((M, x_0), (S^1, 1))^G$ is the set of G-invariant based gauge transformations for a base point $x_0 \in M$. When M/G is disconnected, more base points should be assigned so that $\mathcal{G}_o^G \subset \mathcal{G}^G$ is a maximal subgroup acting freely on $\mathcal{A}(W_+)^G$. Thus the number of base points is exactly the number of connected components of M/G.

Let $A_0 \in \mathcal{A}(W_+)^G$. Just as

$$Pic(M) := (A_0 + i \ker d)/\mathcal{G}_o$$

for $\mathcal{G}_o := Map((M, x_0), (S^1, 1))$ is a $b_1(M)$-dimensional torus, one gets the quotient

$$Pic^G(M) := (A_0 + i \ker d)^G/\mathcal{G}_o^G\ .$$

Since $\mathcal{G}^G := Map(M, S^1)^G$ is equal to $\mathcal{G}_o^G \times S^1$, and any constant gauge transformation acts trivially on connections, $Pic^G(M)$ is well-defined independently of the choice of the base point x_0.

LEMMA 2.1. *$Pic^G(M)$ is diffeomorphic to a torus $T^{b_1(M)^G}$ of dimension $b_1(M)^G :=$ $\dim H^1(M; \mathbf{R})^G$, and also covers a torus $T^{b_1(M)^G}$ embedded in $Pic(M)$.*

PROOF. Here we need the condition that G is finite. Let $A_0 + \alpha \in (A_0 + i \ker d)^G$. If $\alpha \in \operatorname{Im} d$, namely $\alpha = df$ for some $f \in Map(M, i\mathbf{R})$, then

$$\alpha = df = d\left(\frac{\sum_{h\in G} h^* f}{|G|}\right),$$

and hence $\alpha \in d \ln \mathcal{G}_o^G$.

If $[\alpha]$ defines a nonzero element in $H^1(M; i\mathbf{Z})$, then write $\alpha = d \ln \mathfrak{g}$ for $\mathfrak{g} \in \mathcal{G}_o$, and

$$|G|\alpha = \sum_{h \in G} h^* d \ln \mathfrak{g} = d \ln \prod_{h \in G} h^* \mathfrak{g} \in d \ln \mathcal{G}_o^G .$$

Let $\{[\alpha_i] | i = 1, \ldots, b_1(M)^G\}$ be a basis for a lattice $H^1(M; i\mathbf{Z})^G \simeq \mathbf{Z}^{b_1(M)^G}$. For each $[\alpha_i]$, let n_i be the smallest positive number such that $n_i \alpha_i \in d \ln \mathcal{G}_o^G$. In fact, n_i must be an integer, and we let $n = \prod_{i=1}^{b_1(M)^G} n_i$.

If $b_1(M)^G \neq 0$, then $Pic^G(M)$ is the obvious n-fold covering of the subtorus generated by those $[\alpha_i]$'s, and moreover the subtorus is embedded in $Pic(M) = H^1(M; \mathbf{R})/H^1(M; \mathbf{Z})$, because it is compact, and hence a closed subgroup. If $b_1(M)^G = 0$, then obviously $Pic^G(M)$ is a point embedded in $Pic(M)$. □

Define infinite-dimensional Hilbert bundles $\mathcal{E}^G$ and $\mathcal{F}^G$ over $Pic^G(M)$ by

$$\mathcal{E}^G := \tilde{\mathcal{E}}^G / \mathcal{G}_o^G , \quad \text{and} \quad \mathcal{F}^G := \tilde{\mathcal{F}}^G / \mathcal{G}_o^G ,$$

where

$$\tilde{\mathcal{E}}^G := (A_0 + i \ker d)^G \times (\Gamma(W_+)^G \oplus \Gamma(\Lambda^1 M)^G \oplus H^0(M)^G) ,$$
$$\tilde{\mathcal{F}}^G := (A_0 + i \ker d)^G \times \mathcal{U}^G$$

for

$$\mathcal{U}^G := \Gamma(W_-)^G \oplus \Gamma(\Lambda_+^2 M)^G \oplus \Gamma(\Lambda^0 M)^G \oplus H^1(M)^G ,$$

and $\mathcal{G}_o^G$ are endowed with appropriate Sobolov norms, and $\mathcal{G}_o^G$ acts nontrivially on the connection part $(A_0 + i \ker d)^G$ and the spinor parts. Since $\mathcal{G}_o^G$ actions are free, $\mathcal{E}^G$ and $\mathcal{F}^G$ are smooth Hilbert manifolds still endowed with (non-free) S^1-actions.

The G-monopole map $\mu^G : \mathcal{E}^G \to \mathcal{F}^G$ is an S^1-equivariant continuous fiber-preserving map defined as

$$[A, \Phi, a, f] \mapsto [A, D_{A+ia}\Phi, F_{A+ia}^+ - \Phi \otimes \Phi^* + \frac{|\Phi|^2}{2}\mathrm{Id}, d^*a + f, a^{harm}] ,$$

which is fiberwisely the sum of a linear Fredholm operator denoted by $\mathfrak{L}^G$ and a (quadratic) compact operator. Note that

$$(\mu^G)^{-1}(\text{zero section of } \mathcal{F}^G)/S^1$$

is exactly the G-monopole moduli space. The important property that the inverse image of any bounded set in $\mathcal{F}^G$ is bounded follows directly from the corresponding boundedness property of the ordinary monopole map $\mu : \mathcal{E} \to \mathcal{F}$ with linear part $\mathfrak{L}$.

Expressing the G-monopole map as an S^1-equivariant stable cohomotopy class is almost verbatim the same as ordinary Bauer-Furuta invariant, and we will omit the proof. The virtual index bundle ind $\mathfrak{L}^G$ over $Pic^G(M)$ is

$$\ker(D)^G - \operatorname{coker}(D)^G - \underline{H^2_+(M)^G} \in KO(Pic^G(M))\,,$$

where D is the Spin^c Dirac operator, and $\underline{H^2_+(M)^G}$ is the trivial bundle of rank $b_2^+(M)^G := \dim H^2_+(M;\mathbf{R})^G$. Note that ind $\mathfrak{L}^G$ can be represented as

$$E - F \in KO(Pic^G(M))$$

for some finite-dimensional vector bundles

$$F := Pic^G(M) \times V \subset Pic^G(M) \times \mathcal{U}^G\,, \quad \text{and} \quad E := (\mathfrak{L}^G)^{-1}(F)\,,$$

where we used a Hilbert bundle isomorphism

$$\mathcal{F}^G \simeq Pic^G(M) \times \mathcal{U}^G$$

over a compact space $Pic^G(M)$.

With TH denoting the Thom space of a vector bundle H, define an S^1-equivariant stable cohomotopy group

$$\pi^0_{S^1,\mathcal{U}^G}(Pic^G(M);\, \text{ind}\, \mathfrak{L}^G) \tag{2.1}$$

as

$$\operatorname{colim}_{U \subset \mathcal{U}^G}[S^U \wedge TE\,,\, S^U \wedge TV]^{S^1}\,,$$

where U runs all finite dimensional real vector subspaces of $\mathcal{U}^G$ transversal to V, and $S^U \wedge$ denotes the smash product with the one-point compactification of a vector space U.

Then our G-monopole map gives an element $\overline{BF}^G_{M,\mathfrak{s}}$ in the above stable cohomotopy group, and let us call it "*G-invariant Bauer-Furuta invariant*" of a G-space $(M,\mathfrak{s})$. When G is the trivial group $\{1\}$, $\overline{BF}^{\{1\}}_{M,\mathfrak{s}}$ is just equal to the ordinary Bauer-Furuta invariant $BF_{M,\mathfrak{s}}$ in [1, 2]. Just as $BF_{M,\mathfrak{s}}$, it is also independent of choice of a Riemannian metric on M and a base point x_0. Indeed for a one parameter family of base point x_0, there is an isotopy of $\mathcal{G}^G_o$ in $\mathcal{G}^G$ so that the homotopy class $\overline{BF}^G_{M,\mathfrak{s}}$ remains the same.

Also in the same way as $BF_{M,\mathfrak{s}}$, $\overline{BF}^G_{M,\mathfrak{s}}$ can be viewed as the S^1-equivariant homotopy class of μ^G in the set of the S^1-equivariant continuous fiber-preserving maps which differ from μ^G by the fiberwise compact perturbations and have bounded inverse image for any bounded subset in $\mathcal{F}^G$. (See [3].) An important fact for our purpose is the following:

THEOREM 2.2. *If $\overline{BF}^G_{M,\mathfrak{s}}$ is nontrivial, then $c_1(\mathfrak{s})$ is a G-monopole class.*

PROOF. This is a consequence of facts from functional analysis, and one can take the proof in [5, Proposition 6] verbatim, which proves that $c_1(\mathfrak{s})$ is a monopole class, if ordinary Bauer-Furuta invariant $BF_{M,\mathfrak{s}} \neq 0$. □

The G-equivariant Bauer-Furuta invariant $BF^G_{M,\mathfrak{s}}$ first introduced by M. Szymik [9] (in case of $b_1(M) = 0$) is a little different. (See also [6].) For this we need the condition that $M^G \neq \emptyset$ or $b_1(M) = 0$, which will be always assumed whenever $BF^G_{M,\mathfrak{s}}$ appears.

In the first case, we take the base point x_0 in M^G. Then the induced G-action is well-defined on $\mathcal{E} := \tilde{\mathcal{E}}/\mathcal{G}_o$ and $\mathcal{F} := \tilde{\mathcal{F}}/\mathcal{G}_o$, where

$$\tilde{\mathcal{E}} := (A_0 + i \ker d) \times (\Gamma(W_+) \oplus \Gamma(\Lambda^1 M) \oplus H^0(M))$$

and

$$\tilde{\mathcal{F}} := (A_0 + i \ker d) \times \mathcal{U}$$

for

$$\mathcal{U} := \Gamma(W_-) \oplus \Gamma(\Lambda^2_+ M) \oplus \Gamma(\Lambda^0 M) \oplus H^1(M) .$$

The ordinary monopole map $\mu : \mathcal{E} \to \mathcal{F}$ is $(G \times S^1)$-equivariant, and one takes its class in the $(G \times S^1)$-equivariant stable homotopy group

$$\pi^0_{G\times S^1,\mathcal{U}}(Pic(M); \operatorname{ind} \mathfrak{L}) \tag{2.2}$$

to get $BF^G_{M,\mathfrak{s}}$ by using a trivialization $\mathcal{F} \simeq Pic(M) \times \mathcal{U}$.

If $b_1(M) = 0$, then $Pic(M)$ is a point, and hence regardless of the choice of x_0, $\mathcal{E}$ and $\mathcal{F}$ are canonically isomorphic to $\Gamma(W_+) \oplus \Gamma(\Lambda^1 M) \oplus H^0(M)$ and $\mathcal{U}$ respectively, on which the G-action is well-defined, and hence enables us to get $(G \times S^1)$-equivariant stable homotopy element $BF^G_{M,\mathfrak{s}}$.

There is the obvious forgetful map from (2.2) to

$$\pi^0_{S^1,\mathcal{U}}(Pic(M); \operatorname{ind} \mathfrak{L}) ,$$

under which $BF^G_{M,\mathfrak{s}}$ gets mapped to $BF_{M,\mathfrak{s}}$.

LEMMA 2.3. *Let $\tilde{p}_1 : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}}/\mathcal{G}_o$ and $\tilde{p}_2 : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}}/\mathcal{G}_o$ be the associated quotient maps. If $M^G \neq \emptyset$ or $b_1(M)^G = 0$, then the obvious maps*

$$p_1 : \mathcal{E}^G \to \tilde{p}_1(\tilde{\mathcal{E}}^G) \quad \text{and} \quad p_2 : \mathcal{F}^G \to \tilde{p}_2(\tilde{\mathcal{F}}^G)$$

are bijective, and $Pic^G(M)$ is a submanifold of $Pic(M)$.

PROOF. Since $\mathcal{G}^G_o$ is a subgroup of $\mathcal{G}_o$, p_1 and p_2 are obviously surjective.

To show that p_1 is injective, suppose that $[A_1, \Phi_1, a_1, f_1]$ and $[A_2, \Phi_2, a_2, f_2]$ in $\mathcal{E}^G$ are equivalent under $\gamma \in \mathcal{G}_o$. Then

$$A_1 = A_2 - 2d \ln \gamma\,, \quad \text{and} \quad \Phi_1 = \gamma \Phi_2\,.$$

By the first equality, $d \ln \gamma$ is G-invariant.

Let's first consider the case when $M^G \neq \emptyset$. Let S be the subset of M where γ is G-invariant. By the continuity of γ, S must be a closed subset. Since S contains $M^G \neq \emptyset$, S is nonempty. It suffices to show that S is open. Let $x_0 \in S$. Then we have that for any $g \in G$,

$$g^* \ln \gamma(x_0) = \ln \gamma(x_0)\,, \quad \text{and} \quad g^* d \ln \gamma = d \ln \gamma\,,$$

which implies that $g^* \ln \gamma = \ln \gamma$ on an open neighborhood of x_0 on which $g^* \ln \gamma$ and $\ln \gamma$ are well-defined. By the compactness of G, there exists an open neighborhood of x_0 on which $g^* \ln \gamma$ is well-defined for all $g \in G$, and $\ln \gamma$ is G-invariant. This proves the openness of S.

In case when $b_1(M)^G = 0$, a G-invariant closed 1-form $d \ln \gamma$ can be written as df for $f \in Map(M, i\mathbf{R})$. Again using the compactness of G, $df = d\left(\frac{\sum_{h \in G} h^* f}{|G|}\right)$, and so $\gamma \in \mathcal{G}_o^G$. In the same way, one can show that p_2 is injective.

Now it follows that $Pic^G(M)$ becomes a submanifold of $Pic(M)$. Namely, the n in Lemma 2.1 is 1. □

Thus if $M^G \neq \emptyset$ or $b_1(M) = 0$, then $\mathcal{E}^G$ and $\mathcal{F}^G$ are subsets of $\mathcal{E}$ and $\mathcal{F}$ respectively so that we can think of the restriction of μ to $\mathcal{E}^G$, which is equal to μ^G. Letting ρ be the map from (2.2) to (2.1) induced by restricting to its G-fixed point set, we have:

THEOREM 2.4. *If $M^G \neq \emptyset$ or $b_1(M) = 0$, then*

$$\rho(BF^G_{M,\mathfrak{s}}) = \overline{BF}^G_{M,\mathfrak{s}}\,,$$

and hence when $\overline{BF}^G_{M,\mathfrak{s}}$ is nontrivial, so is $BF^G_{M,\mathfrak{s}}$.

As observed in [9], ρ is not injective in general.

When the G-action on M is free, $\overline{BF}^G_{M,\mathfrak{s}}$ is equal to $BF_{M/G,\mathfrak{s}'}$, where $\mathfrak{s}'$ is the Spin^c structure on M/G induced from $\mathfrak{s}$ and its G-action. Under the further assumption that $b_1(M) = 0$, $|G|$ is prime, and the dimension of Seiberg-Witten moduli space is zero, $BF^G_{M,\mathfrak{s}}$ may be expressed as $BF_{M,\mathfrak{s}}$ and $BF_{M/G,\mathfrak{s}''}$ for all $\mathfrak{s}''$ lifting to $\mathfrak{s}$. (See [9].) In general, it is difficult to compute $BF^G_{M,\mathfrak{s}}$ as well as $BF_{M,\mathfrak{s}}$ itself. Therefore it is quite worthwhile to compute $\overline{BF}^G_{M,\mathfrak{s}}$ when the G-action is not free.

3. Main Theorem

THEOREM 3.1. *Let M and N be smooth closed oriented connected 4-manifolds satisfying $b_2^+(M) > 1$ and $b_2^+(N) = 0$, and $\bar{M}_k$ for any $k \geq 2$ be the connected sum $M\#\ldots\#M\#N$ where there are k summands of M.*

Suppose that a finite group G with $|G| = k$ acts effectively on N in a smooth orientation-preserving way, and that N admits a Riemannian metric of positive scalar curvature invariant under the G-action and a G-equivariant Spin^c structure $\mathfrak{s}_N$ with $c_1^2(\mathfrak{s}_N) = -b_2(N)$.

Define a G-action on $\bar{M}_k$ induced from that of N permuting k summands of M glued along a free orbit in N, and let $\bar{\mathfrak{s}}$ be the Spin^c structure on $\bar{M}_k$ obtained by gluing $\mathfrak{s}_N$ and a Spin^c structure $\mathfrak{s}$ of M.

Then for any G-action on $\bar{\mathfrak{s}}$ induced from the above G-action on $\bar{M}_k$,

$$\overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} = BF_{M,\mathfrak{s}} \wedge \overline{BF}^G_{N,\mathfrak{s}_N} ,$$

and when $b_1(N)^G = 0$,

$$\overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} = BF_{M,\mathfrak{s}} .$$

If $BF_{M,\mathfrak{s}}$ is nontrivial, so is $\overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}}$.

PROOF. Let $\tilde{M}_k = N \cup \amalg_{i=1}^k (M \cup S^4)$ be the disjoint union of N and k-copies of $M \cup S^4$, and endow it with a Spin^c structure $\tilde{\mathfrak{s}}$ which is $\mathfrak{s}_N$ on N, $\mathfrak{s}$ on each M, and the trivial Spin^c structure $\mathfrak{s}_0$ on each S^4. Then $(\tilde{M}_k, \tilde{\mathfrak{s}})$ has an obvious G-action induced from the G-action on $\bar{\mathfrak{s}}$ in a unique way. (Here G acts on $\amalg_{i=1}^k S^4$ by the obvious permutation, and on its Spin^c structure as induced from the action on $\bar{\mathfrak{s}}$ over the cylindrical gluing regions.)

Just as the ordinary monopole maps shown in [2], the stable cohomotopy class of the disjoint union of G-monopole maps is equal to the smash product $\wedge$ of those, and hence

$$\begin{aligned}
\overline{BF}^G_{\tilde{M}_k,\tilde{\mathfrak{s}}} &= \overline{BF}^G_{\amalg_{i=1}^k (M\cup S^4), \amalg_{i=1}^k (\mathfrak{s}\amalg\mathfrak{s}_0)} \wedge \overline{BF}^G_{N,\mathfrak{s}_N} \\
&= BF_{M\cup S^4,\mathfrak{s}\amalg\mathfrak{s}_0} \wedge \overline{BF}^G_{N,\mathfrak{s}_N} \\
&= BF_{M,\mathfrak{s}} \wedge BF_{S^4,\mathfrak{s}_0} \wedge \overline{BF}^G_{N,\mathfrak{s}_N} \\
&= BF_{M,\mathfrak{s}} \wedge \overline{BF}^G_{N,\mathfrak{s}_N} ,
\end{aligned}$$

where we used the fact that $BF_{S^4,\mathfrak{s}_0}$ is just $[id] \in \pi^0_{S^1}(\mathrm{pt}) \cong \mathbf{Z}$, which was shown in [2].

A surgery following S. Bauer [2] turns $\tilde{M}_k$ into the union of $\bar{M}_k$ and k-copies of $S^4 \amalg S^4$. In the notations of [2], for $X = X_1 \cup X_2 \cup X_3 = \tilde{M}_k$, we take

$$\begin{aligned}
X_1 &= N = (N - \amalg_{i=1}^k D^4) \cup (\amalg_{i=1}^k D^4) , \\
X_2 &= \amalg_{i=1}^k M = (\amalg_{i=1}^k D^4) \cup (\amalg_{i=1}^k (M - D^4)) , \\
X_3 &= \amalg_{i=1}^k S^4 = (\amalg_{i=1}^k D^4) \cup (\amalg_{i=1}^k D^4) ,
\end{aligned}$$

and

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},$$

where X_3 is needed to make τ an even permutation so that "the gluing map V" of Hilbert bundles along the necks is well-defined continuously. After interchanging the second half parts of X_i's by τ, we get

$$X^\tau = X_1^\tau \cup X_2^\tau \cup X_3^\tau = \bar{M}_k \cup (\amalg_{i=1}^k S^4) \cup (\amalg_{i=1}^k S^4)$$

as desired.[1]

Most importantly, we perform the above surgery from $\tilde{M}_k$ to $\bar{M}_k \cup \amalg_{i=1}^{2k} S^4$ in a G-invariant way, and also "the gluing map V" from the Hilbert bundles $\mathcal{E}^G$, $\mathcal{F}^G$ on $Pic^G(\tilde{M}_k)$ to the Hilbert bundles on $Pic^G(\bar{M}_k \cup \amalg_{i=1}^{2k} S^4)$ in a G-invariant way. The homotopy of the ordinary monopole map of $\tilde{M}_k$ shown in [2] can also be done in a G-invariant way. Then those G-monopole maps of $\tilde{M}_k$ and $\bar{M}_k \cup \amalg_{i=1}^{2k} S^4$ are conjugate via "the gluing map V" up to G-invariant homotopy. Therefore their stable cohomotopy classes are equal so that

$$\begin{aligned} \overline{BF}^G_{\tilde{M}_k,\tilde{\mathfrak{s}}} &= \overline{BF}^G_{\bar{M}_k \cup \amalg_{i=1}^{2k} S^4, \bar{\mathfrak{s}} \amalg \mathfrak{s}_0} \\ &= \overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} \wedge \overline{BF}^G_{\amalg_{i=1}^{2k} S^4, \mathfrak{s}_0} \\ &= \overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} \wedge BF_{S^4 \amalg S^4, \mathfrak{s}_0} \\ &= \overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} \wedge BF_{S^4,\mathfrak{s}_0} \wedge BF_{S^4,\mathfrak{s}_0} \\ &= \overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}}, \end{aligned}$$

where we again used that $BF_{S^4,\mathfrak{s}_0} = [id]$.

Therefore we obtained

$$\overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}} = BF_{M,\mathfrak{s}} \wedge \overline{BF}^G_{N,\mathfrak{s}_N},$$

and it gets equal to $BF_{M,\mathfrak{s}}$ in case of $b_1(N)^G = 0$ by the following lemma:

LEMMA 3.2. *If $b_1(N)^G = 0$, then $\overline{BF}^G_{N,\mathfrak{s}_N}$ is the class of the identity map*

$$[id] \in \pi^0_{S^1}(pt) \cong \mathbf{Z}.$$

PROOF. The method of proof is basically the same as the ordinary Bauer-Furuta invariant in [2].

First, we need to show that the G-index of the Spin^c Dirac operator is zero. Take a G-invariant metric of positive scalar curvature on N. Using the homotopy invariance of a

[1] The gluing theorem 2.1 of [2] was stated when each X_i is connected with one gluing neck, but the proof also works well without this assumption. For more details, readers are referred to [2].

G-index, we compute the index at a G-invariant connection A_0 whose curvature 2-form is harmonic and hence anti-self-dual.

Applying the Weitzenböck formula with the fact that the scalar curvature of N is positive, and the curvature 2-form is anti-self-dual, we get zero kernel. Now then from the vanishing of the ordinary index given by $(c_1^2 - \tau(N))/8$, the cokernel must be also zero. In particular, we have vanishing of G-invariant kernel and cokernel, implying that the G-index is zero.

Then along with $b_1(N)^G = b_2^+(N)^G = 0$, we conclude that $\overline{BF}^G_{N,\mathfrak{s}_N}$ belongs to $\pi^0_{S^1}(\text{pt})$ which is isomorphic to $\pi^0_{st}(\text{pt}) = \mathbf{Z}$ by the isomorphism induced by restriction to the S^1-fixed point set on which the G-monopole map is just the linear isomorphism:

$$L^2_{m+1}(\Lambda^1 N)^G \times H^0(N)^G \to L^2_m(\Lambda^2_+ N)^G \times L^2_m(\Lambda^0 N)^G \times H^1(N)^G$$
$$(a, c) \mapsto (d^+a, d^*a + c, a^{harm}),$$

because it has no kernel and cokernel. This completes the proof. □

Now let's consider the case of $b_1(N)^G \geq 1$. Again the G-index bundle of the Spin^c Dirac operator over $Pic^G(N)$ is zero so that $\overline{BF}^G_{N,\mathfrak{s}_N}$ belongs to $\pi^0_{S^1}(T^{b_1(N)^G})$.

Following [5], we consider the restriction map

$$\sigma : \pi^0_{S^1}(T^{b_1(N)^G}) \to \pi^0_{S^1}(\text{pt})$$

to the fiber over a point in $Pic^G(N)$. By the same method as the above lemma, $\sigma(\overline{BF}^G_{N,\mathfrak{s}_N})$ is just the identity map. Then the restriction of $\overline{BF}^G_{\tilde{M}_k,\bar{\mathfrak{s}}} = \overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}}$ to

$$Pic^G(\amalg_{i=1}^k (M \cup S^4)) \times \{\text{pt}\} \subset Pic^G(\tilde{M}_k) = Pic^G(\bar{M}_k)$$

is given by

$$BF_{M,\mathfrak{s}} \wedge \sigma(\overline{BF}^G_{N,\mathfrak{s}_N}) = BF_{M,\mathfrak{s}}\,.$$

It is obvious that $\overline{BF}^G_{\bar{M}_k,\bar{\mathfrak{s}}}$ is nontrivial, when $\sigma(\overline{BF}^G_{\tilde{M}_k,\bar{\mathfrak{s}}})$ is nontrivial, which completes the proof. □

COROLLARY 3.3. *Let M be a smooth closed oriented* 4*-manifold with $b_1(M) = 0$ and $\bar{M}_k$ for $k \geq 2$ be the k-fold connected sum $M\#\ldots\#M$. Suppose that $BF_{M,\mathfrak{s}}$ is nontrivial for a Spin^c structure $\mathfrak{s}$, and $\bar{\mathfrak{s}}$ denotes the glued Spin^c structure on $\bar{M}_k$ as before. Then there exists an integer $K > 0$ such that for any integer $k \geq K$, $BF_{\bar{M}_k,\bar{\mathfrak{s}}}$ is trivial but $BF^{\mathbf{Z}_k}_{\bar{M}_k,\bar{\mathfrak{s}}}$ is not trivial for a smooth $\mathbf{Z}_k$-action.*

PROOF. As shown in Theorem 4.4 of [8], S^4 admits a smooth $\mathbf{Z}_k$-action with nonempty fixed point set, which satisfies the conditions for N in Theorem 3.1. Thus by applying the above theorem, we have that $\overline{BF}^{\mathbf{Z}_k}_{\bar{M}_k,\bar{\mathfrak{s}}}$ for any integer $k > 0$ is nontrivial, and so is $BF^{\mathbf{Z}_k}_{\bar{M}_k,\bar{\mathfrak{s}}}$ by Theorem 2.4.

But by the result of [4], there must exist an integer $K > 0$ such that for any integer $k \geq K$, $BF_{\bar{M}_k,\bar{\mathfrak{s}}}$ is trivial. □

REMARK. For example, one can take M in the above corollary to be

$$lX\#m\overline{\mathbf{C}P}_2\,,$$

where X is a K3 surface, and integers l and m satisfy that $1 \leq l \leq 4$, and $m \geq 0$. □

4. Examples of N

More examples of such N in Theorem 3.1 are given in Theorem 4.4 of [8], where those examples have $\mathbf{Z}_k$-actions which are free or have fixed points. Here, we will present some non-cyclic actions.

Take N to be $S^1 \times S^3$, and G to be any finite group with a smooth orientation-preserving effective action on S^3. By the spherical space form conjecture which follows from the geometrization theorem proven by G. Perelman, such G action is conjugate to a subgroup of $SO(4)$, and so S^3 has a G-invariant metric of positive scalar curvature. Since the frame bundle of N is trivial, the G action can be lifted to its spin bundle which is also trivial. Such trivial Spin^c structure obviously satisfy $c_1^2 = -b_2(N) = 0$. One can take those G actions either free or with fixed points.

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