TOKYO J. MATH.
VOL. 39, NO. 2, 2016

Reidemeister Torsion and Dehn Surgery on Twist Knots

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(Communicated by Y. Yokota)

Abstract. We compute the Reidemeister torsion of the complement of a twist knot in S^3 and that of the 3-manifold obtained by a $\frac{1}{q}$-Dehn surgery on a twist knot.

1. Main results

In a recent paper Kitano [Ki1] gives a formula for the Reidemeister torsion of the 3-manifold obtained by a $\frac{1}{q}$-Dehn surgery on the figure eight knot. In this paper we generalize his result to all twist knots. Specifically, we will compute the Reidemeister torsion of the complement of a twist knot in S^3 and that of the 3-manifold obtained by a $\frac{1}{q}$-Dehn surgery on a twist knot.

Let $J(k, l)$ be the knot/link in Figure 1, where k, l denote the numbers of half twists in the boxes. Positive (resp. negative) numbers correspond to right-handed (resp. left-handed) twists. Note that $J(k, l)$ is a knot if and only if kl is even. If kl is odd, then $J(k, l)$ is a two-component link. The knot $J(2, 2n)$, where $n \neq 0$, is known as a twist knot. For more information on $J(k, l)$, see [HS].

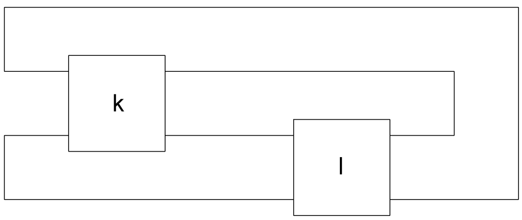

FIGURE 1. The knot/link $J(k, l)$

Received June 16, 2015; revised September 17, 2015
2010 *Mathematics Subject Classification*: 57N10 (Primary), 57M25 (Secondary)
Key words and phrases: Dehn surgery, nonabelian representation, Reidemeister torsion, twist knot

In this paper we fix $K = J(2, 2n)$. Let E_K be the complement of K in S^3. The fundamental group of E_K has a presentation $\pi_1(E_K) = \langle a, b \mid w^n a = b w^n \rangle$ where a, b are meridians and $w = ba^{-1}b^{-1}a$. A representation $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is called nonabelian if the image of ρ is a nonabelian subgroup of $SL_2(\mathbf{C})$. Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ -u & s^{-1} \end{bmatrix},$$

where $(s, u) \in (\mathbf{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s, u)$, see [Ri].

Let $x := \operatorname{tr} \rho(a) = s + s^{-1}$ and $z := \operatorname{tr} \rho(w) = u^2 - (x^2 - 4)u + 2$. Let $S_k(z)$ be the Chebyshev polynomials of the second kind defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = z S_{k-1}(z) - S_{k-2}(z)$ for all integers k.

THEOREM 1. *Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. If $x \neq 2$ then the Reidemeister torsion of E_K is given by*

$$\tau_\rho(E_K) = (2 - x)\frac{S_n(z) - S_{n-2}(z) - 2}{z - 2} + x S_{n-1}(z)\,.$$

Now let M be the 3-manifold obtained by a $\frac{1}{q}$-surgery on the twist knot K. The fundamental group $\pi_1(M)$ has a presentation

$$\pi_1(M) = \langle a, b \mid w^n a = b w^n, a\lambda^q = 1 \rangle\,,$$

where λ is the canonical longitude corresponding to the meridian $\mu = a$.

THEOREM 2. *Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbf{C})$. If $x \notin \{0, 2\}$ then the Reidemeister torsion of M is given by*

$$\tau_\rho(M) = \left((x - 2)\frac{S_n(z) - S_{n-2}(z) - 2}{z - 2} - x S_{n-1}(z)\right)\left(u^{-2}(u + 1)(x^2 - 4) - 1\right)x^{-2}\,.$$

REMARK 1.1. (1) One can see that the expression $(S_n(z) - S_{n-2}(z) - 2)/(z - 2)$ is actually a polynomial in z.

(2) Theorem 2 generalizes the formula for the Reidemeister torsion of the 3-manifold obtained by a $\frac{1}{q}$-surgery on the figure eight knot by Kitano [Ki1].

EXAMPLE 1.2. (1) If $n = 1$, then $K = J(2, 2)$ is the trefoil knot. In this case the Riley polynomial is $\phi_K(s, u) = u - (x^2 - 3)$, and hence

$$\tau_\rho(M) = -2\left(u^{-2}(u + 1)(x^2 - 4) - 1\right)x^{-2} = \frac{2}{x^2(x^2 - 3)^2}\,.$$

(2) If $n = -1$, then $K = J(2, -2)$ is the figure eight knot. In this case the Riley polynomial is $\phi_K(s, u) = u^2 - (u + 1)(x^2 - 5)$, and hence

$$\tau_\rho(M) = (2x - 2)\left(u^{-2}(u + 1)(x^2 - 4) - 1\right)x^{-2} = \frac{2x - 2}{x^2(x^2 - 5)}.$$

The paper is organized as follows. In Section 2 we review the Chebyshev polynomials of the second kind and their properties. In Section 3 we give a formula for the Riley polynomial of a twist knot, and compute the trace of a canonical longitude. In Section 4 we review the Reidemeister torsion of a knot complement and its computation using Fox's free calculus. We prove Theorems 1 and 2 in Section 5.

The author would like to thank the referee for helpful comments and suggestions.

2. Chebyshev polynomials

Recall that $S_k(z)$ are the Chebyshev polynomials defined by $S_0(z) = 1$, $S_1(z) = z$ and $S_k(z) = zS_{k-1}(z) - S_{k-2}(z)$ for all integers k. The following lemma is elementary.

LEMMA 2.1. *One has* $S_k^2(z) - zS_k(z)S_{k-1}(z) + S_{k-1}^2(z) = 1$.

Let $P_k(z) := \sum_{i=0}^{k} S_i(z)$.

LEMMA 2.2. *One has* $P_k(z) = \frac{S_{k+1}(z) - S_k(z) - 1}{z-2}$.

PROOF. We have

$$\begin{aligned} zP_k(z) &= z\sum_{i=0}^{k} S_i(z) = \sum_{i=0}^{k} \left(S_{i+1}(z) + S_{i-1}(z)\right) \\ &= \left(P_k(z) + S_{k+1}(z) - S_0(z)\right) + \left(P_k(z) - S_k(z) + S_{-1}(z)\right) \\ &= 2P_k(z) + S_{k+1}(z) - S_k(z) - 1\,. \end{aligned}$$

The lemma follows. □

LEMMA 2.3. *One has* $P_k^2(z) + P_{k-1}^2(z) - zP_k(z)P_{k-1}(z) = P_k(z) + P_{k-1}(z)$.

PROOF. Let $Q_k(z) = \left(P_k^2(z) + P_{k-1}^2(z) - zP_k(z)P_{k-1}(z)\right) - \left(P_k(z) + P_{k-1}(z)\right)$. We have

$$Q_{k+1}(z) - Q_k(z) = \left(P_{k+1}(z) - P_{k-1}(z)\right)\left(P_{k+1}(z) + P_{k-1}(z) - zP_k(z) - 1\right).$$

Since $zP_k(z) = \sum_{i=0}^{k} \left(S_{i+1}(z) + S_{i-1}(z)\right) = P_{k+1}(z) - 1 + P_{k-1}(z)$, we obtain $Q_{k+1}(z) = Q_k(z)$ for all integers k. Hence $Q_k(z) = Q_1(z) = 0$. □

PROPOSITION 2.4. *Suppose* $V = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbf{C})$. *Then*

$$V^k = \begin{bmatrix} S_k(t) - dS_{k-1}(t) & bS_{k-1}(t) \\ cS_{k-1}(t) & S_k(t) - aS_{k-1}(t) \end{bmatrix}, \tag{2.1}$$

$$\sum_{i=0}^{k} V^i = \begin{bmatrix} P_k(t) - dP_{k-1}(t) & bP_{k-1}(t) \\ cP_{k-1}(t) & P_k(t) - aP_{k-1}(t) \end{bmatrix}, \tag{2.2}$$

where $t := \operatorname{tr} V = a + d$. *Moreover, one has*

$$\det\left(\sum_{i=0}^{k} V^i\right) = \frac{S_{k+1}(t) - S_{k-1}(t) - 2}{t - 2}. \tag{2.3}$$

PROOF. Since $\det V = 1$, by the Cayley-Hamilton theorem we have $V^2 - tV + I = 0$. This implies that $V^k - tV^{k-1} + V^{k-2} = 0$ for all integers k. Hence, by induction on k, one can show that $V^k = S_k(t)I - S_{k-1}(t)V^{-1}$. Since $V^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, (2.1) follows.

Since $P_k(t) = \sum_{i=0}^{k} S_i(t)$, (2.2) follows directly from (2.1). By Lemma 2.3 we have

$$\begin{aligned}
\det\left(\sum_{i=0}^{k} V^i\right) &= P_k^2(t) + (ad - bc)P_{k-1}^2(t) - (a + d)P_k(t)P_{k-1}(t) \\
&= P_k^2(t) + P_{k-1}^2(t) - tP_k(t)P_{k-1}(t) \\
&= P_k(t) + P_{k-1}(t).
\end{aligned}$$

Then (2.3) follows from Lemma 2.2. □

3. Nonabelian representations

In this section we give a formula for the Riley polynomial of a twist knot. This formula was already obtained in [DHY, Mo]. We also compute the trace of a canonical longitude.

3.1. Riley polynomial. Recall that $K = J(2, 2n)$ and $E_K = S^3 \setminus K$. The fundamental group of E_K has a presentation $\pi_1(E_K) = \langle a, b \mid w^n a = bw^n \rangle$ where a, b are meridians and $w = ba^{-1}b^{-1}a$. Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is a nonabelian representation. Up to conjugation, we may assume that

$$\rho(a) = \begin{bmatrix} s & 1 \\ 0 & s^{-1} \end{bmatrix} \quad \text{and} \quad \rho(b) = \begin{bmatrix} s & 0 \\ -u & s^{-1} \end{bmatrix},$$

where $(s, u) \in (\mathbf{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s, u)$.

We now compute $\phi_K(s,u)$. Since

$$\rho(w)=\begin{bmatrix} 1-s^2u & s^{-1}-s-su \\ (s-s^{-1})u+su^2 & 1+(2-s^{-2})u+u^2 \end{bmatrix},$$

by Proposition 2.4 we have

$$\rho(w^n)=\begin{bmatrix} S_n(z)-\big(1+(2-s^{-2})u+u^2\big)S_{n-1}(z) & (s^{-1}-s-su)S_{n-1}(z) \\ \big((s-s^{-1})u+su^2\big)S_{n-1}(z) & S_n(z)-(1-s^2u)S_{n-1}(z) \end{bmatrix},$$

where $z=\operatorname{tr}\rho(w)=2+(2-s^2-s^{-2})u+u^2$. Hence, by a direct computation we have

$$\rho(w^na-bw^n)=\begin{bmatrix} 0 & \phi_K(s,u) \\ u\phi_K(s,u) & 0 \end{bmatrix}$$

where

$$\phi_K(s,u)=S_n(z)-\big(u^2-(u+1)(s^2+s^{-2}-3)\big)S_{n-1}(z)\,.$$

3.2. Trace of the longitude. It is known that the canonical longitude corresponding to the meridian $\mu=a$ is $\lambda=\overleftarrow{w}^n w^n$, where $\overleftarrow{w}$ is the word in the letters a,b obtained by writing w in the reversed order. We now compute its trace. This computation will be used in the proof of Theorem 2.

LEMMA 3.1. *One has* $S_{n-1}^2(z)=\frac{1}{(u+2-s^2-s^{-2})\left(u^2-(s^2+s^{-2}-2)(u+1)\right)}$.

PROOF. Since $(s,u)\in(\mathbf{C}^*)^2$ is a root of the Riley polynomial $\phi_K(s,u)$, we have $S_n(z)=\big(u^2-(u+1)(s^2+s^{-2}-3)\big)\,S_{n-1}(z)$. Lemma 2.1 then implies that

$$\begin{aligned}1&=S_n^2(z)-zS_n(z)S_{n-1}(z)+S_{n-1}^2(z)\\&=\Big(\big(u^2-(u+1)(s^2+s^{-2}-3)\big)^2-z\big(u^2-(u+1)(s^2+s^{-2}-3)\big)+1\Big)S_{n-1}^2(z)\,.\end{aligned}$$

By replacing $z=2+(2-s^2-s^{-2})u+u^2$ into the first factor of the above expression, we obtain the desired equality. □

PROPOSITION 3.2. *One has* $\operatorname{tr}\rho(\lambda)-2=\frac{u^2(s^2+s^{-2}+2)}{(u+1)(s^2+s^{-2}-2)-u^2}$.

PROOF. Since

$$\rho(\overleftarrow{w})=\begin{bmatrix} 1+(2-s^2)u+u^2 & s-s^{-1}-s^{-1}u \\ (s^{-1}-s)u+s^{-1}u^2 & 1-s^{-2}u \end{bmatrix},$$

by Proposition 2.4 we have

$$\rho(\overleftarrow{w}^n)=\begin{bmatrix} S_n(z)-(1-s^{-2}u)S_{n-1}(z) & (s-s^{-1}-s^{-1}u)S_{n-1}(z) \\ \big((s^{-1}-s)u+s^{-1}u^2\big)S_{n-1}(z) & S_n(z)-\big(1+(2-s^2)u+u^2\big)S_{n-1}(z) \end{bmatrix}.$$

Hence, by a direct calculation we have

$$\begin{aligned}\operatorname{tr}\rho(\lambda) &= \operatorname{tr}(\rho(\overleftarrow{w}^n)\rho(w))\\ &= 2S_n^2(z) - 2zS_n(z)S_{n-1}(z) + \big(2 + (s^4 + s^{-4} - 2)u^2 - (s^2 + s^{-2} + 2)u^3\big)S_{n-1}^2(z)\\ &= 2 + u^2(s^2 + s^{-2} + 2)(s^2 + s^{-2} - 2 - u)S_{n-1}^2(z)\,.\end{aligned}$$

The proposition then follows from Lemma 3.1. □

4. Reidemeister torsion

In this section we briefly review the Reidemeister torsion of a knot complement and its computation using Fox's free calculus. For more details on the Reidemeister torsion, see [Jo, Mi1, Mi2, Mi3, Tu].

4.1. Torsion of a chain complex. Let C be a chain complex of finite dimensional vector spaces over $\mathbf{C}$:

$$C = \Big(0 \to C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \to 0\Big)$$

such that for each $i = 0, 1, \dots, m$ the followings hold

- the homology group $H_i(C)$ is trivial, and
- a preferred basis c_i of C_i is given.

Let $B_i \subset C_i$ be the image of ∂_{i+1}. For each i choose a basis b_i of B_i. The short exact sequence of $\mathbf{C}$-vector spaces

$$0 \to B_i \longrightarrow C_i \xrightarrow{\partial_i} B_{i-1} \to 0$$

implies that a new basis of C_i can be obtained by taking the union of the vectors of b_i and some lifts $\tilde{b}_{i-1}$ of the vectors b_{i-1}. Define $[(b_i \cup \tilde{b}_{i-1})/c_i]$ to be the determinant of the matrix expressing $(b_i \cup \tilde{b}_{i-1})$ in the basis c_i. Note that this scalar does not depend on the choice of the lift $\tilde{b}_{i-1}$ of b_{i-1}.

DEFINITION 4.1. The *torsion* of C is defined to be

$$\tau(C) := \prod_{i=0}^{m} [(b_i \cup \tilde{b}_{i-1})/c_i]^{(-1)^{i+1}} \in \mathbf{C} \setminus \{0\}\,.$$

REMARK 4.2. Once a preferred basis of C is given, $\tau(C)$ is independent of the choice of $b_0, \dots, b_m$.

4.2. Reidemeister torsion of a CW-complex. Let M be a finite CW-complex and $\rho : \pi_1(M) \to SL_2(\mathbf{C})$ a representation. Denote by $\tilde{M}$ the universal covering of M. The fundamental group $\pi_1(M)$ acts on $\tilde{M}$ as deck transformations. Then the chain complex $C(\tilde{M}; \mathbf{Z})$ has the structure of a chain complex of left $\mathbf{Z}[\pi_1(M)]$-modules.

Let V be the 2-dimensional vector space $\mathbf{C}^2$ with the canonical basis $\{e_1, e_2\}$. Using the representation ρ, V has the structure of a right $\mathbf{Z}[\pi_1(M)]$-module which we denote by V_ρ. Define the chain complex $C(M; V_\rho)$ to be $C(\tilde{M}; \mathbf{Z}) \otimes_{\mathbf{Z}[\pi_1(M)]} V_\rho$, and choose a preferred basis of $C(M; V_\rho)$ as follows. Let $\{u^i_1, \dots, u^i_{m_i}\}$ be the set of i-cells of M, and choose a lift $\tilde{u}^i_j$ of each cell. Then $\{\tilde{u}^i_1 \otimes e_1, \tilde{u}^i_1 \otimes e_2, \dots, \tilde{u}^i_{m_i} \otimes e_1, \tilde{u}^i_{m_i} \otimes e_2\}$ is chosen to be the preferred basis of $C_i(M; V_\rho)$.

A representation ρ is called *acyclic* if all the homology groups $H_i(M; V_\rho)$ are trivial.

DEFINITION 4.3. The Reidemeister torsion $\tau_\rho(M)$ is defined as follows:

$$\tau_\rho(M) = \begin{cases} \tau(C(M; V_\rho)) & \text{if } \rho \text{ is acyclic}, \\ 0 & \text{otherwise}. \end{cases}$$

4.3. Reidemeister torsion of a knot complement and Fox's free calculus. Let L be a knot in S^3 and E_L its complement. We choose a Wirtinger presentation for the fundamental group of E_L:

$$\pi_1(E_L) = \langle a_1, \dots, a_l \mid r_1, \dots, r_{l-1} \rangle .$$

Let $\rho : \pi_1(E_L) \to SL_2(\mathbf{C})$ be a representation. This map induces a ring homomorphism $\tilde{\rho} : \mathbf{Z}[\pi_1(E_L)] \to M_2(\mathbf{C})$, where $\mathbf{Z}[\pi_1(E_L)]$ is the group ring of $\pi_1(E_L)$ and $M_2(\mathbf{C})$ is the matrix algebra of degree 2 over $\mathbf{C}$. Consider the $(l-1) \times l$ matrix A whose (i, j)-component is the 2×2 matrix

$$\tilde{\rho}\left(\frac{\partial r_i}{\partial a_j}\right) \in M_2(\mathbf{C}),$$

where $\partial/\partial a$ denotes the Fox calculus. For $1 \le j \le l$, denote by A_j the $(l-1) \times (l-1)$ matrix obtained from A by removing the jth column. We regard A_j as a $2(l-1) \times 2(l-1)$ matrix with coefficients in $\mathbf{C}$. Then Johnson showed the following.

THEOREM 4.4 ([Jo]). *Let $\rho : \pi_1(E_L) \to SL_2(\mathbf{C})$ be a representation such that $\det(\tilde{\rho}(a_1) - I) \neq 0$. Then the Reidemeister torsion of E_L is given by*

$$\tau_\rho(E_L) = \frac{\det A_1}{\det(\tilde{\rho}(a_1) - I)} .$$

5. Proof of main results

5.1. Proof of Theorem 1. We will apply Theorem 4.4 to calculate the Reidemeister torsion of the complement E_K of the twist knot $K = J(2, 2n)$.

Recall that $\pi_1(E_K) = \langle a, b \mid w^n a = b w^n \rangle$. We have $\det(\tilde{\rho}(b) - I) = 2 - (s + s^{-1}) = 2 - x$. Let $r = w^n a w^{-n} b^{-1}$. By a direct computation we have

$$\begin{aligned}\frac{\partial r}{\partial a} &= w^n \left(1 + (1-a)(w^{-1} + \cdots + w^{-n})\frac{\partial w}{\partial a}\right) \\ &= w^n \big(1 + (1-a)(1 + w^{-1} + \cdots + w^{-(n-1)})a^{-1}(1-b)\big)\,.\end{aligned}$$

Suppose $x \neq 2$. Then $\det(\tilde{\rho}(b) - I) \neq 0$ and hence

$$\tau_\rho(E_K) = \det\tilde{\rho}\left(\frac{\partial r}{\partial a}\right)\Big/\det(\tilde{\rho}(b) - I) = \det\tilde{\rho}\left(\frac{\partial r}{\partial a}\right)\Big/(2-x)\,.$$

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Let $\Delta = \tilde{\rho}(1 + w^{-1} + \cdots + w^{-(n-1)})$ and $\Omega = \tilde{\rho}\big(a^{-1}(1-b)(1-a)\big)\Delta$. Then

$$\det\tilde{\rho}\left(\frac{\partial r}{\partial a}\right) = \det(I + \Omega) = 1 + \det\Omega + \operatorname{tr}\Omega\,.$$

LEMMA 5.1. *One has* $\det\Omega = (2-x)^2\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right)$.

PROOF. Since $\operatorname{tr}\tilde{\rho}(w^{-1}) = \operatorname{tr}\tilde{\rho}(w) = z$, by Proposition 2.4 we have $\det\Delta = \frac{S_n(z) - S_{n-2}(z) - 2}{z-2}$. The lemma follows, since $\det\Omega = \det\tilde{\rho}(a^{-1}(1-a)(1-b))\det\Delta = (2-x)^2\det\Delta$. □

LEMMA 5.2. *One has* $\operatorname{tr}\Omega = x(2-x)S_{n-1}(z) - 1$.

PROOF. Since $\tilde{\rho}(w^{-1}) = \begin{bmatrix} 1 + (2 - s^{-2})u + u^2 & s - s^{-1} + su \\ (s^{-1} - s)u - su^2 & 1 - s^2 u \end{bmatrix}$, by Proposition 2.4 we have

$$\Delta = \begin{bmatrix} P_{n-1}(z) - (1 - s^2 u)P_{n-2}(z) & (s - s^{-1} + su)P_{n-2}(z) \\ \big((s^{-1} - s)u - su^2\big)P_{n-2}(z) & P_{n-1}(z) - \big(1 + (2 - s^{-2})u + u^2\big)P_{n-2}(z) \end{bmatrix}.$$

By a direct computation we have

$$\tilde{\rho}(a^{-1}(1-b)(1-a)) = \begin{bmatrix} s + s^{-1} - 2 + (s-1)u & s^{-1} - s^{-2} + u \\ su - s^2 u & s + s^{-1} - 2 - su \end{bmatrix}.$$

Hence

$$\begin{aligned}\operatorname{tr}\Omega &= \operatorname{tr}\big(\tilde{\rho}\big(a^{-1}(1-b)(1-a)\big)\Delta\big) \\ &= (2s + 2s^{-1} - 4 - u)P_{n-1}(z) + \big(4 - 2s - 2s^{-1} + (3 - s^2 - s^{-2})u + u^2\big)P_{n-2}(z) \\ &= (2s + 2s^{-1} - 4 - u)\big(P_{n-1}(z) - P_{n-2}(z)\big) + ((2 - s^2 - s^{-2})u + u^2)P_{n-2}(z) \\ &= (2s + 2s^{-1} - 4 - u)S_{n-1}(z) + (z-2)P_{n-2}(z) \\ &= (2s + 2s^{-1} - 4 - u)S_{n-1}(z) + S_{n-1}(z) - S_{n-2}(z) - 1\,.\end{aligned}$$

Since (s,u) satisfies $\phi_K(s,u) = 0$, we have $S_n(z) = \big(u^2 - (u+1)(s^2 + s^{-2} - 3)\big)S_{n-1}(z)$. This implies that $S_{n-2}(z) = zS_{n-1}(z) - S_n(z) = (s^2 + s^{-2} - 1 - u)S_{n-1}(z)$. Hence

$$\operatorname{tr}\Omega = \big(2s + 2s^{-1} - s^2 - s^{-2} - 2\big)S_{n-1}(z) - 1\,.$$

The lemma follows since $2s + 2s^{-1} - s^2 - s^{-2} - 2 = x(2-x)$. $\square$

We now complete the proof of Theorem 1. Lemmas 5.1 and 5.2 imply that

$$\det\tilde{\rho}\left(\frac{\partial r}{\partial a}\right) = 1 + \det\Omega + \operatorname{tr}\Omega = (2-x)^2\left(\frac{S_n(z) - S_{n-2}(z) - 2}{z-2}\right) + x(2-x)S_{n-1}(z)\,.$$

Since $\tau_\rho(E_K) = \det\tilde{\rho}\left(\frac{\partial r}{\partial a}\right)\big/(2-x)$, we obtain the desired formula for $\tau_\rho(E_K)$.

REMARK 5.3. In [Mo], Morifuji proved a similar formula for the twisted Alexander polynomial of twist knots for nonabelian representations.

5.2. Proof of Theorem 2. Suppose $\rho : \pi_1(E_K) \to SL_2(\mathbf{C})$ is a nonabelian representation which extends to a representation $\rho : \pi_1(M) \to SL_2(\mathbf{C})$. Recall that λ is the canonical longitude corresponding to the meridian $\mu = a$. If $\operatorname{tr}\rho(\lambda) \neq 2$, then by [Ki1] (see also [Ki2, Ki3]) the Reidemeister torsion of M is given by

$$\tau_\rho(M) = \frac{\tau_\rho(E_K)}{2 - \operatorname{tr}\rho(\lambda)}\,. \tag{5.1}$$

By Theorem 1 we have $\tau_\rho(E_K) = (2-x)\frac{S_n(z) - S_{n-2}(z) - 2}{z-2} + xS_{n-1}(z)$ if $x \neq 2$. By Proposition 3.2 we have $\operatorname{tr}\rho(\lambda) - 2 = \frac{x^2}{u^{-2}(u+1)(x^2-4) - 1}$. Theorem 2 then follows from (5.1).

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