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On Hausdorff Dimension of Certain Sets Arising from Diophantine Approximations for Complex Numbers

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Abstract. We discuss the Hausdorff dimension of certain sets related to Diophantine approximations over an imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. We show that, for an infinite subset $\mathcal{A}$ of $\mathbb{Z}[\omega]\backslash\{0\}$, the set of $z \in \mathbb{C}$ with $|z - a/r| < 1/|r|^{1+\rho}$ having infinitely many solutions of $a \in \mathbb{Z}[\omega]$ and $r \in \mathcal{A}$ with some $\rho > 0$ has Hausdorff dimension $2(1+\gamma)/(1+\rho)$, where γ is the sup of h such that $\sum_{r\in\mathcal{A}} 1/(|r|^2)^h$ diverges. This result is a version of a result by G. Harman for complex numbers without the coprime condition. In particular, this result implies a version of the classical Jarnik-Besicovitch result when we take $\mathcal{A} = \mathbb{Z}[\omega]\backslash\{0\}$. We also discuss the Hausdorff dimension of the set of complex numbers which have infinitely many solutions to the Diophantine inequality concerning the Duffin-Schaeffer conjecture over $\mathbb{Q}(\sqrt{d})$.

1. Introduction

In the theory of Diophantine approximations, we usually use the Hausdorff dimension to measure the size of the exceptional sets. In 1929, V. Jarnik [5] proved that the Hausdorff dimension of the set of $r \in \mathbb{R}$ such that the inequality

$$\left|r - \frac{m}{n}\right| < \frac{1}{n^q}$$

has infinitely many solutions of rational numbers m/n is $2/q$ for $q > 2$, and also in 1934 A. S. Besicovitch [1] proved the same result. G. Harman [4] then showed a more general result that the Hausdorff dimension of the set of $\alpha \in \mathbb{R}$ such that the inequality $|q\alpha - p| < q^{-\rho}$ with $(p, q) = 1$ and $\gamma = \sup\{0 \leqslant h : \sum_{n\in\mathcal{A}} n^{-h}$ diverges$\}$ for some infinite set $\mathcal{A}$ of positive integers has infinitely many solutions of rational numbers p/q, equals to $(1+\gamma)/(1+\rho)$. We note that V. Jarnik and A. S. Besicovitch's results can be followed as its corollary. G. Harman also proved that the Hausdorff dimension of the set of real numbers those have infinitely many solutions to the Diophantine inequality concerning the Duffin-Schaeffer conjecture [3] is 1 by using this result.

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In this paper, we show a similar result for the imaginary quadratic fields. For a given square-free negative integer d, we define

$$\omega = \begin{cases} (1+\sqrt{d})/2, & \text{if } d \equiv 1 \pmod 4, \\ \sqrt{d}, & \text{if } d \equiv 2, 3 \pmod 4, \end{cases}$$

and denote by $\mathbb{Z}[\omega]$ the ring of integers of $\mathbb{Q}(\sqrt{d})$. In order to avoid the problem of different prime factor decompositions of an integer in $\mathbb{Z}[\omega]$, we consider ideals for the uniqueness of the prime factor decomposition.

We define the set

$$\Sigma = \left\{(a, r) : a, r \in \mathbb{Z}[\omega], r \neq 0, \frac{a}{r} \in \mathbb{F}\right\}$$

and

$$\mathbb{F} = \{x + y\omega : x, y \in \mathbb{R}, 0 \leqslant x, y < 1\}.$$

Our main result is the following, which is a complex number version of Theorem 10.6 in [4].

THEOREM 1. *For an infinite subset $\mathcal{A}$ of $\mathbb{Z}[\omega]\backslash\{0\}$, let*

$$\nu = \sup\left\{h \geqslant 0 : \sum_{r \in \mathcal{A}} \left(\frac{1}{|r|^2}\right)^h = \infty\right\}.$$

For a real number ρ with $\rho > \nu$, define the set

$$D = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| < |r|^{-(1+\rho)} \ has\ infinitely\ many\ (a, r) \in \Sigma\ with\ r \in \mathcal{A}\right\}.$$

Then we have $\dim_H D = \frac{2(1+\nu)}{1+\rho}$.

If the class number of $\mathbb{Q}(\sqrt{d})$ is 1 and $\mathcal{A} = \mathbb{Z}[\omega]\backslash\{0\}$, then we have $\nu = 1$ and we see that for any $z \in D$ there exist infinitely many pairs of a and r in $\mathbb{Z}[\omega]$ with $r \neq 0$ such that $|z - a/r| < |r|^{-(1+\rho)}$ holds and $(a, r) = (1)$, where $(a, r) = (1)$ means that the ideals (a) and (r) are coprime. This is because of the following: (i) if $a'/r' = a/r$, $|z - a'/r'| < |r'|^{-(1+\rho)}$ and $|r'| > |r|$ hold, then $|z - a/r| < |r|^{-(1+\rho)}$ also holds; (ii) there are at most finitely many pairs of a' and r' with $a'/r' = a/r$ such that $|z - a'/r'| < |r'|^{-(1+\rho)}$ holds. Thus, in this case, there is no difference between the inequality with and without the coprime condition on a and r. This situation is the same as V. Jarnik and A. S. Besicovitch's result for real numbers. However, it seems to be not obvious if the class number is not 1.

COROLLARY 1. *Suppose that the class number of $\mathbb{Q}(\sqrt{d})$ is 1 and put*

$$D_0 = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| < |r|^{-(1+\rho)} \ has\ infinitely\ many\ (a, r) \in \Sigma\ with\ (a, r) = (1)\right\}.$$

then $\dim_H D_0 = \frac{4}{1+\rho}$ *for* $\rho > 1$.

We also consider the set of solutions related to the Duffin-Schaeffer conjecture for complex numbers from Theorem 1. Following the Duffin-Schaeffer conjecture for real numbers in [3], we can state it as follows in the case of complex numbers (see [2]).

CONJECTURE 1 (a complex version of the Duffin-Schaeffer conjecture). *Suppose that* $\Psi((r))$ *is a non-negative function with*

$$\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Phi((r))\frac{\Psi^2((r))}{|r|^2}$$

diverges. Define the set

$$D_1 = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|} \; has \; infinitely \; many \; (a,r) \in \Sigma \; with \; (a,r) = (1) \right\} .$$

Then D_1 *has full Lebesgue measure in* $\mathbb{F}$.

Here $\Phi((r))$ is the Euler function for ideals, that is, it denotes the number of reduced residue classes modulo (r). Toward this conjecture, we show the following theorem without assuming $(a,r) = (1)$.

THEOREM 2. *Suppose that* $\Psi((r))$ *is a non-negative function such that*

$$\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Phi((r))\frac{\Psi^2((r))}{|r|^2}$$

diverges. Define the set

$$D_2 = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|} \; has \; infinitely \; many \; (a,r) \in \Sigma \right\} .$$

Then we have $\dim_H D_2 = 2$.

REMARK 1. Recently, the author proved that if $\Psi((r)) = \mathcal{O}(|r|^{-1})$ then D_1 has full Lebesgue measure (see [2]). The author believes that Theorems 1 and 2 hold with the coprime condition $(a,r) = (1)$. However, the distribution of $\frac{a}{r}$ with $(a,r) = (1)$ in the fundamental region $\mathbb{F}$ is not uniform for some $r \in \mathbb{Z}[\omega]\setminus\{0\}$ and this fact makes difficulty to prove them.

REMARK 2. In 1991, H. Nakada and G. Wagner [7] showed Gallagher's 0-1 laws over the complex numbers, that is, either D_1 or its complement is a set of Lebesgue measure 0 even if

$$\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Psi^2((r)) = \infty . \tag{1}$$

If $\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Phi((r))\Psi^2((r))|r|^{-2} < \infty$, then the normalized Lebesgue measure of D_1 is 0 due to the Borel-Cantelli lemma. We can not ignore the possibility that the measure of the set D_2 equals to 0 under the condition (1). In the last section of this paper, by following an idea of Duffin and Schaeffer [3], we construct a counter example by giving a sequence of $\Psi((r))$ which satisfies (1) but the measure of D_2 under our choice of $\{\Psi((r))\}$ is not 1.

2. Proof of the results

First, we define closed disc $\mathcal{I}$ in the complex plane with

$$\mathcal{I} = \left\{ z \in \mathbb{C} : \left| z - \frac{a}{r} \right| \leqslant \delta \right\} ,$$

where δ is a positive real number and $a, r \in \mathbb{Z}[\omega]$ with $r \neq 0$. We adopt the following as the definition of the Hausdorff dimension of a subset of complex numbers (see [1] and G. Harman [4] chapter 10).

DEFINITION 1. Suppose that D is a set of complex numbers. The Hausdorff dimension of D is equal to d ($\dim_H D = d$) if it satisfies the next two conditions:
(1) For any $\beta > d$ and any $\varepsilon > 0$, there exist a sequence of closed discs in the complex plane of $\{\mathcal{I}_j\}_{j=1}^{\infty}$, such that
(a) $D \subset \cup_{j=1}^{\infty} \mathcal{I}_j$,
(b) $\displaystyle\sum_{j=1}^{\infty} (\mathrm{diam}(\mathcal{I}_j))^{\beta} < 1$, where $\mathrm{diam}(\cdot)$ denotes the diameter of the closed disc,
(c) $\mathrm{diam}(\mathcal{I}_j) < \varepsilon$, for any $j \in \mathbb{N}$.
(2) For any $\beta < d$, there exists $\varepsilon > 0$, such that there is no sequence of closed discs in the complex plane satisfies all of the above (a), (b) and (c).

Before we prove Theorem 1, we first give two lemmas which will be used later.

Let δ be a positive real number. For any $a, r \in \mathbb{Z}[\omega]$ with $r \neq 0$, put

$$\mathcal{I}_0(a, r, \delta) := \left\{ z \in \mathbb{C} : \left| z - \frac{a}{r} \right| \leqslant \delta \right\} .$$

Moreover, for any $r \in \mathbb{Z}[\omega]$ with $r \neq 0$ and any closed disc $\mathcal{I}$ in $\mathbb{C}$, we denote by $N(r, \mathcal{I})$ (resp. $N'(r, \mathcal{I})$) the number of $a \in \mathbb{Z}[\omega]$ satisfying $\mathcal{I}_0 \cap \mathcal{I} \neq \phi$ (resp. $\mathcal{I}_0(a, r, \delta) \subset \mathcal{I}$).

LEMMA 1. *Let $\mathcal{I}$ be a closed disc with diameter ζ and δ, η real numbers with $0 < \delta < \frac{1}{4}\zeta$ and $0 < \eta < 1$. Then there exist positive constants $c_1(d, \eta)$, $c_2(d, \eta)$ and $R_0(d, \eta)$, depending only on d and η, satisfying the following: for any $r \in \mathbb{Z}[\omega] \backslash \{0\}$ with $\zeta > |r|^{\eta - 1}$ and $|r| > R_0(d, \eta)$, we have*

$$N(r, \mathcal{I}) \leqslant c_1(d, \eta) \zeta^2 |r|^2 ,$$

$$N'(r, \mathcal{I}) \geqslant c_2(d, \eta) \zeta^2 |r|^2 .$$

PROOF. We only consider the case of $d \equiv 1 \pmod 4$. In fact, we can prove the case of $d \equiv 2, 3 \pmod 4$ in the same way. Suppose $z_0 \in \mathbb{C}$ is the center of $\mathcal{I}$ i.e. $\mathcal{I} = \{z \in \mathbb{C} :$

$|z - z_0| \leqslant \frac{\zeta}{2}\}$. If $\mathcal{I}_0(a, r, \delta)$ intersects $\mathcal{I}$, then we consider the bigger disc $\mathcal{I}' = \{z \in \mathbb{C} : |z - z_0| \leqslant \frac{\zeta}{2} + \delta\}$ and count the number of lattice points of $a \in \mathbb{Z}[\omega]$ with $a/r \in \mathcal{I}'$ for a fixed $r \in \mathbb{Z}[\omega]\backslash\{0\}$ to estimate $N(r, \mathcal{I})$. Let $c_1(d) = \frac{\sqrt{9-d}}{2}$ be the diameter and $c_2(d) = \frac{\sqrt{-d}}{2}$ be the area of the parallelgram $\mathbb{F}$. Then we have

$$\begin{aligned} N(r, \mathcal{I}) &\leqslant \frac{\pi\left(\left(\frac{\zeta}{2} + \delta\right) + \frac{c_1(d)}{|r|}\right)^2}{\frac{c_2(d)}{|r|^2}} \\ &\leqslant \frac{\pi}{c_2(d)}(\zeta|r| + c_1(d))^2 \\ &= \frac{\pi}{c_2(d)}(\zeta^2|r|^2 + 2c_1(d)\zeta|r| + c_1^2(d)) . \end{aligned}$$

Since $\zeta > |r|^{\eta-1}$, $\zeta^{-1}|r|^{-1} \to 0$ as $|r|$ tends to ∞. So we have that for $|r| > R_0(d, \eta)$ with some large $R_0(d, \eta)$, there is some $c_1(d, \eta) > 0$ such that

$$N(r, \mathcal{I}) \leqslant c_1(d, \eta)\zeta^2|r|^2 .$$

Similarly we count the number of lattice points in a smaller disc to estimate $N'(r, \mathcal{I})$ as follows:

$$\begin{aligned} N'(r, \mathcal{I}) &\geqslant \frac{\pi\left(\left(\frac{\zeta}{2} - \delta\right) - \frac{c_1(d)}{|r|}\right)^2}{\frac{c_2(d)}{|r|^2}} \\ &\geqslant \frac{\pi}{c_2(d)}\left(\frac{\zeta}{4}|r| - c_1(d)\right)^2 \\ &= \frac{\pi}{c_2(d)}\left(\frac{\zeta^2}{16}|r|^2 - \frac{c_1(d)}{2}\zeta|r| + c_1^2(d)\right) . \end{aligned}$$

So for $|r| > R_0(d, \eta)$, there is some $c_2(d, \eta) > 0$ such that

$$N'(r, \mathcal{I}) \geqslant c_2(d, \eta)\zeta^2|r|^2 .$$

□

The next lemma gives the estimate of the number of two different closed discs which intersect each other described in Lemma 1.

LEMMA 2. *Given a positive integer Q. For $\delta > 0$ and $a, r \in \mathbb{Z}[\omega]$ which $r \neq 0$ and $a/r \in \mathbb{F}$, put*

$$\mathcal{I}(a, r, \delta) = \left\{z \in \mathbb{F} : \left|z - \frac{a}{r}\right| \leqslant \delta\right\} .$$

Consider

$$\mathcal{G} = \{\mathcal{I}(a, r, \delta) : (a, r) \in \Sigma, r \in \mathcal{C}\}$$

for any subset $\mathcal{C}$ of $\{r \in \mathcal{A} : |r|^2 \in (0, Q]\}$, where $\mathcal{A}$ is any infinite subset of $\mathbb{Z}[\omega]\backslash\{0\}$. Then there is some constant $k'(d) > 0$ depending on d such that

$$\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G}\\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1\right) \leqslant 4N_d k'(d)\delta^2 Q^2 |\mathcal{C}|^2 \tag{2}$$

where N_d is the number of units of $\mathbb{Q}(\sqrt{d})$.

PROOF. We have

$$\begin{aligned}\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G}\\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1\right) &\leqslant \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,b\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<\left|\frac{a}{r}-\frac{b}{s}\right|\leqslant 2\delta}} 1 = \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,b\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|as-br|\leqslant\delta|rs|}} 1 \\ &\leqslant \sum_{r,s\in\mathcal{C}} \sum_{\substack{a,k\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|k|\leqslant 2\delta Q\\ as\equiv k \pmod{(r)}}} 1\,,\end{aligned} \tag{3}$$

for $k = as - br$. Let $U = (r, s)$ and then there are ideals R' and S' such that $(r) = UR'$ and $(s) = US'$. First, we consider the number of k with

$$U \mid (k) \ \text{ and } \ 1 \leqslant |k|^2 \leqslant 4\delta^2 Q^2\,. \tag{4}$$

Let us denote by $T(t)$ the number of ideals whose norms are smaller than or equal to $t > 0$ and by $\mathrm{N}(\cdot)$ the norm of ideal. Put $(k) = UU'$ with an ideal U', then the number of (k) which satisfies (4) equals to the number of U' with $\mathrm{N}(U') \leqslant 4\delta^2 Q^2/\mathrm{N}(U)$ which is smaller than $T(4\delta^2 Q^2/\mathrm{N}(U))$. Fix one $k \in \mathbb{Z}[\omega]$ which satisfies (4) and suppose that $a_0, b_0 \in \mathbb{Z}[\omega]$ and $a_1, b_1 \in \mathbb{Z}[\omega]$ are two different pairs of integers with $k = a_0 s - b_0 r = a_1 s - b_1 r$. Then we have $(a_0 - a_1)S' = (b_0 - b_1)R'$ which shows that a_0 and a_1 are in the same residue class modulo the ideal R'. Since the number of residue class modulo the ideal R' is $\mathrm{N}(R')$ and the number of $a \in \mathbb{Z}[\omega]$ with $a/r \in \mathbb{F}$ is $|r|^2$ and these integers a are all in different residue classes modulo the ideal (r), the number of pairs of $a, b \in \mathbb{Z}[\omega]$ with $k = as - br$ is $|r|^2\mathrm{N}^{-1}(R') = \mathrm{N}(U)$ for fixed $k \in \mathbb{Z}[\omega]$. Thus we have

$$\begin{aligned}\sum_{\substack{a,k\in\mathbb{Z}[\omega]\\ \frac{a}{r},\frac{b}{s}\in\mathbb{F}\\ 0<|k|\leqslant 2\delta Q\\ as\equiv k \pmod{(r)}}} 1 &\leqslant N_d \cdot T\left(\frac{4\delta^2 Q^2}{\mathrm{N}(U)}\right) \cdot \mathrm{N}(U) \leqslant 4N_d k'(d)\frac{\delta^2 Q^2}{\mathrm{N}(U)} \cdot \mathrm{N}(U) \\ &= 4N_d k'(d)\delta^2 Q^2\,,\end{aligned}$$

with some $k'(d) > 0$. Note that N_d is always a constant. The constant $k'(d)$, depending on d, exists since the number of units in an imaginary quadratic field is finite and the sequence $\{T(n)/n\}$ converges to some constant depending on d by Theorem 1.114 in [6]. By the above result and inequality (3) we have

$$\left(\sum_{\substack{\mathcal{I},\mathcal{J}\in\mathcal{G} \\ \mathcal{I}\neq\mathcal{J},\mathcal{I}\cap\mathcal{J}\neq\phi}} 1 \right) \leqslant \sum_{r,s\in\mathcal{C}} 4N_d k'(d)\delta^2 Q^2 = 4N_d k'(d)\delta^2 Q^2 |\mathcal{C}|^2 .$$

$\square$

Now we will give the proof of Theorem 1.

PROOF OF THEOREM 1. First, we show (1) of Definition 1 holds for the set D. For any $\beta > 2(1+\nu)/(1+\rho)$ and any $\varepsilon > 0$, we can choose a sufficiently large $X > 0$ with

$$\frac{2}{(X)^{\frac{1+\rho}{2}}} < \varepsilon \quad \text{and} \quad \sum_{\substack{r\in\mathcal{A} \\ |r|^2>X}} \frac{2^\beta}{(|r|^2)^{(\frac{\rho\beta+\beta}{2}-1)}} < 1 .$$

This is possible since $(\rho\beta+\beta)/2 - 1 > \nu$, which means

$$\sum_{r\in\mathcal{A}} \left(\frac{1}{|r|^2}\right)^{\frac{\rho\beta+\beta}{2}-1} < \infty .$$

We denote by $\{\mathcal{I}_1, \mathcal{I}_2, \ldots\}$ the collection of the discs of the form $\mathcal{I}_0(a, r, |r|^{-1-\rho})$, where $a \in \mathbb{Z}[\omega]$, $r \in \mathcal{A}$, $|r|^2 > X$, and $a/r \in \mathbb{F}$. Then the set D can be covered by the union of $\{\mathcal{I}_j\}_{j=1}^{\infty}$ and this satisfies condition (a) in Definition 1. Next, we have

$$\sum_{j=1}^{\infty} \big(\operatorname{diam}(\mathcal{I}_j)\big)^\beta = \sum_{\substack{(a,r)\in\Sigma \\ r\in\mathcal{A} \\ |r|^2>X}} \left(\frac{2}{|r|^{1+\rho}}\right)^\beta = \sum_{\substack{r\in\mathcal{A} \\ |r|^2>X}} \frac{2^\beta}{(|r|^2)^{\frac{\rho\beta+\beta}{2}-1}} < 1 ,$$

which satisfies condition (b) in Definition 1. Condition (c) holds for our choice of the closed discs with $|r|^2 > X$, which satisfies

$$\operatorname{diam}(\mathcal{I}_j) = \frac{2}{|r|^{1+\rho}} < \frac{2}{X^{\frac{1+\rho}{2}}} < \varepsilon$$

for all $j \in \mathbb{N}$. Thus we see that the set D satisfies (1) of Definition 1, i.e., $\dim_H D \leqslant 2(1+\nu)/(1+\rho)$ holds.

Next, we show that the set D satisfies (2) of Definition 1, i.e., $\dim_H D \geqslant 2(1+\nu)/(1+\rho)$.

Pick some g with $0 \leqslant g \leqslant \nu$ such that

$$\sum_{r \in \mathcal{A}} (|r|^2)^{-g} = \infty\,.$$

Then there are infinitely many integers of K satisfying

$$\sum_{\substack{r \in \mathcal{A} \\ \frac{1}{2}K \leqslant |r|^2 < K}} 1 > \frac{K^g}{\log^2 K}\,. \tag{5}$$

We show this by a contradiction. Suppose there are only finitely many rational integers of $\{K_1, K_2, \ldots, K_N\}$ which satisfies (5) with some $N \in \mathbb{N}$. Let $\frac{1}{2}K_0 = \max(K_1, K_2, \ldots, K_N)$, then we have

$$\sum_{\substack{r \in \mathcal{A} \\ |r|^2 < \frac{1}{2}K_0}} \left(\frac{1}{|r|^2}\right)^g < \infty\,.$$

For any $K \geqslant K_0$ we have

$$\sum_{\substack{r \in \mathcal{A} \\ \frac{1}{2}K \leqslant |r|^2 < K}} 1 \leqslant \frac{K^g}{\log^2 K}\,.$$

This shows

$$\begin{aligned}
\sum_{\substack{r \in \mathcal{A} \\ |r|^2 \geqslant \frac{1}{2}K_0}} \left(\frac{1}{|r|^2}\right)^g &= \sum_{\substack{r \in \mathcal{A} \\ \frac{1}{2}K_0 \leqslant |r|^2 < K_0}} \frac{1}{|r|^{2g}} + \sum_{\substack{r \in \mathcal{A} \\ K_0 \leqslant |r|^2 < 2K_0}} \frac{1}{|r|^{2g}} + \cdots \\
&\leqslant \left(\frac{2}{K_0}\right)^g \frac{(K_0)^g}{\log^2(K_0)} + \left(\frac{1}{K_0}\right)^g \frac{(2K_0)^g}{\log^2(2K_0)} + \cdots \\
&= 2^g \left(\frac{1}{\log^2(K_0)} + \frac{1}{\log^2(2K_0)} + \frac{1}{\log^2(2^2K_0)} + \cdots\right) \\
&= 2^g \sum_{m=0}^{\infty} \frac{1}{(k_0 + mk')^2} < \infty
\end{aligned} \tag{6}$$

with $k_0 = \log(K_0)$ and $k' = \log 2$. Hence we have

$$\sum_{r \in \mathcal{A}} (|r|^2)^{-g} < \infty\,,$$

which gives the contradiction.

Next, let $\beta < 2(1+g)/(1+\rho)$ and choose $\eta > 0$ for Lemma 1 with

$$\eta \leqslant \min\left(\frac{1}{4}(\rho - g), \frac{1}{4}\left(\frac{1+g}{1+\rho} - \frac{\beta}{2}\right)\right).$$

Choose a sequence of integers of $\{K_j\}_{j=0}^{\infty}$ satisfying the following conditions:

$$\begin{cases} \text{(i) } K_0 = 1, \\ \text{(ii) } K_1 > \max\{2R_0^2(d,\eta),\ (4N_d k'(d))^{\frac{1}{2\eta}},\ 2\cdot 4^{\frac{1}{1-\eta}},\ (\frac{8}{2^{\eta}c_2(d,\eta)})^{\frac{1}{\eta}},\ 64^{\frac{1}{1+\rho}}\}, \\ \text{(iii) } 2\log^2(2|r|^2) < |r|^{2\eta} \text{ for all } r \in \mathbb{Z}[\omega]\backslash\{0\} \text{ with } |r|^2 \geqslant K_1, \\ \text{(iv) } K_j^{1-\eta} > K_{j-1}^{1+\rho} \text{ and } K_j > 4K_{j-1} \text{ for all } j \geqslant 1, \\ \text{(v) } \displaystyle\sum_{\substack{r\in\mathcal{A} \\ \frac{1}{2}K_j \leqslant |r|^2 < K_j}} 1 > \frac{(K_j)^g}{\log^2(K_j)} \text{ and } (K_j)^g\left(1 - \frac{1}{\log^2 K_j}\right) \geqslant 2 \text{ for all } j \geqslant 1, \end{cases} \tag{7}$$

where $c_2(d,\eta)$ and $R_0(d,\eta)$ are from Lemma 1 and $k'(d)$ is the constant from Lemma 2. Let $D' = D \cap \mathbb{F}'$, where $\mathbb{F}'$ is a subset of $\mathbb{F}$ defined by

$$\mathbb{F}' = \left\{z \in \mathbb{C} : \left|z - \frac{1+\omega}{2}\right| \leqslant \frac{1}{4}\right\}.$$

Since $\dim_H D' \leqslant \dim_H D$, it is enough to show that $\dim_H D' \geqslant \frac{2(1+\nu)}{1+\rho}$ by checking (2) of Definition 1. Put $\varepsilon = 2K_2^{-1/2}$ and we will show that for any sequence of closed discs of $\{\mathcal{I}_j\}_{j=1}^{\infty}$ which satisfies conditions (b) and (c) in Definition 1 does not satisfy (a), that is, if

$$\sum_{j=1}^{\infty}(\mathrm{diam}(\mathcal{I}_j))^{\beta} < 1 \tag{8}$$

and

$$\mathrm{diam}(\mathcal{I}_j) < \varepsilon = 2\left(\frac{1}{K_2}\right)^{\frac{1}{2}} \text{ for all } j \in \mathbb{N}$$

hold, then $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. We construct a collection of nested sets $\{\mathcal{J}_j\}_{j=1}^{\infty}$ with $\mathcal{J}_1 \supset \mathcal{J}_2 \supset \mathcal{J}_3 \supset \cdots$ so that $\mathcal{J} = \cap_{j=1}^{\infty}\mathcal{J}_j \subset D'$ and $\mathcal{J} \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. Then we have $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$ which completes our proof.

To do this, we define a sequence of positive real numbers $\{\varepsilon_j\}_{j=0}^{\infty}$ with $\varepsilon_j = 2(K_j)^{-\frac{1+\rho}{2}}$ for any $j \geqslant 0$. We construct the nested sets $\{\mathcal{J}_j\}_{j=1}^{\infty}$ by induction such that it satisfies the following four properties:

(P1) $\mathcal{J}_j$ is a union of M_j disjoint closed discs with diameters $\varepsilon_j = 2(K_j)^{-\frac{1+\rho}{2}}$.
(P2) For any $\mathcal{I}_m$ with diameter between ε_j and ε_{j-1}, we have $\mathcal{I}_m \cap \mathcal{J}_j = \phi$.
(P3) For any $z \in \mathcal{J}_j$, there exist $a \in \mathbb{Z}[\omega]$ and $r \in \mathcal{A}$ with $(1/2)K_j \leqslant |r|^2 < K_j$ such that

$|z - a/r| \leqslant (K_j)^{-\frac{1+\rho}{2}}$ with $\frac{a}{r} \in \mathbb{F}'$;

(P4) $M_j \geqslant (K_j)^{1+g-2\eta}$.

By (P3), we have $\mathcal{J} \subset D'$. Since $\mathcal{J}_j$ is compact for all $j \in \mathbb{N}$, $\mathcal{J} = \cap_{j=1}^{\infty} \mathcal{J}_j \neq \phi$. By (P2), for any $a \in \mathcal{J}$ we have $a \notin \mathcal{I}_j$ for all $j \in \mathbb{N}$, so $a \notin \cup_{j=1}^{\infty} \mathcal{I}_j$ which shows $\mathcal{J} \not\subset \cup_{j=1}^{\infty} \mathcal{I}_j$. Thus it is enough to construct $\{\mathcal{J}_j\}_{j=1}^{\infty}$ with the above four properties to show $D' \not\subset \cup_{j=1}^{\infty} \mathcal{I}_j$.

By (7), we can choose a set $\mathcal{C}_1 \subset \{r \in \mathcal{A} : K_1/2 \leqslant |r|^2 < K_1\}$ such that

$$\frac{(K_1)^g}{\log^2 K_1} \leqslant |\mathcal{C}_1| \leqslant (K_1)^g \tag{9}$$

where $|\mathcal{C}_1|$ denotes the cardinal of the set $\mathcal{C}_1$. Then we construct $\mathcal{J}_1$ by using the closed discs centered at $a/r \in \mathbb{F}'$ with $r \in \mathcal{C}_1$ and their radius are $\varepsilon_1/2$ which are wholly within $\mathbb{F}'$. By Lemma 1, the number of closed discs we could choose is more than $\frac{c_2(d,\eta)}{4} \sum_{r \in \mathcal{C}_1} |r|^2$. It's obvious that these closed discs all satisfy the property (P3). By the choice of ε we have chosen, they also satisfy the property (P2). By Lemma 2 for $\delta = \varepsilon_1/2$, the number of pairs of discs intersect to each other is at most $4N_d k'(d)(K_1)^{1-\rho}|\mathcal{C}_1|^2$. Remove one disc from each pairs of discs intersect to each other and denote by M_1 the number of the left closed discs such that property (P1) holds. Now we confirm that M_1 satisfies the property (P4). Indeed we have

$$\begin{aligned} M_1 &\geqslant \frac{c_2(d,\eta)}{4} \sum_{r \in \mathcal{C}_1} |r|^2 - 4N_d k'(d)(K_1)^{1-\rho}|\mathcal{C}_1|^2 \\ &> \frac{c_2(d,\eta)}{4} \sum_{r \in \mathcal{C}_1} 2(|r|^2)^{1-\eta} \log^2(2|r|^2) - 4N_d k'(d)(K_1)^{-2\eta}(K_1)^{1+g-2\eta} \\ &> \frac{2^\eta c_2(d,\eta)}{4}(K_1)^{1-\eta} \log^2(K_1)|\mathcal{C}_1| - (K_1)^{1+g-2\eta} \\ &> \left(\frac{2^\eta c_2(d,\eta) K_1^\eta}{4} - 1 \right) (K_1)^{1+g-2\eta} \\ &> (K_1)^{1+g-2\eta}. \end{aligned}$$

The above discussion implies that $\mathcal{J}_1$ can actually be constructed. Suppose $\mathcal{J}_j$ has already been constructed and now we will construct $\mathcal{J}_{j+1}$. Similarly to the choice of $\mathcal{C}_1$, we can find $\mathcal{C}_{j+1} \subset \{r \in \mathcal{A} : K_{j+1}/2 \leqslant |r|^2 < K_{j+1}\}$ which satisfies

$$\frac{(K_{j+1})^g}{\log^2 K_{j+1}} \leqslant |\mathcal{C}_{j+1}| \leqslant (K_{j+1})^g. \tag{10}$$

We only use the closed discs of $\{z \in \mathbb{C} : |z - a/r| \leqslant \varepsilon_{j+1}/2\}$ with $a/r \in \mathbb{F}'$ and $r \in \mathcal{C}_{j+1}$ which are wholly within $\mathcal{J}_j \subset \mathcal{J}_1 \subset \mathbb{F}'$ to construct $\mathcal{J}_{j+1}$ so that $\mathcal{J}_{j+1}$ satisfies (P3). The steps of our construction of $\mathcal{J}_{j+1}$ are as follows:

(step 1) Choose all the closed discs $\{z \in \mathbb{F}' : |z - a/r| \leqslant \varepsilon_{j+1}/2\}$ which are wholly within

$\mathcal{J}_j$.
(step 2) Remove the closed discs which intersect to each other such that all the left closed discs are all disjoint.
(step 3) Remove all the closed discs which intersect some closed discs in $\{\mathcal{I}_j\}_{j=1}^{\infty}$ whose diameter is between ε_{j+1} and ε_j.
(step 4) Confirm the number of closed discs, that is, whether $M_{j+1} \geqslant (K_{j+1})^{1+g-2\eta}$ or not.
(step 5) If (step 4) satisfies property (P4), then define $\mathcal{J}_{j+1}$ as the union of the left closed discs.

Let $\zeta = \varepsilon_j$ and $\delta = \varepsilon_{j+1}/2 = (K_{j+1})^{-\frac{1+\rho}{2}}$. By our choice of $\{K_j\}$ in (7) we have $\delta < (4K_j)^{-\frac{1+\rho}{2}} < (1/4)\varepsilon_j = (1/4)\zeta$. From our choice of K_j in (7) with $(K_{j+1})^{1-\eta} > (K_j)^{1+\rho}$, the number of closed discs which are wholly within $\mathcal{J}_j$ is more than

$$c_2(d,\eta) M_j \varepsilon_j^2 \sum_{r \in \mathcal{C}_{j+1}} |r|^2 \tag{11}$$

by using Lemma 1. By Lemma 2 for $\delta = \varepsilon_{j+1}/2$, we have that the number of pairs of closed discs which intersect to each other is less than

$$4N_d k'(d) \left(\frac{\varepsilon_{j+1}}{2}\right)^2 (K_{j+1})^2 |\mathcal{C}_{j+1}|^2 = 4N_d k'(d)(K_{j+1})^{1-\rho} |\mathcal{C}_{j+1}|^2 . \tag{12}$$

Define

$$\mathcal{F}_j = \{\mathcal{I} \in \{\mathcal{I}_j\}_{j=1}^{\infty} : \varepsilon_{j+1} \leqslant \operatorname{diam}(\mathcal{I}) < \varepsilon_j\},$$

and put

$$\mathcal{F}_j^{(1)} = \left\{\mathcal{I} \in \mathcal{F}_j : 2\left(\frac{1}{K_{j+1}}\right)^{\frac{1-\eta}{2}} \leqslant \operatorname{diam}(\mathcal{I}) < \varepsilon_j\right\},$$

$$\mathcal{F}_j^{(2)} = \left\{\mathcal{I} \in \mathcal{F}_j : \varepsilon_{j+1} \leqslant \operatorname{diam}(\mathcal{I}) < 2\left(\frac{1}{K_{j+1}}\right)^{\frac{1-\eta}{2}}\right\}.$$

By Lemma 1, we have that the number of closed discs in $\mathcal{J}_{j+1}$ which intersect some closed discs in $\mathcal{F}_j$ is less than

$$\sum_{\mathcal{I} \in \mathcal{F}_j^{(1)}} \sum_{r \in \mathcal{C}_{j+1}} c_1(d,\eta)(\operatorname{diam}(\mathcal{I}))^2 |r|^2 + \sum_{\mathcal{I} \in \mathcal{F}_j^{(2)}} \sum_{r \in \mathcal{C}_{j+1}} \left(\frac{5}{2}(\operatorname{diam}(\mathcal{I}) + \varepsilon_{j+1})|r|\right)^2 \tag{13}$$

for some $c_1(d,\eta) > 0$. From (8) we see

$$\sum_{\mathcal{I} \in \mathcal{F}_j^{(1)}} c_1(d,\eta)(\operatorname{diam}(\mathcal{I}))^2 \sum_{r \in \mathcal{C}_{j+1}} |r|^2 = \sum_{\mathcal{I} \in \mathcal{F}_j^{(1)}} c_1(d,\eta)(\operatorname{diam}(\mathcal{I}))^{2-\beta} (\operatorname{diam}(\mathcal{I}))^{\beta} \sum_{r \in \mathcal{C}_{j+1}} |r|^2$$

$$\leqslant c_1(d,\eta)(\varepsilon_j)^{2-\beta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right).$$

Since

$$(\varepsilon_j)^{2-\beta}=2^{2-\beta}\left(\frac{1}{K_j}\right)^{\frac{(1+\rho)}{2}(2-\beta)}<4\left(\frac{1}{K_j}\right)^{\frac{(1+\rho)}{2}(2-\beta)-4\eta\rho}\leqslant 4\left(\frac{1}{K_j}\right)^{\rho-g+4\eta},$$

we have

$$\sum_{\mathcal{I}\in\mathcal{F}_j^{(1)}}c_1(d,\eta)(\mathrm{diam}(\mathcal{I}))^2\sum_{r\in\mathcal{C}_{j+1}}|r|^2<4c_1(d,\eta)(K_j)^{g-\rho-4\eta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right). \tag{14}$$

The estimate of the second sum in (13) is

$$\begin{aligned}\sum_{\mathcal{I}\in\mathcal{F}_j^{(2)}}\sum_{r\in\mathcal{C}_{j+1}}\left(\frac{5}{2}(\mathrm{diam}(\mathcal{I})+\varepsilon_{j+1})|r|\right)^2&<100\left(\frac{1}{K_{j+1}}\right)^{\frac{(1-\eta)}{2}(2-\beta)}\sum_{r\in\mathcal{C}_{j+1}}|r|^2\\&<100\left(\frac{1}{K_{j+1}}\right)^{3\eta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right),\end{aligned} \tag{15}$$

since

$$\frac{(1-\eta)}{2}(2-\beta)-3\eta=1-4\eta-\frac{\beta}{2}+\frac{\beta\eta}{2}\geqslant\frac{\rho-g}{1+\rho}+\frac{\beta\eta}{2}>0.$$

Finally, we estimate M_{j+1} in (step 4). From (11), (12), (14), and (15) we have

$$\begin{aligned}M_{j+1}\geqslant{}&c_2(d,\eta)M_j\varepsilon_j^2\sum_{r\in\mathcal{C}_{j+1}}|r|^2-4N_dk'(d)(K_{j+1})^{1-\rho}|\mathcal{C}_{j+1}|^2\\&-4c_1(d,\eta)(K_j)^{g-\rho-4\eta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right)-100(K_{j+1})^{-3\eta}\left(\sum_{r\in\mathcal{C}_{j+1}}|r|^2\right).\end{aligned} \tag{16}$$

By (7) and $4\eta\leqslant\rho-g$, we have

$$\begin{aligned}&4N_dk'(d)(K_{j+1})^{1-\rho}|\mathcal{C}_{j+1}|^2\\&\leqslant 4N_dk'(d)(K_{j+1})^{1-\rho}(K_{j+1})^{2g}\\&\leqslant 4N_dk'(d)(K_{j+1})^{1+g-4\eta}\\&=4N_dk'(d)(K_{j+1})^{g}(K_{j+1})^{1-\eta}(K_{j+1})^{-3\eta}\\&=4N_dk'(d)\frac{1}{2}\cdot 2^{1-\eta}\cdot(K_{j+1})^{-3\eta}\frac{(K_{j+1})^g}{\log^2(K_{j+1})}\cdot 2\left(\frac{1}{2}K_{j+1}\right)^{1-\eta}\log^2(K_{j+1})\end{aligned}$$

$$< 4N_d k'(d)(K_{j+1})^{-3\eta} \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) . \tag{17}$$

From (16) and (17), we get

$$\begin{aligned} &M_{j+1} \\ &> \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) \Big(c_2(d,\eta) M_j \varepsilon_j^2 - 4c_1(d,\eta)(K_j)^{g-\rho-4\eta} - (4N_d k'(d) + 100)(K_{j+1})^{-3\eta} \Big) \\ &\geqslant \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) \Big(4c_2(d,\eta)(K_j)^{g-\rho-2\eta} - 4c_1(d,\eta)(K_j)^{g-\rho-4\eta} - (4N_d k'(d) + 100)(K_{j+1})^{-3\eta} \Big) \end{aligned}$$

Here, we can add some more conditions to our choice of $\{K_j\}$ for all $j \geqslant 1$:

$$K_1 > \left(\frac{2c_1(d,\eta)}{c_2(d,\eta)} + (2N_d k'(d) + 50) \right)^{\frac{1}{\eta}} , \tag{18}$$

$$K_j > \left(\frac{(K_{j-1})^{\rho+2\eta-g}}{2c_2(d,\eta)} \right)^{\frac{1}{\eta}} . \tag{19}$$

By (19), we have

$$(K_{j+1})^{-3\eta} < (K_j)^{-2\eta} \cdot (K_{j+1})^{-\eta} < (K_j)^{-2\eta} \cdot 2c_2(d,\eta)(K_j)^{g-\rho-2\eta}$$

and then we see

$$\begin{aligned} M_{j+1} &> \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) 2c_2(d,\eta)(K_j)^{g-\rho-2\eta} \left(2 - \left(\frac{2c_1(d,\eta)}{c_2(d,\eta)} + (2N_d k'(d) + 50) \right) (K_j)^{-2\eta} \right) \\ &> \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) 2c_2(d,\eta)(K_j)^{g-\rho-2\eta} . \end{aligned}$$

Since

$$\begin{aligned} \left(\sum_{r \in \mathcal{C}_{j+1}} |r|^2 \right) &> \sum_{r \in \mathcal{C}_{j+1}} 2 \cdot (|r|^2)^{1-\eta} \log^2(2|r|^2) \\ &\geqslant \frac{(K_{j+1})^g}{\log^2(K_{j+1})} \cdot 2 \left(\frac{1}{2} K_{j+1} \right)^{1-\eta} \log^2(K_{j+1}) \\ &= 2^\eta (K_{j+1})^{1-\eta+g} \end{aligned}$$

and by (19), we have

$$(K_j)^{g-\rho-2\eta} > \frac{(K_{j+1})^{-\eta}}{2c_2(d,\eta)}.$$

This gives

$$\begin{aligned} M_{j+1} &> 2^{\eta}(K_{j+1})^{1-\eta+g} \cdot 2c_2(d,\eta) \cdot \frac{(K_{j+1})^{-\eta}}{2c_2(d,\eta)} \\ &= 2^{\eta}(K_{j+1})^{1+g-2\eta} > (K_{j+1})^{1+g-2\eta} \end{aligned}$$

which satisfies the property (P4). So we can actually construct $\mathcal{J}_{j+1}$ from $\mathcal{J}_j$. By this construction, we have $D' \not\subset \cup_{j=1}^{\infty}\mathcal{I}_j$. Thus we see that $\dim_H D' \geqslant \frac{2(1+\nu)}{1+\rho}$, which completes the proof of Theorem 1. □

Next, we give the proof of Theorem 2 by using Theorem 1.

PROOF OF THEOREM 2. From Theorem 1.1 in [2] we see if $\Psi((r)) = \mathcal{O}(|r|^{-1})$ then D_2 has the full Lebesgue measure which also means $\dim_H D_2 = 2$. Thus it is enough to only consider the case of $\Psi((r)) = \mathcal{O}(|r|^{-1})$ doesn't hold, i.e., there are infinitely many $r \in \mathbb{Z}[\omega]\backslash\{0\}$ such that $\Psi((r)) > |r|^{-1}$. Let's define

$$\hat{\Psi}((r)) = \begin{cases} \Psi((r)), & \text{if } \Psi((r)) > |r|^{-1}, \\ 0, & \text{otherwise,} \end{cases}$$

and put $\mathcal{A}' = \{r \in \mathbb{Z}[\omega]\backslash\{0\} : \hat{\Psi}((r)) \neq 0\}$. If $\sum_{r\in\mathcal{A}'} \Phi((r))\hat{\Psi}^2((r))|r|^{-2}$ converges, then $\sum_{r\notin\mathcal{A}'} \Phi((r))\Psi^2((r))|r|^{-2}$ diverges. By Theorem 1.1 in [2] again, the Hausdorff dimension of the set D_2 is 2 for the sequence $\{\Psi((r))\}$. Now let's consider the case of $\sum_{r\in\mathcal{A}'} \Phi((r))\hat{\Psi}^2((r))|r|^{-2}$ diverges. In this case, it is enough to prove it with $\{\hat{\Psi}((r))\}$ instead of $\{\Psi((r))\}$.

We restrict $\hat{\Psi}((r)) \leqslant 1$ for all $r \in \mathbb{Z}[\omega]\backslash\{0\}$ without loss of generality. For any given $\varepsilon > 0$, let

$$\mathcal{A}(m) = \{r \in \mathcal{A}' : |r|^{-(m+1)\varepsilon} < \hat{\Psi}((r)) \leqslant |r|^{-m\varepsilon}\}$$

for $0 \leqslant m < [\varepsilon^{-1}]$ and put

$$\mathcal{A}([\varepsilon^{-1}]) = \{r \in \mathcal{A}' : |r|^{-1} < \hat{\Psi}((r)) \leqslant |r|^{-[\varepsilon^{-1}]\varepsilon}\}.$$

Since

$$\sum_{r\in\mathcal{A}'} \hat{\Psi}^2((r)) = \sum_{m=0}^{\left[\frac{1}{\varepsilon}\right]} \sum_{r\in\mathcal{A}(m)} \hat{\Psi}^2((r)) = \infty,$$

there is at least one m with $0 \leqslant m \leqslant [\varepsilon^{-1}]$ such that

$$\sum_{r \in \mathcal{A}(m)} \hat{\Psi}^2((r)) = \infty \tag{20}$$

with $|\mathcal{A}(m)| = \infty$. By (20) and $(|r|^2)^{-m\varepsilon} \leqslant 1$, there exists a sequence of $\{\mathcal{B}_n\}$ of pairwise disjoint nonempty subsets of $\mathcal{A}(m)$ satisfying the following conditions:
(1) $\mathcal{A} = \cup_{n=1}^{\infty} \mathcal{B}_n$.
(2) Let $n \leqslant n'$ be any positive integers. Then, for any $r \in \mathcal{B}_n$ and $r' \in \mathcal{B}_{n'}$, we have $|r| \leqslant |r'|$.
(3) For any positive integer n, we have

$$1 \leqslant \sum_{r \in \mathcal{B}_n} \left(\frac{1}{|r|^2}\right)^{m\varepsilon} \leqslant 2\,.$$

For any $n \in \mathbb{N}$, put $\eta_n = 2^{-n}$. Then there exists $k_n \in \mathbb{N}$ such that

$$\sum_{r \in \mathcal{B}_k} \left(\frac{1}{|r|^2}\right)^{m\varepsilon+\eta_n} < \frac{1}{2^{n-1}} \tag{21}$$

holds for any $k \geqslant k_n$. So we have a sequence $\{k_n\}$ with $k_1 < k_2 < k_3 < \cdots$ which satisfies (21). Put $\mathcal{B} = \cup_{j=1}^{\infty} \mathcal{B}_{k_j}$, then $\mathcal{B}$ is an infinite subset of $\mathcal{A}(m)$ and obviously satisfies

$$\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{m\varepsilon} = \infty\,.$$

For any $h > m\varepsilon$, there exists some $n_0 \in \mathbb{N}$ with $h > m\varepsilon + \eta_n$ for all $n \geqslant n_0$, which shows

$$\begin{aligned}
\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{h} &< \sum_{r \in \cup_{j=1}^{n_0-1} B_{k_j}} \left(\frac{1}{|r|^2}\right)^{h} + \sum_{j=n_0}^{\infty} \sum_{r \in \mathcal{B}_{k_j}} \left(\frac{1}{|r|^2}\right)^{m\varepsilon+\eta_j} \\
&< \sum_{r \in \cup_{j=1}^{n_0-1} B_{k_j}} \left(\frac{1}{|r|^2}\right)^{h} + \sum_{j=n_0}^{\infty} \frac{1}{2^{j-1}} < \infty\,.
\end{aligned}$$

Thus $\mathcal{B}$ is an infinite subset of $\mathcal{A}(m)$ satisfies

$$\begin{cases} \displaystyle\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{h} = \infty, & \text{if } h \leqslant m\varepsilon, \\ \displaystyle\sum_{r \in \mathcal{B}} \left(\frac{1}{|r|^2}\right)^{h} < \infty, & \text{if } h > m\varepsilon. \end{cases}$$

Let

$$D_2' = \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{1}{|r|^{1+(m+1)\varepsilon}} \ has\ infinitely\ many\ (a, r) \in \Sigma\ with\ r \in \mathcal{B} \right\}.$$

Then we have $\dim_H D_2 \geqslant \dim_H D_2'$ since $D_2' \subset D_2$. Let $\nu = m\varepsilon$ and $\rho = (m+1)\varepsilon$. By Theorem 1 we see

$$2 \geqslant \dim_H D_2 \geqslant \dim_H D_2' = \frac{2(1+\nu)}{1+\rho} = \frac{2(1+m\varepsilon)}{1+m\varepsilon+\varepsilon} > 2 - 2\varepsilon\,.$$

Since $\varepsilon > 0$ is arbitrary, we have $\dim_H D_2 = 2$. □

3. An example

In this section, we show a counter example stated in Remark 2 following the example of [3]. We denote by λ the normalized Lebesgue measure of $\mathbb{F}$, i.e., $\lambda(\mathbb{F}) = 1$, and give a sequence $\{\Psi((r))\}$ with $\sum_{r\in\mathbb{Z}[\omega]\setminus\{0\}} \Psi^2((r)) = \infty$ such that $\lambda(D_2) < 1$. First, we give the complex version of Lemma V in [3] as follows:

LEMMA 3. *Let R and ε be given positive numbers. There is an infinite sequence $\{\Psi((r))\}$ of non-negative numbers with $\Psi((r)) = 0$ for all but finitely many r such that*

$$\sum \Psi^2((r)) > 1, \quad \sum \Phi((r))\frac{\Psi^2((r))}{|r|^2} < c_d\varepsilon, \quad \Psi((r)) = 0 \text{ whenever } |r| \leqslant R\,,$$

where c_d is some constant depending on d, but for $z \in \mathbb{F}$ the inequality

$$\left|z - \frac{a}{r}\right| < \frac{\Psi((r))}{|r|}$$

for some $a, r \in \mathbb{Z}[\omega]$ can be satisfied only in a set of λ-measure smaller than ε.

PROOF. Let N_d be the number of units of the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$. Fix some $\alpha > 0$ with $\alpha < \frac{\sqrt{-d}}{2N_d k'(d)\pi}\varepsilon$ and we can choose prime numbers $p_1, p_2, \ldots, p_k$ such that

$$\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right) > 1 + \frac{1}{N_d\alpha}$$

where $p_i > R$ for $1 \leqslant i \leqslant k$, since $\displaystyle\sum_{p:prime}\frac{1}{p}$ diverges. Denote by (u) a principal ideal as $(u) = (p_1)(p_2)\cdots(p_k) = \displaystyle\prod_{P|(u)} P^c$ where P denotes the prime ideal and $c \geqslant 0$. Note that here we do not need (p_i) are all prime ideals, and for any ideal U with $U \mid (u)$ it can be denoted by $U = \displaystyle\prod_{P|U} P^{c'}$ with $0 \leqslant c' \leqslant c$. We define $\Psi((r))$ as follows:

$$\Psi((r)) = \begin{cases} \frac{\alpha^{1/2}|r|^{1/2}}{|u|^{1/2}}, & \text{if } |r| > 1 \text{ and } (r) \mid (u) \\ 0, & \text{otherwise.} \end{cases}$$

Define the set

$$E_{(r)} = \bigcup_{\substack{|a|^2 \leqslant |r|^2 \\ a \in \mathbb{Z}[\omega]}} \left\{ z \in \mathbb{F} : \left| z - \frac{a}{r} \right| < \frac{\Psi((r))}{|r|} \right\}$$

and put

$$E = \bigcup_{\substack{(r)|(u) \\ (r) \neq (1)}} E_{(r)} .$$

Since $E_{(r)} \subset E_{(u)}$ for all (r) with $(r)|(u)$, we have

$$\lambda(E) = \lambda(E_{(u)}) \leqslant \pi \frac{N_d \alpha}{|u|^2} \cdot \frac{2}{\sqrt{-d}} \cdot k'(d)|u|^2 = \frac{2N_d k'(d)\pi}{\sqrt{-d}} \alpha < \varepsilon .$$

Also we have

$$\begin{aligned} \sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} \Psi^2((r)) = \frac{\alpha}{|u|} \sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} |r| \geqslant N_d \frac{\alpha}{|u|} \left(\prod_{i=1}^{k} (1 + p_i) - 1 \right) \\ \geqslant N_d \alpha \left(\prod_{i=1}^{k} \left(1 + \frac{1}{p_i} \right) - 1 \right) > 1 \end{aligned}$$

and

$$\begin{aligned} \sum_{\substack{r \in \mathbb{Z}[\omega] \setminus \{0\} \\ (r)|(u) \\ (r) \neq (1)}} \Phi((r)) \frac{\Psi^2((r))}{|r|^2} &= \frac{N_d \alpha}{|u|} \sum_{\substack{(r)|(u) \\ (r) \neq (1)}} \frac{\Phi((r))}{|r|} \leqslant \frac{N_d \alpha}{|u|} \sum_{\substack{U: ideals \\ U|(u)}} \frac{\Phi(U)}{(\mathrm{N}(U))^{1/2}} \\ &= \frac{N_d \alpha}{|u|} \sum_{\substack{U: ideals \\ U|(u)}} \frac{1}{(\mathrm{N}(U))^{1/2}} \prod_{P|U} \Phi(P^{c'}) \\ &= \frac{N_d \alpha}{|u|} \prod_{P|(u)} \left(1 + \frac{\Phi(P)}{(\mathrm{N}(P))^{1/2}} + \frac{\Phi(P^2)}{(\mathrm{N}(P^2))^{1/2}} + \cdots + \frac{\Phi(P^c)}{(\mathrm{N}(P^c))^{1/2}} \right) \\ &< \frac{2N_d \alpha}{|u|} \prod_{P|(u)} (\mathrm{N}(P))^{c/2} = 2N_d \alpha < \frac{\sqrt{-d}}{k'(d)\pi} \varepsilon . \end{aligned}$$

Thus, we see that the sequence $\{\Psi((r))\}$ with $\Psi((r))$ defined above is the required finite sequence. □

Now let $R_1 = 1$ and we have a sequence $\{\Psi^{(1)}((r))\}$ which satisfies Lemma 3 with $R = R_1$ and $\varepsilon = 2^{-1}$. Then for some R_2 with $\Psi^{(1)}((r)) = 0$ for all $|r| \geqslant R_2$, let $R = R_2$ and $\varepsilon = 2^{-2}$ and we have another sequence $\{\Psi^{(2)}((r))\}$ which satisfies Lemma 3. We do this process infinitely many times and obtain infinitely many sequences of $\{\Psi^{(1)}((r))\}, \{\Psi^{(2)}((r))\}, \ldots, \{\Psi^{(n)}((r))\}, \ldots$. Let $\Psi((r)) = \sum_{k=1}^{\infty} \Psi^{(k)}((r))$ for all $r \in \mathbb{Z}[\omega]\backslash\{0\}$, then we see

$$\sum_{r \in \mathbb{Z}[\omega]\backslash\{0\}} \Psi^2((r)) = \infty ,$$

whereas

$$\sum_{r \in \mathbb{Z}[\omega]\backslash\{0\}} \Phi((r)) \frac{\Psi^2((r))}{|r|^2} < \infty .$$

However, λ-measure of the set of $z \in \mathbb{F}$ satisfies inequality $|z - a/r| < \Psi((r))/|r|$ is smaller than 1 by our choice of $\{\Psi((r))\}$, which means $\lambda(D_2) < 1$. Thus even $\sum_{r \in \mathbb{Z}[\omega]\backslash\{0\}} \Psi^2((r)) = \infty$, we cannot ignore the possibility of the case $\lambda(D_2) = 0$, and from our choice of $\{\Psi((r))\}$ we see $\sum \Phi((r))\Psi^2((r))|r|^{-2} < \infty$ in this case.

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