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The Capitulation Problem for Certain Cyclic Quartic Number Fields

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Abstract. Let K be a cyclic quartic number field such that its 2-class group is of type $(2,4)$, $K_2^{(1)}$ be the Hilbert 2-class field of K, $K_2^{(2)}$ be the Hilbert 2-class field of $K_2^{(1)}$ and $G = \mathrm{Gal}(K_2^{(2)}/K)$ be the Galois group of $K_2^{(2)}/K$. Our goal is to study the capitulation problem of 2-ideal classes of K and to determine the structure of G.

1. Introduction

Let K be a number field of finite degree over $\mathbb{Q}$. We denote by $\mathcal{O}_K$, E_K and C_K, the ring of integers, the unit group and the ideal class group of K, respectively. For a prime number p, let $C_{K,p}$ be the p-class group and $K_p^{(1)}$ the Hilbert p-class field of K. Further, we define $K_p^{(n)}$, for an integer $n \geqslant 0$, by $K_p^{(0)} = K$ and $K_p^{(n+1)} = (K_p^{(n)})^{(1)}$. So we have the sequence

$$K \subseteq K_p^{(1)} \subseteq \cdots \subseteq K_p^{(n)} \subseteq \cdots$$

that is called the p-class field tower of K. We know that it is finite if and only if there exists a finite p-extension of K whose p-class number is equal to 1. It is well-known that if $C_{K_p^{(1)},p}$ is cyclic then $C_{K_p^{(2)},p}$ is trivial, implying that $K_p^{(2)} = K_p^{(3)}$ (Taussky [17]).

A fractional ideal $\mathcal{A}$ of K is said to capitulate in an extension L/K if $\mathcal{A}\mathcal{O}_L = \alpha\mathcal{O}_L$ for some $\alpha \in L$.

Let L/K be a cyclic unramified extension and $j = j_{L/K} : C_K \longrightarrow C_L$ the conorm of L/K for ideal classes. Taussky defined:

- the extension L/K is said of type (A) iff $|\mathrm{Ker}(j) \cap N_{L/K}(C_L)| > 1$;
- the extension L/K is said of type (B) iff $|\mathrm{Ker}(j) \cap N_{L/K}(C_L)| = 1$.

Note that $\mathrm{Ker}(j)$ is the set of all the ideal classes of K which capitulate in L.

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DEFINITION 1. Let G be a group. We say that G is metacyclic if there exist a normal cyclic subgroup H of G such that the quotient group G/H is cyclic.

REMARK 1. If G is a metagroup, then the commutator group G' is cyclic.

Let K be a number field such that its 2-class group $C_{K,2}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (i.e. is of type $(2, 4)$) and G be the Galois group of $K_2^{(2)}/K$. By class field theory, G/G' is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Then, the following diagram shows all the unramified subextensions of $K_2^{(1)}/K$:

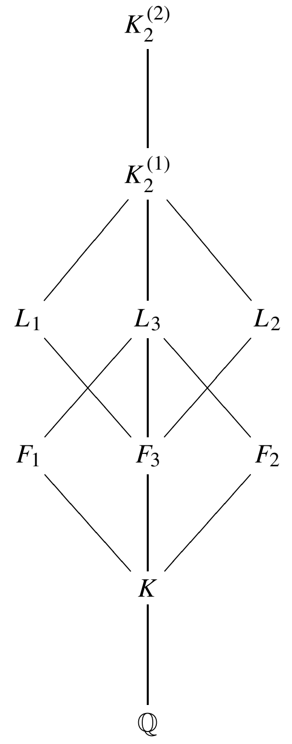

Diagram 1

where F_i's and L_i's are the extensions over K of degree 2 and 4, respectively.

THEOREM 1. *Let the notation be as above. Then the following assertions are equivalent*:

1. *The group G is non-metacyclic*;
2. *The* 2*-class group of* F_3 *is of type* $(2, 2, 2)$;
3. *The* 2*-rank of the* 2*-class group of* F_3 *is equal to* 3.

PROOF. See [3]. □

In [3], the authors have proved with the help of the transfer (Verlagerung) the following remark:

REMARK 2.

1. If G is abelian, then four 2-ideal classes of K capitulate in F_i for each i and the 2-class group $C_{K,2}$ of K capitulates in L_i for each i.
2. If G is non-metacyclic, then the capitulation of 2-ideal classes of K in F_3 is of type $2A$ (i.e. two 2-ideal classes of K capitulate in F_3 and F_3/K is of type (A)).

The aim of this work is the study of the capitulation problem of the 2-ideal classes of an imaginary cyclic quartic number field K with 2-class group of type $(2, 4)$, and we determine the structure of $G = \mathrm{Gal}(K_2^{(2)}/K)$. Let $K = k(\sqrt{-n\varepsilon\sqrt{l}})$ with $k = \mathbb{Q}(\sqrt{l})$, ε the fundamental unit of k, l a prime number and n a square free positive integer prime to l. According to E. Brown and C. J. Parry [6] and [7], $C_{K,2}$ the 2-class group of K is of type $(2, 4)$ in the following cases:

1. $l \equiv 5 \bmod 8$, $n = p \equiv 1 \bmod 4$ and $\left(\frac{p}{l}\right)_4 = -\left(\frac{l}{p}\right)_4 = 1$, where p is a prime number;
2. $l = 2$, $n = p \equiv 1 \bmod 16$ and $\left(\frac{2}{p}\right)_4 = -1$, where p is a prime number;
3. $l \equiv 9 \bmod 16$, $n = 1$ and $\left(\frac{2}{l}\right)_4 = 1$.

We denote by $K^{(*)}$ the genus field of K. Our two main theorems are the following:

THEOREM A. *Let $K = k(\sqrt{-p\varepsilon\sqrt{l}})$ with $k = \mathbb{Q}(\sqrt{l})$, ε the fundamental unit of k, l and p two distinct primes satisfying one of the following forms*:

1. $l \equiv 5 \bmod 8$, $p \equiv 1 \bmod 4$ *and* $\left(\frac{p}{l}\right)_4 = -\left(\frac{l}{p}\right)_4 = 1$;
2. $l = 2$, $p \equiv 1 \bmod 16$ *and* $\left(\frac{2}{p}\right)_4 = -1$.

Then the 2-class field tower of K stops at $K_2^{(1)}$, i.e. the group G is abelian. Moreover, four 2-ideal classes of K which capitulate in F_i for each i and the 2-class group $C_{K,2}$ of K capitulates in L_i for each i.

THEOREM B. *Let $K = k(\sqrt{-\varepsilon\sqrt{l}})$ with $k = \mathbb{Q}(\sqrt{l})$, ε the fundamental unit of k, and l a prime number satisfying $l \equiv 9 \bmod 16$ and $\left(\frac{2}{l}\right)_4 = 1$. Then,*

1. *The group G is non-metacyclic. Moreover, the capitulation of 2-ideal classes of K in $F_3 = K^{(*)}$ is of type $2A$;*
2. $F_1 = K(\sqrt{\varepsilon})$, $F_2 = K(\sqrt{\varepsilon'})$ *and* $L_3 = K^{(*)}(\sqrt{\varepsilon})$, *where ε' denotes the conjugate of ε.*

2. Preliminary results

This section is reserved for some useful results in the rest of this paper.

LEMMA 1. *Let $K = k(\sqrt{-p\varepsilon\sqrt{l}})$, where ε is the fundamental unit of $k = \mathbb{Q}(\sqrt{l})$ with l a prime number such that $l = 2$ or $l \equiv 5 \bmod 8$ and p a prime number different to l such that $p \equiv 1 \bmod 4$. Then $K^{(*)} = K(\sqrt{p})$.*

PROOF. See [2]. □

LEMMA 2. *Let $K = k(\sqrt{-\varepsilon\sqrt{l}})$, where ε is the fundamental unit of $k = \mathbb{Q}(\sqrt{l})$ with l a prime number such that $l \equiv 1 \bmod 8$. Then $K^{(*)} = K(\sqrt{-1})$.*

PROOF. As l is the unique odd prime of $\mathbb{Q}$ which ramifies in K, of ramification index $e_l = 4$; then, according to [10, Theorem 4, p. 48–49], we have $K^{(*)} = M_l K$ where M_l is the unique subfield of the l-th cyclotomic field of degree $e_l = 4$. Moreover, it is known that $M_l = \mathbb{Q}(\sqrt{\varepsilon\sqrt{l}})$. Thus $K^{(*)} = K(\sqrt{-1})$. □

THEOREM 2. *Let p and l be two distinct prime numbers such that $l = 2$ or $l \equiv 1 \bmod 4$, $p \equiv 1 \bmod 4$, $h(K_0)$ (respectively $h(lp)$) be the class number of $K_0 = \mathbb{Q}(\sqrt{l}, \sqrt{p})$ (respectively $\mathbb{Q}(\sqrt{lp})$) and e be the norm of the fundamental unit of $\mathbb{Q}(\sqrt{lp})$.*

1. *If $\left(\frac{l}{p}\right) = -1$, then $h(K_0)$ is odd, $h(lp) \equiv 2 \bmod 4$ and $e = -1$.*
2. *If $\left(\frac{l}{p}\right) = 1$, so we have:*
   - (a) *If $\left(\frac{l}{p}\right)_4 \neq \left(\frac{p}{l}\right)_4$, then $h(K_0)$ is odd, $h(lp) \equiv 2 \bmod 4$ and $e = 1$.*
   - (b) *If $\left(\frac{l}{p}\right)_4 = \left(\frac{p}{l}\right)_4 = -1$, then $h(K_0)$ is even, $h(lp) \equiv 4 \bmod 8$ and $e = -1$.*
   - (c) *If $\left(\frac{l}{p}\right)_4 = \left(\frac{p}{l}\right)_4 = 1$, then $h(K_0)$ is even and $h(lp) \equiv 0 \bmod 4$. Moreover, if $e = -1$, then $h(lp) \equiv 0 \bmod 8$.*

PROOF. See [13]. □

PROPOSITION 1. *Let L/M be a normal biquadratic extension of Galois group of type $(2, 2)$. Then L/M has three intermediate fields N_1, N_2, N_3 and*

$$h(L) = \frac{2^{d-\kappa-2-v} q(L/M) h(N_1) h(N_2) h(N_3)}{h(M)^2},$$

where $q(L/M) = [E_L : E_{N_1} E_{N_2} E_{N_3}]$ is the unit index of L/M, d is the number of infinite places ramified in L/M, κ is the $\mathbb{Z}$-rank of E_M, and $v = 1$ or 0 according to whether $L \subset M(\sqrt{E_M})$ or not.

PROOF. See [14]. □

LEMMA 3. *Let $K = k(\sqrt{-n\varepsilon\sqrt{d}})$ be a cyclic quartic number field, where ε is the fundamental unit of $k = \mathbb{Q}(\sqrt{d})$ with co-prime square free positive integers d and n. Then $\{\varepsilon\}$ is the fundamental system of units of K.*

PROOF. See [2]. □

THEOREM 3. *Let p be a prime number such that $p \equiv 1 \bmod 4$, $K_0 = \mathbb{Q}(\sqrt{2}, \sqrt{p})$, ε_1 (respectively ε_2, ε_3) be the fundamental unit of $k_1 = \mathbb{Q}(\sqrt{2})$ (respectively $k_2 = \mathbb{Q}(\sqrt{p})$, $k_3 = \mathbb{Q}(\sqrt{2p})$) and $F = K_0(\sqrt{-\varepsilon_1\sqrt{2}})$.*

1. *If ε_3 is of norm 1, then $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of K_0 and of F.*
2. *Else, $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$ is a fundamental system of units of K_0 and of F.*

PROOF. See [1]. □

THEOREM 4. *Let $K_0 = \mathbb{Q}(\sqrt{l}, \sqrt{p})$ where p and l are two distinct primes such that $l \equiv p \equiv 1 \bmod 4$, ε_1 (respectively ε_2, ε_3) be the fundamental unit of $k_1 = \mathbb{Q}(\sqrt{l})$ (respectively $k_2 = \mathbb{Q}(\sqrt{p})$, $k_3 = \mathbb{Q}(\sqrt{lp})$) and $F = K_0(\sqrt{-n\varepsilon_1\sqrt{l}})$ where n is a square free positive integer.*

1. *If ε_3 is of norm 1, then $\{\varepsilon_1, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of K_0 and of F.*
2. *Else, $\{\sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3\}$ is a fundamental system of units of K_0 and of F.*

PROOF. See [1]. □

3. Proof of Theorem A

The proof of Theorem A is based on the following result:

PROPOSITION 2. *Let M be a number field with $C_{M,2}$ the 2-class group of M of type $(2^m, 2^n)$. If there is an unramified quadratic extension of M with 2-class number equal to 2^{m+n-1}; then all the three unramified quadratic extensions of M have 2-class number equal to 2^{m+n-1}, and the 2-class field tower of M terminates at $M_2^{(1)}$.*

PROOF. See [5]. □

In particular, let K be a cyclic quartic number field with $C_{K,2}$ the 2-class group of K of type $(2, 4)$. If there is an unramified quadratic extension of K with 2-class number equal to 4, then all the three unramified quadratic extensions of K have 2-class number equal to 4, and the 2-class field tower of K terminates at $K_2^{(1)}$.

THEOREM 5. *Let $K = k(\sqrt{-p\varepsilon\sqrt{l}})$ with $k = \mathbb{Q}(\sqrt{l})$, ε the fundamental unit of k, l and p two distinct primes satisfying one of the following forms:*

1. $l \equiv 5 \bmod 8$, $p \equiv 1 \bmod 4$ *and* $\left(\frac{p}{l}\right)_4 = -\left(\frac{l}{p}\right)_4 = 1$;
2. $l = 2$, $p \equiv 1 \bmod 16$ *and* $\left(\frac{2}{p}\right)_4 = -1$.

Then $h_2(K^{(*)})$, the 2-class number of $K^{(*)}$, is equal to 4.

PROOF. By Lemma 1, $K^{(*)} = K(\sqrt{p})$. Then $K^{(*)}/k$ is a normal biquadratic extension of Galois group of type $(2, 2)$, with quadratic subextensions K, $K' = k(\sqrt{-\varepsilon\sqrt{l}})$ and $K_0 = \mathbb{Q}(\sqrt{l}, \sqrt{p})$. According to Proposition 1, we have

$$h_2(K^{(*)}) = \frac{1}{2} q(K^{(*)}/k) h_2(K) h_2(K') h_2(K_0),$$

because $h_2(k) = 1$ (see [12]), $d = 2$, $\kappa = 1$ and $v = 0$. As $K_0/\mathbb{Q}(\sqrt{lp})$ is an unramified extension and the 2-class group of $\mathbb{Q}(\sqrt{lp})$ is cyclic (see [12]), we have $h_2(K_0) = \frac{1}{2}h_2(lp)$, where $h_2(lp)$ is the 2-class number of $\mathbb{Q}(\sqrt{lp})$. Moreover, $h_2(K) = 8$ (see [6], [7]) and $h_2(K') = 1$ (see [9]), which give $h_2(K^{(*)}) = 2q(K^{(*)}/k)h_2(lp)$. Also we have $\{\varepsilon\}$ is a fundamental system of units of K and of K' (Lemma 3), and from the Theorems 2, 3 and 4, $\{\varepsilon, \varepsilon_2, \sqrt{\varepsilon_3}\}$ is a fundamental system of units of $K^{(*)}$ and of K_0, since $\left(\frac{p}{l}\right)_4 \neq \left(\frac{l}{p}\right)_4$ for (1) and $\left(\frac{p}{2}\right)_4 = (-1)^{\frac{p-1}{8}} = 1 \neq \left(\frac{2}{p}\right)_4$ for (2), thus $q(K^{(*)}/k) = 1$. Finally, $h_2(K^{(*)}) = 2h_2(lp) = 4$, where $h_2(lp) = 2$ (Theorem 2). □

3.1. Proof of Theorem A. According to E. Brown and C. J. Parry [6] and [7], $C_{K,2}$ the 2-class group of K is of type $(2, 4)$. By Theorem 5, $K^{(*)}$ is the unramified quadratic extension of K with 2-class number equal to 4. Then all the three unramified quadratic extensions of K have 2-class number equal to 4, and the 2-class field tower of K terminates with $K_2^{(1)}$. On the other hand, by Remark 2, we have four 2-ideal classes of K which capitulate in F_i for each i and the 2-class group $C_{K,2}$ of K capitulates in L_i for each i. This completes the proof.

EXAMPLE 1. Let $K = \mathbb{Q}(\sqrt{-29\varepsilon\sqrt{13}})$ where $\varepsilon = \frac{3+\sqrt{13}}{2}$. As $13 \equiv 5 \bmod 8$, $29 \equiv 1 \bmod 4$ and $\left(\frac{29}{13}\right)_4 = -\left(\frac{13}{29}\right)_4 = 1$, the group G is abelian and $C_{K^{(*)},2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

EXAMPLE 2. Let $K = \mathbb{Q}(\sqrt{-17\varepsilon\sqrt{2}})$ where $\varepsilon = 1 + \sqrt{2}$. As $17 \equiv 1 \bmod 16$ and $\left(\frac{2}{17}\right)_4 = -1$, the group G is abelian and $C_{K^{(*)},2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

4. Proof of Theorem B.

In this section, we want to prove the second main theorem.

LEMMA 4. *Let p be a prime number such that $p \equiv 1 \bmod 8$, then*

$$p \neq x^2 + 32y^2 \iff \left(\frac{2}{p}\right)_4 = -\left(\frac{p}{2}\right)_4.$$

PROOF. See [4]. □

THEOREM 6. *Let $K = k(\sqrt{-\varepsilon\sqrt{l}})$ with $k = \mathbb{Q}(\sqrt{l})$, ε the fundamental unit of k, and l a prime number satisfying $l \equiv 9 \bmod 16$ and $\left(\frac{2}{l}\right)_4 = 1$. Then $h_2(K^{(*)})$ is equal to $2h_2(-l)$, where $h_2(-l)$ is the 2-class number of $\mathbb{Q}(\sqrt{-l})$. Moreover, $h_2(K^{(*)}) = 8$.*

PROOF. By Lemma 2, $K^{(*)} = K(\sqrt{-1})$. So we have that $K^{(*)}/k$ is a normal bi-quadratic extension of Galois group of type $(2, 2)$, with quadratic subextensions K, $L = k(\sqrt{\varepsilon\sqrt{l}})$ and $K_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-1})$. Therefore, by Proposition 1, we have

$$h_2(K^{(*)}) = \frac{1}{2}q(K^{(*)}/k)h_2(K)h_2(L)h_2(K_0),$$

because $h_2(k) = 1$ (see [12]), $d = 2$, $\kappa = 1$ and $v = 0$. As $K_0/\mathbb{Q}(\sqrt{-l})$ is an unramified extension and the 2-class group of $\mathbb{Q}(\sqrt{-l})$ is cyclic (see [11]), we have $h_2(K_0) = \frac{1}{2}h_2(-l)$. Moreover, $h_2(K) = 8$ (see [7]) and $h_2(L) = 1$ (see [18]), which give $h_2(K^{(*)}) = 2q(K^{(*)}/k)h_2(-l)$. Also we have $q(K^{(*)}/k) = 1$ (see [16, p.84]). Thus $h_2(K^{(*)}) = 2h_2(-l) = 8$, since $h_2(-l) = 4$ and $\left(\frac{l}{2}\right)_4 = (-1)^{\frac{l-1}{8}} = -1 = -\left(\frac{2}{l}\right)_4$ (see [4] and Lemma 4). □

REMARK 3. The group G is non-abelian because $K_2^{(1)} \neq K_2^{(2)}$.

4.1. Proof of Theorem B. (1) From the last proof, we see that $K^{(*)}$ is a CM-field with its maximal real subfield $L = k(\sqrt{\varepsilon\sqrt{l}})$ of odd class number (see [18]). Therefore, by [15],

$$\operatorname{rank} C_{K^{(*)},2} = t - 1 + \operatorname{rank}(E_L \cap N_{K^{(*)}/L}(K^{(*)})/E_L^2),$$

where t is the number of finite prime ideals ramifying in $K^{(*)}/L$. Using a result in [16], $\left(\frac{\varepsilon\sqrt{l}}{2_1}\right) = \left(\frac{\varepsilon\sqrt{l}}{2_2}\right) = \left(\frac{2}{l}\right)_4 = 1$ where 2_1 and 2_2 are the prime ideals in k above 2, we see that 2 splits completely in L. Thus exactly 4 prime ideals are ramified in $K^{(*)}/L$. It follows from $\operatorname{rank}(E_L \cap N_{K^{(*)}/L}(K^{(*)})/E_L^2) = 0$ (see [16, p.84]) that $\operatorname{rank} C_{K^{(*)},2} = 3$. Further $h_2(K^{(*)}) = 8$ by Theorem 6. Then, since $C_{K^{(*)},2}$ is of type $(2, 2, 2)$, we have $F_3 = K^{(*)}$. On the other hand , $C_{K,2}$ is of type $(2, 4)$ by Brown and Parry [7], while G is non-metacyclic by Theorem 1. Combining them, we can use Remark 2 to see that the capitulation of 2-ideal classes of K in $K^{(*)}$ is of type $2A$.

(2) We know by Cohn [8], that if $l \equiv 1 \bmod 8$, then $F = K_0(\sqrt{\varepsilon})$ is an unramified quadratic extension over K_0 where $K_0 = \mathbb{Q}(\sqrt{l}, \sqrt{-1})$. It is easy to see that $\sqrt{\varepsilon} \notin F_3$. Therefore $KF = F_3(\sqrt{\varepsilon})$ is an unramified quadratic extension over $KK_0 = F_3$.

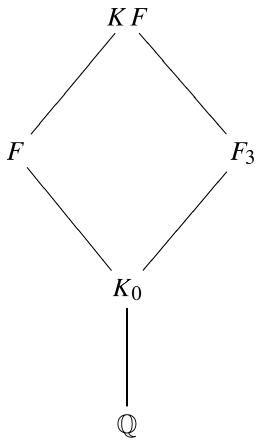

Diagram 2

Moreover, we have that F_3 is unramified over K. Thus KF is unramified over K and the Galois group of KF/K is of type $(2, 2)$. Consequently, $F_1 = K(\sqrt{\varepsilon})$, $F_2 = K(\sqrt{\varepsilon'})$ and $L_3 = K^{(*)}(\sqrt{\varepsilon})$.

EXAMPLE 3. Let $K = \mathbb{Q}(\sqrt{-\varepsilon\sqrt{73}})$ where $\varepsilon = 1068 + 125\sqrt{73}$. As $73 \equiv 9 \bmod 16$ and $\left(\frac{2}{73}\right)_4 = 1$, the group G is non-metacyclic, $C_{K^{(*)},2} \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $K^{(*)}/K$ is of type $2A$ where $K^{(*)} = K(\sqrt{-1})$.

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