TOKYO J. MATH.
VOL. 39, NO. 3, 2017

On Mono-nodal Trees and Genus One Dessins of Pakovich-Zapponi Type

Dedicated to Professor Ken-ichi SHINODA

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(Communicated by N. Suwa)

Abstract. In this paper, we classify Grothendieck dessins of X-shaped plane trees defined over the rationals.

1. Introduction

In [16], F. Pakovich studied special families of plane trees which featured the beautiful arithmetic nature of Grothendieck dessins d'enfants. Among others, he closely looked at trees with a single node joining three or four branches, and figured out how those Y-shaped, X-shaped trees arise from Pell solutions of quartic models of elliptic curves. These trees raised Grothendieck dessins of genus one, and their explicit description in terms of theta functions have also been studied in loc. cit. and by L. Zapponi [21] in depth. A particularly interesting result found in [16] Théorème 5 classifies all Y-shaped $\mathbf{Q}$-rational trees into the mono-nodal trivalent trees with branch length multiples of $\{1, 1, 1\}$, $\{1, 1, 2\}$, $\{1, 2, 2\}$ and $\{1, 1, 3\}$, after elaborate combination of classics of Abel's elliptic function theory and a modern theorem of B. Mazur on rational torsion of elliptic curves. The proof given in loc. cit., however, partly relied on an unpublished work of M. Magot still staying unavailable in literatures (e.g., [10] 2.5.6; [17] p.150). This situation urged us to execute such a process of computation through our own individual resources.

The purpose of this paper is to report our line work using mainly Maple [11], which yielded results for both Y- and X-shaped cases. Especially for the latter shape, we obtained the following

THEOREM 1.1. *The X-shaped plane trees defined over* $\mathbf{Q}$ *are only mono-nodal trees with* 4 *branches having length multiples of* $(1, 1, 1, 1)$, $(1, 1, 1, 2)$, $(1, 1, 1, 3)$, $(1, 1, 2, 2)$, $(1, 2, 1, 2)$, $(1, 2, 1, 5)$, $(1, 2, 2, 2)$, $(1, 2, 4, 2)$, $(1, 3, 1, 3)$, $(1, 5, 1, 5)$ *and* $(1, 5, 13, 5)$ *up to cyclic permutations.*

Received September 30, 2014; revised October 29, 2015
Mathematics Subject Classification: 11G32, 14H57, 14H30, 12Y05
Key words and phrases: Grothendieck dessin, Belyi function, Shabat polynomial

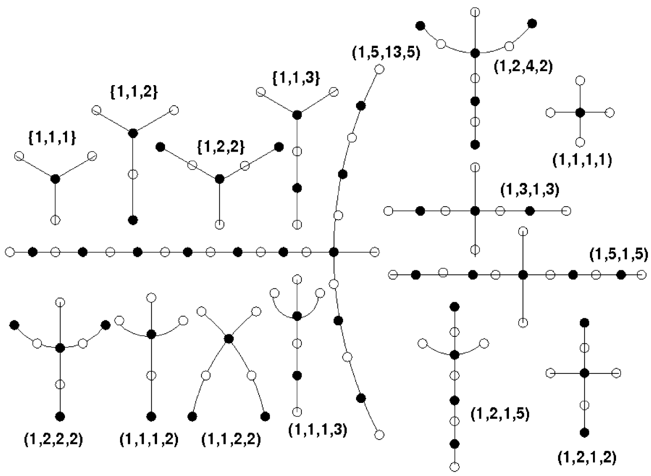

Figure A

We shall say a mono-nodal tree to be *primitive* if the branch lengths have no nontrivial common divisors. Figure A illustrates our "child's drawings" (following spirit of [5]) for all primitive **Q**-rational mono-nodal trees of X- or Y-shape.

The construction of this paper is as follows: In §2, we review a basic diagram that describes relationship between X-, Y-shaped Shabat polynomials and Belyi functions on quartic affine models of elliptic curves. In §3, we summarize a continued fraction algorithm computing units of the affine rings of elliptic curves, and in §4, we discuss conditions for those units to supply our desired dessins. In §5, we search all possible moduli parameters to satisfy that condition and conclude the above result. We also complement a remark that the compositions with Chebyshev polynomials cover all X-, Y-shaped trees defined over **Q**.

2. Pakovich diagram

We here quickly review the notion of Belyi functions and Shabat polynomials. A *Belyi function* on a nonsingular projective curve X over $\mathbf{C}$ is a finite morphism $\beta : X \to \mathbf{P}^1_{\mathbf{C}}$ whose branch locus BR_β contains at most three points of $\mathbf{P}^1(\mathbf{C})$. We say that β has genus g if the genus of X is g. Once BR_β is normalized to lie in $\{0, 1, \infty\}$, the inverse image $D_\beta := \beta^{-1}([0, 1])$ of the unit segment $[0, 1] \subset \mathbf{P}^1(\mathbf{C})$ gives rise to a certain connected graph on the Riemann surface $X(\mathbf{C})$, which is called the (Grothendieck) dessin associated to β. It is known that the topological type of the dessin $D_\beta \subset X(\mathbf{C})$ determines the equivalence class of β as a cover over $\mathbf{P}^1$. Given a topological graph on an oriented surface, it is, in general, difficult to obtain an explicit algebraic model of a Belyi function $\beta : X \to \mathbf{P}^1$ supplying the graph as a dessin. If β has genus zero and is totally ramified over $\infty \in \mathsf{BR}_\beta$, then

$\beta : \mathbf{P}^1_x \to \mathbf{P}^1_t$ is realized by a polynomial $P(x) \in \mathbf{C}[x]$ (called the Shabat polynomial) so that $t = \beta(x) = P(x)$. In this case, the dessin D_β is a tree on the complex x-plane. By definition, a *Shabat polynomial* is a polynomial in one variable with at most two critical values. We are also interested in rationality questions on Grothendieck dessins, e.g., to estimate (resp. to determine) fields of definition (resp. the field of moduli) of a given Belyi pair (X, β), or to know which type of graph on a topological surface can be realized by such a pair defined over $\mathbf{Q}$. There are a number of interesting works by many authors concerning these questions (e.g., [4], [10], [15], [16], [17], [21]). In particular, it is worth mentioning that, in the tree case considered in this paper, the problem of distinction between the field of moduli and minimal fields of definition does not occur according to a result of J.-M. Couveignes [4] (cf. [10, Theorem 2.4.12]).

Below, we will mainly consider Shabat polynomials normalized to have critical values in $\{\pm 1\}$ (occasionally in $\{\pm t_0\}$ for some $t_0 \in \mathbf{Q}$) and relate them to Belyi functions of genus one whose critical values are normalized in $\{0, u_0, \infty\}$ for some $u_0 \in \mathbf{Q}$.

Let $P(x) \in \mathbf{Q}[x]$ be a Shabat polynomial of degree N with critical values ± 1 such that the inverse image of $[-1, 1]$ forms a mono-nodal tree of X- or Y-shape. Regard $t = P(x)$ as representing a cover $P : \mathbf{P}^1_x \to \mathbf{P}^1_t$, and introduce a double cover $\varpi : \mathbf{P}^1_u \to \mathbf{P}^1_t$ by $t = \varpi(u) := \frac{1}{2}\left(\frac{u}{u_0} + \frac{u_0}{u}\right)$ branching at $u = \pm u_0 (\in \mathbf{Q}^\times)$. A basic observation of [16] Théorème 2 (in the genus one case) is that the fiber product of P and ϖ produces an elliptic curve $E : y^2 = D(x) = x^4 + ax^3 + bx^2 + cx + d$ and a Belyi function $\beta : E \to \mathbf{P}^1_u$, both defined over $\mathbf{Q}$ fitting in the commutative diagram:

$$\begin{tikzcd} \mathbf{P}^1_x \arrow[d, "P"'] & E \arrow[l, "x"'] \arrow[d, "\beta"] \\ \mathbf{P}^1_t & \mathbf{P}^1_u \arrow[l, "\varpi"] \end{tikzcd} \tag{1}$$

with the following properties:

(1) The Belyi function $u = \beta(x, y) = p(x) + q(x)y$ has critical values $u = 0, \infty, u_0$;
(2) β is a unit of the affine ring $\mathbf{Q}[x, y]/(y^2 - D(x))$ and gives a solution to the Pell equation $p(x)^2 - D(x)q(x)^2 = u_0^2$;
(3) If ∞_+ and ∞_- are the two infinity points of E, then the difference $\infty_+ - \infty_-$ gives a torsion of E of order N.

One easily sees that $P(x) = u_0^{-1} p(x)$ holds by combining the commutativity of (1) and the property (2).

So our problem is reduced to finding a pair (E, β) with the above properties (1), (2), (3) so that everything is defined over $\mathbf{Q}$.

3. Continued fractions

Let us now start with a quartic $D(x) = x^4 + ax^3 + bx^2 + cx + d \in \mathbf{Q}[x]$ without multiple zeros and an elliptic curve $E : y^2 = D(x)$ such that $\infty_+ - \infty_-$ is of order $N > 1$. According to Mazur's theorem [12], we may assume $2 \leq N \leq 12$ and $N \neq 11$. To find a solution to the Pell property $p(x)^2 - D(x)q(x)^2 \in \mathbf{Q}^\times$, one can apply a continued fraction algorithm which has explicitly been discussed, for example, in papers [1] (1980), [2] (1990) and [18] (2000). It is convenient to work in the field $\mathbf{Q}(\!(x^{-1})\!)$ of Laurent power series

$$f(x) = \lfloor f(x) \rfloor + \sum_{n=1}^{\infty} \frac{c_n}{x^n} \quad (c_n \in \mathbf{Q})$$

where $\lfloor f(x) \rfloor \in \mathbf{Q}[x]$ denotes the polynomial part of $f(x)$. Define the degree function $\deg : \mathbf{Q}(\!(x^{-1})\!) \to \mathbf{N} \cup \{\infty\}$ by associating to f the usual degree of $\lfloor f \rfloor$ (as an element of $\mathbf{Q}[x]$). It induces a standard metric on $\mathbf{Q}(\!(x^{-1})\!)$ by the norm $|f| = 2^{\deg(f)}$ (cf. [1] p.492). Set

$$\delta(x) = \lfloor \sqrt{D(x)} \rfloor = x^2 + \frac{1}{2}ax + \frac{1}{2}\left(b - \frac{a^2}{4}\right),$$

and construct a sequence of partial quotients $a_0 = 2\delta(x)$, $a_{i+1} = 1/(a_i - \lfloor a_i \rfloor)$ for $i \geq 0$ so as to obtain the identity:

$$\delta(x) + \sqrt{D(x)} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

For practical computation, it is also useful to introduce more sequences of complete quotients $\{F_h\}$ as well as of the associated quantities $\{P_h\}$, $\{Q_h\}$ $(h \geq 0)$ in the form

$$F_h = a_h + \cfrac{1}{a_{h+1} + \cfrac{1}{\cdots}} =: \frac{P_h + \sqrt{D}}{Q_h} \qquad \left(Q_h | (D - P_h^2)\right),$$

which are inductively produced from $(P_0, Q_0, a_0) = (\delta(x), 1, 2\delta(x))$ and the rule

$$\begin{cases} P_{h+1} &= a_h Q_h - P_h \,, \\ Q_{h+1} &= \frac{D(x) - P_{h+1}^2}{Q_h} \,, \\ a_{h+1} &= \left\lfloor \frac{P_{h+1} + P_0}{Q_{h+1}} \right\rfloor . \end{cases} \tag{2}$$

It follows from [18] Proposition 3.1, that

$$\deg(P_h) = 2, \quad \deg(Q_h) \leq 1 \,. \tag{3}$$

These quantities are also related with the continuants $(p_h, q_h) \in \mathbf{Q}[x]^2$, which are defined by

$$\begin{pmatrix} p_0 \\ q_0 \end{pmatrix} = \begin{pmatrix} \delta \\ 1 \end{pmatrix}, \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \begin{pmatrix} 1+\delta a_1 \\ a_1 \end{pmatrix}, \begin{pmatrix} p_{h+2} \\ q_{h+2} \end{pmatrix} = a_{h+2}\begin{pmatrix} p_{h+1} \\ q_{h+1} \end{pmatrix} + \begin{pmatrix} p_h \\ q_h \end{pmatrix}$$

and satisfy the relations:

$$p_h^2 - Dq_h^2 = (-1)^{h+1} Q_{h+1}, \tag{4}$$

$$p_h q_{h-1} - q_h p_{h-1} = (-1)^{h-1}, \tag{5}$$

$$\begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix} = \begin{pmatrix} \delta & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_h & 1 \\ 1 & 0 \end{pmatrix}. \tag{6}$$

Let $n_0 := N-1$, and set $n_1 := n_0$ if N is odd, and $n_1 := 2n_0$ if N is even. Then, the sequence $\{a_0, a_1, \dots\}$ has a quasi-period n_0 and a period n_1 with palindromic symmetry as shown in [1] Corollary 4.4 and (4.6) (cf. also [18] Prop. 3.4). This means that $p_{n_0-1}^2 - Dq_{n_0-1}^2 = \kappa_0^{-1}$ ($\kappa_0 \in \mathbf{Q}^\times$), $p_{n_1-1}^2 - Dq_{n_1-1}^2 = 1$ and $a_h = a_{n_1-h}$ $(0 \le h \le n_1)$. One can then easily show by induction that

$$p_{kn_1-1} + q_{kn_1-1}\sqrt{D} = (p_{n_1-1} + q_{n_1-1}\sqrt{D})^k \tag{7}$$

$$p_{n_0+kn_1-1} + q_{n_0+kn_1-1}\sqrt{D} = (p_{n_0-1} + q_{n_0-1}\sqrt{D})(p_{n_1-1} + q_{n_1-1}\sqrt{D})^k \tag{8}$$

for each $k \ge 1$. From this, it also follows that

$$\deg(p_{kn_0-1}) = k \cdot \deg(p_{n_0-1}) \qquad (k \ge 1). \tag{9}$$

LEMMA 3.1. *If $p, q \in \mathbf{Q}[x]$ are solutions to $p^2 - Dq^2 = \kappa^{-1}$ for some constant $\kappa \in \mathbf{Q}^\times$, then $p = cp_n$, $q = \varepsilon c q_n$ for some $n \ge 1$, $c \in \mathbf{Q}^\times$ and $\varepsilon \in \{\pm 1\}$.*

PROOF. Suppose first that $\deg(p - \sqrt{D}q) \le \deg(p + \sqrt{D})$, i.e., $|p - \sqrt{D}q| \le |p + \sqrt{D}|$. Then, $|q| < |2\sqrt{D}q| \le \max\{|p + \sqrt{D}q|, |p - \sqrt{D}|\} = |p + \sqrt{D}|$ so that $|p - \sqrt{D}q| = |p + \sqrt{D}q|^{-1} < \frac{1}{|q|}$. The usual approximation theorem (cf. [1] (5.14)) implies then that $p = cp_n$, $q = cq_n$ for some $n \ge 1$, $c \in \mathbf{Q}^\times$. If $\deg(p - \sqrt{D}q) > \deg(p + \sqrt{D})$, we may argue in the same way with q replaced by $-q$, hence conclude the assertion. □

When N is even so that $n_1 = 2n_0$, the equality $p_{n_0-1}^2 - Dq_{n_0-1}^2 = \kappa_0^{-1}$ implies $\{\kappa_0(p_{n_0-1}^2 + Dq_{n_0-1}^2)\}^2 - D\{2\kappa_0 p_{n_0-1}q_{n_0-1}\}^2 = 1$. Then, using Lemma 3.1 and comparing degrees (9), we find $p_{n_1-1} = \pm\kappa_0(p_{n_0-1}^2 + Dq_{n_0-1}^2)$, $q_{n_1-1} = \pm 2\varepsilon\kappa_0 p_{n_0-1}q_{n_0-1}$ ($\varepsilon \in \{\pm 1\}$), or equivalently $p_{n_1-1} + q_{n_1-1}\sqrt{D} = \pm\kappa_0(p_{n_0-1} + \varepsilon q_{n_0-1}\sqrt{D})^2$. Therefore, (7), (8) may be summarized as:

$$p_{kn_0-1} + q_{kn_0-1}\sqrt{D} = c_k(p_{n_0-1} + q_{n_0-1}\sqrt{D})^{\pm k} \quad (k \ge 1,\ \exists c_k \in \mathbf{Q}^\times). \tag{10}$$

The following lemma will be useful. Recall that Chebyshev polynomials $\{T_k(X)\}$, $\{U_k(X)\}$ of the 1st and 2nd kind are defined by $\cos kz = T_k(\cos z)$, $\frac{\sin(k+1)z}{\sin z} = U_k(\cos z)$.

LEMMA 3.2. *If $p^2 - Dq^2 = 1$, then $(p + q\sqrt{D})^k = T_k(p) + U_{k-1}(p)q\sqrt{D}$.*

PROOF. Straightforward by induction, using classical identities of Chebyshev polynomials such as $T_{n+1}(x) = xT_n(x) - (1 - x^2)U_{n-1}(x)$, $U_n(x) = xU_{n-1}(x) + T_n(x)$ for $n = 1, 2, \dots$. □

REMARK 3.3. The Chebyshev polynomial $T_k(x)$ is a Shabat polynomial whose dessin $T_k^{-1}([-1, 1])$ is a segment graph $[-1, 1]$ divided into k smaller segments. From this, we easily see that the composition $T_k(P(x))$ of any Shabat polynomial P with T_k replaces each edge of the dessin of P by a concatenation of k edges, as long as the critical values of P are normalized to $\{\pm 1\}$.

4. From Pell solution to Belyi function

Following the notation introduced in the previous section, let E be the elliptic curve defined by $y^2 = D(x)$ with the difference of the two infinity points $\infty_+ - \infty_-$ is torsion of order N. For any given solution $(p(x), q(x)) \in \mathbf{Q}[x]^2$ to the Pell property $p(x)^2 - D(x)q(x)^2 \in \mathbf{Q}^\times$, the function $u = p(x) + q(x)y$ is a unit on $E \setminus \{\infty_\pm\}$ whose divisor is of the form $\div(u) = d[\infty_+] - d[\infty_-]$ (the integer d being the degree of u). It follows that d is a multiple of N. By Lemma 3.1, $p(x) = cp_n(x)$, $q(x) = \varepsilon c q_n(x)$ for some $n \geq 1$, $c \in \mathbf{Q}^\times$ and $\varepsilon \in \{\pm 1\}$. The degree d of $u = p(x) + q(x)y$ is then equal to $\deg(p_n)$.

We first claim the following

PROPOSITION 4.1. *The above n is of the form $kn_0 - 1$ $(k > 0)$.*

PROOF. By the periodicity and the palindromic symmetry of the partial quotients $\{a_h\}_{h \geq 0}$, this claim is equivalent to saying that $Q_1, \dots, Q_{n_0-1}$ are non-constant. Now, suppose that the Pell property $p_h^2 - Dq_h^2 = (-1)^{h+1}Q_{h+1} \in \mathbf{Q}^\times$ holds for some $h \geq 1$. Then, considering the divisor of $p_h + \sqrt{D}q_h$ on E, we have $N | \deg(p_h)$ as above. Assume moreover that $h > 0$ is the smallest index giving $p_h^2 - Dq_h^2 \in \mathbf{Q}^\times$. Then, by (4), $Q_1, \dots, Q_h$ are non-constant; therefore follows from (2), (3) that $\deg(a_1), \dots, \deg(a_h) \leq 1$. As $\deg(a_0) = 2$, (6) implies that $\deg(p_h) \leq 2 + h$. Therefore, we see $N \leq 2 + h$, i.e., $n_0 - 1 \leq h$. This concludes the claim $Q_1, \dots, Q_{n_0-1}$ are not constants. □

From the above discussion, we are now able to characterize the degree of initial Pell solution $(p_{n_0-1}, q_{n_0-1}) = (p_{N-2}, q_{N-2})$ in terms of the order N of infinity points of E:

COROLLARY 4.2. $\deg(p_{n_0-1}) = N$, $\deg(q_{n_0-1}) = N - 2$.

PROOF. By Riemann-Roch, we know there does exist a unit of degree N which must be attained by the least degree of $\deg(p_{kn_0-1})$ $(k > 0)$. □

Returning to any Pell solution $(p(x), q(x)) \in \mathbf{Q}[x]^2$, we see that the quotient $p'(x)/q(x)$ is a linear polynomial. Indeed, this follows from $\deg(p) = \deg(q) + 2$ and $2pp' = q(2q'D + qD')$ which implies that $q(x) | p'(x)$. Define $x_0 \in \mathbf{Q}$ to be a unique solution to $(p'/q)(x_0) = 0$.

LEMMA 4.3. *The rational number x_0 defined above only depends on $D(x)$ and not on the choice of any solution $(p(x), q(x))$ to $p(x)^2 - D(x)q(x)^2 \in \mathbf{Q}^\times$.*

PROOF. By virtue of Proposition 4.1, the invariance of x_0 is reduced to (7), (8) and (10). $\square$

The divisor $\mathrm{div}(p'/q)$ is of the form $[Q_1] + [Q_2] - [\infty_+] - [\infty_-]$, where $Q_1 = Q_2$ if and only if $D(x_0) = 0$. This, applied to the differential $du = (p'/q) \cdot u \cdot (dx/y)$, leads to the identity:

$$\mathrm{div}(du) = (d-1)[\infty_+] - (d+1)[\infty_-] + [Q_1] + [Q_2]\,.$$

Thus, in order for the function $u = \beta(x, y) = p(x) + q(x)y$ to be a Belyi function, it is necessary and sufficient that one of the following conditions holds:

(I): $D(x_0) = 0$;

(II): $D(x_0) \neq 0$ and $q(x_0) = 0$.

We easily see that $u = p(x) + q(x)y$ gives a Belyi function on E with branch type $[d, d, 3 \cdot 1^{d-3}]$ in Case I and $[d, d, 2^2 \cdot 1^{d-4}]$ in Case II, and that $t = p(x)$ gives a Shabat polynomial of shape Y in Case I, and of shape X in Case II.

The following lemma gives a criterion for these cases occur, especially a finite criterion even for case (II):

LEMMA 4.4. *Case* (I), (II) *occur only when $p_{n_1-1}(x_0) + q_{n_1-1}(x_0)\sqrt{D(x_0)}$ is a root of unity in a quadratic field.*

PROOF. Supposing $p^2 - Dq^2 = \kappa^{-1} \in \mathbf{Q}^\times$, set $\tilde{p} = \kappa(p^2 + Dq^2)$, $\tilde{q} = 2\kappa pq$ so that $\tilde{p}^2 - D\tilde{q}^2 = 1$. Since $q(x_0)\sqrt{D(x_0)} = 0$ in either case, we find $\tilde{q}(x_0)\sqrt{D(x_0)} = 0$ and $\tilde{p}(x_0)^2 = 1$. On the other hand, by applying Lemma 3.1 and taking into accounts the degrees, we see that $\tilde{p} + \tilde{q}\sqrt{D} = (p_{n_1-1} + q_{n_1-1}\sqrt{D})^k$ for some $k \in \mathbf{Z}$. Hence the assertion follows. $\square$

5. Kubert family

Now, recall that the pairs (E, P) of an elliptic curve E over $\mathbf{Q}$ and an N-torsion $\mathbf{Q}$-rational point P on it ($2 \leq N \leq 12$, $N \neq 11$) are parametrized by $Y_1(N)(\mathbf{Q})$, the $\mathbf{Q}$-rational points of the genus 0 modular curve $Y_1(N)$. A convenient table for $N \geq 4$ giving all such pairs (E, P) by means of a single rational parameter t is given in [6] Table 3 after a preceding work of Kubert [8]. When $N = 2$ (resp. $N = 3$), the modular curve $Y_1(2)$ (resp. $Y_1(3)$) is not a curve (is a stack), and those pairs are given by two rational parameters (say, a and b). (See e.g., [7] §4.2 and §4.4 for an account of derivation of such parametrization from Tate normal forms; we however note that the Hessian family for $N = 3$ presented in loc. cit. parameterizes only those (E, P) with $E : y^2 + a_1xy + a_3y = x^3$, $P = (0, 0)$ and with $a_3 \in \mathbf{Q}^{\times 3}$). Then, the Mordell transformation from cubics to quartics (sending two specific points to infinity; cf., e.g., [14] §5.3) yields a universal family of quartics $D_N(x) = D_N(x; a, b)$ ($N = 2, 3$),

$D_N(x) = D_N(x;t)$ $(N = 4, \ldots, 12,\ N \neq 11)$ with $\infty_+ - \infty_-$ is of order N as follows:

$$D_2(x) := x^4 - 2\,a\,x^2 + a^2 - 4\,b \quad (b \neq 0)\,.$$

$$D_3(x) := x^4 - 8\,a^2\,x^2 + b\,x + 2\,a\,(8\,a^3 - b) \quad (b \neq 0)\,.$$

$$D_4(x) := x^4 + \left(2\,t - \frac{1}{2}\right)x^2 - 4\,tx + t^2 + \frac{3}{2}\,t + \frac{1}{16}\,.$$

$$D_5(x) := x^4 + \left(-\frac{1}{2}\,t^2 + 3\,t - \frac{1}{2}\right)x^2 - 4\,tx + \frac{1}{16}\,(t+1)\left(t^3 - 13\,t^2 + 19\,t + 1\right).$$

$$D_6(x) := x^4 + \left(\frac{3}{2}\,t^2 + 3\,t - \frac{1}{2}\right)x^2 - 4\,t\,(t+1)\,x + \frac{9}{16}\,t^4 + \frac{1}{4}\,t^3 + \frac{15}{8}\,t^2 + \frac{5}{4}\,t + \frac{1}{16}\,.$$

$$D_7(x) := x^4 + \left(-\frac{1}{2}\,t^4 + 3\,t^3 - \frac{3}{2}\,t^2 - t - \frac{1}{2}\right)x^2 - 4\,t^2\,(t-1)\,x$$
$$+ \frac{1}{16}\left(t^2 - 3\,t + 1\right)\left(t^6 - 9\,t^5 + 14\,t^4 - 13\,t^3 - 2\,t^2 + 7\,t + 1\right).$$

$$D_8(x) := x^4 + \frac{\left(4\,t^4 + 4\,t^3 - 16\,t^2 + 8\,t - 1\right)x^2}{2\,t^2} - 4\,(2\,t-1)\,(t-1)\,x$$
$$+ \frac{16\,t^8 - 96\,t^7 + 336\,t^6 - 576\,t^5 + 536\,t^4 - 296\,t^3 + 96\,t^2 - 16\,t + 1}{16\,t^4}\,.$$

$$D_9(x) := x^4 - \left(\frac{t^6 - 6\,t^5 + 9\,t^4 - 10\,t^3 + 6\,t^2 + 1}{2}\right)x^2 - 4\,t^2\,(t-1)\left(t^2 - t + 1\right)x$$
$$+ \frac{1}{16}\left(t^3 - 3\,t^2 + 4\,t - 1\right)\cdot \delta_9(t)\,,$$

where $\delta_9(t) = t^9 - 9\,t^8 + 23\,t^7 - 22\,t^6 + 14\,t^5 - 3\,t^4 - 5\,t^3 + 7\,t^2 - 4\,t - 1\,.$

$$D_{10}(x) := x^4 - \frac{\left(4\,t^6 - 16\,t^5 + 8\,t^4 + 8\,t^3 - 4\,t + 1\right)x^2}{2\left(t^2 - 3\,t + 1\right)^2} - 4\,\frac{t^3\,(2\,t-1)\,(t-1)\,x}{\left(t^2 - 3\,t + 1\right)^2}$$
$$+ \frac{\delta_{12}(t)}{16\left(t^2 - 3\,t + 1\right)^4}\,,$$

where
$$\delta_{12}(t) = 16\,t^{12} - 128\,t^{11} + 448\,t^{10} - 896\,t^9 + 1024\,t^8 - 416\,t^7 - 408\,t^6 + 608\,t^5$$
$$- 304\,t^4 + 48\,t^3 + 16\,t^2 - 8\,t + 1\,.$$

$$D_{12}(x) := x^4 + \frac{\left(12\,t^8 - 120\,t^7 + 336\,t^6 - 468\,t^5 + 372\,t^4 - 168\,t^3 + 36\,t^2 - 1\right)x^2}{2\,(t-1)^6}$$
$$- 4\,\frac{t\,(2\,t-1)\left(3\,t^2 - 3\,t + 1\right)\left(2\,t^2 - 2\,t + 1\right)x}{(t-1)^4} + \frac{\delta_{16}(t)}{16\,(t-1)^{12}}\,,$$

where
$$\delta_{16}(t) = 144\,t^{16} - 576\,t^{15} + 2112\,t^{14} - 9696\,t^{13} + 34016\,t^{12} - 82176\,t^{11} + 141936\,t^{10}$$
$$- 181984\,t^9 + 177240\,t^8 - 132528\,t^7 + 76096\,t^6 - 33208\,t^5 + 10760\,t^4$$
$$- 2480\,t^3 + 376\,t^2 - 32\,t + 1\,.$$

Table B ($N = 2$)

N	b	x_0	[**XY-type**] $j(E)$	Elliptic Curve $E : y^2 = D(x)$ / Shabat polynomial $p(x)$
2	–	0	**X(1,1,1,1)** $j = 1728$	$y^2 = D(x) \quad = x^4 - 4b \quad (b \in \mathbf{Q}^\times)$ $p(x) - 1 = \quad -\frac{1}{2b}x^4$ $p(x) + 1 = \quad -\frac{1}{2b}\left(x^4 - 4b\right)$
2	a^2	0	**X(1,2,1,2)** $j = \frac{2^{11}}{3}$	$y^2 = D(x) \quad = x^4 - 2ax^2 - 3a^2 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 2a = \quad -\frac{\left(x^2+a\right)\left(x^2-2a\right)^2}{a^2}$ $p(x) + 2a = \quad -\frac{x^4\left(x^2-3a\right)}{a^2}$
2	$\frac{a^2}{2}$	0	**X(1,3,1,3)** $j = 2^7$	$y^2 = D(x) \quad = x^4 - 2ax^2 - a^2 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 1 = \quad \frac{2\left(x^4-2ax^2-a^2\right)\left(x^2-a\right)^2}{a^4}$ $p(x) + 1 = \quad \frac{2x^4\left(x^2-2a\right)^2}{a^4}$
2	$\frac{a^2}{3}$	0	**X(1,5,1,5)** $j = 0$	$y^2 = D(x) \quad = x^4 - 2ax^2 - \frac{1}{3}a^2 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 1 = \quad -\frac{27}{2}\,\frac{x^4\left(x^2-2a\right)^2\left(x^2-a\right)^2}{a^6}$ $p(x) + 1 = \quad -\frac{1}{2}\,\frac{\left(3x^4-6ax^2-a^2\right)\left(3x^4-6ax^2+2a^2\right)^2}{a^6}$

For each of the above $D_N(x)$, we can compute quantities associated with the continued fraction expansion of $\sqrt{D_N(x)}$ of §3 so as to obtain Pell solutions (p_{n_0-1}, q_{n_0-1}), (p_{n_1-1}, q_{n_1-1}) and x_0 from the linear equation $(p'_{n_i-1}/q_{n_i-1})(x_0) = 0$ $(i = 0, 1)$ (cf. also Lemma 4.3). Note that these contain the moduli parameter t (or a, b when $N = 2, 3$).

Now, consider the equation $D_N(x_0) = 0$. Using a computer algebra system (e.g., Maple), one can factorize $D_N(x_0)$ as a polynomial of moduli parameter(s) so as to find rational zeros, if any, from linear factors. Each of such a moduli parameter gives a Belyi function of type (I) and hence a mono-nodal tree of shape Y.

To enumerate X-shaped cases, we employ Lemma 4.4 and search for moduli parameters for which $p_{n_1-1}(x_0) + q_{n_1-1}(x_0)\sqrt{D(x_0)} \in \left\{\pm 1, \pm\sqrt{-1}, \frac{\pm 1 \pm \sqrt{-3}}{2}\right\}$ (all possible roots of unity inside the quadratic number fields). In practice, it suffices to solve equations $q_{n_1-1}(x_0) = 0$, $p_{n_1-1}(x_0) = 0$, $p_{n_1-1}(x_0) = \pm\frac{1}{2}$, $p_{n_1-1}(x_0) = \pm 1$ respectively in $\mathbf{Q}[t]$ (or in $\mathbf{Q}[a, b]$ when $N = 2, 3$). The process ends anyway after a finite number of steps and leads to finitely many moduli parameters corresponding to the desired dessins.

Each of Table B ($N = 2$, $N = 3$ and $N \geq 4$) summarizes our computation by Maple for the X- and Y-shaped dessins of the smallest degrees in each moduli parameter distinguished in the above way. We found no rational solutions from $D_N(x)$ for $N = 8, 9, 10, 12$. Among the

Table B ($N = 3$)

N	b	x_0	**[XY-type]** $j(E)$	Elliptic Curve $E : y^2 = D(x)$ / Shabat polynomial $p(x)$
3	–	0	**Y{1,1,1}** $j = 0$	$y^2 = D(x) \quad = x^4 + bx \quad (b \in \mathbf{Q}^\times)$ $p(x) - 1 = \frac{2x^3}{b}$ $p(x) + 1 = \frac{2(x^3+b)}{b}$
3	$\frac{512a^3}{27}$	$\frac{2a}{3}$	**X(1,1,1,3)** $j = \frac{3^3 \cdot 7^3}{2^3}$	$y^2 = D(x) \quad = x^4 - 8a^2x^2 + \frac{512}{27}a^3x - \frac{592}{27}a^4 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 1 = \frac{27(x-2a)(27x^3+54ax^2-108a^2x+296a^3)(x+2a)^2}{32768a^6}$ $p(x) + 1 = \frac{(3x+10a)^2(3x-2a)^4}{32768a^6}$
3	$\frac{1024a^3}{27}$	$\frac{2a}{3}$	**X(1,2,4,2)** $j = \frac{3^2 \cdot 23^3}{2^6}$	$y^2 = D(x) \quad = x^4 - 8a^2x^2 + \frac{1024}{27}a^3x - \frac{1616}{27}a^4 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 1 = \frac{243(x-2a)(x+2a)^2(9x^3+18ax^2-36a^2x+184a^3)^2}{33554432a^9}$ $p(x) + 1 = \frac{(27x^3+54ax^2-108a^2x+808a^3)(3x+10a)^2(3x-2a)^4}{33554432a^9}$
3	$\frac{1024a^3}{81}$	$\frac{2a}{3}$	**X(1,2,1,5)** $j = \frac{3^4 \cdot 5^3}{2^6}$	$: y^2 = D(x) \quad = x^4 - 8a^2x^2 + \frac{1024}{81}a^3x - \frac{752}{81}a^4 \quad (a \in \mathbf{Q}^\times)$ $p(x) - 1 = \frac{729(x-2a)(x+2a)^2(3x+10a)^2(3x-2a)^4}{33554432a^9}$ $p(x) + 1 = \frac{(81x^3+162ax^2-324a^2x+376a^3)\lambda(x)^2}{33554432a^9}$, where $\lambda(x) = 81x^3 + 162ax^2 - 324a^2x - 392a^3$.

produced Belyi functions on elliptic curves, those of degrees($=$ deg $p(x)$) up to 7 in this table coincide with those found in an earliest publication of Birch [3], and in subsequent works by Tsunogai et al. ([19]; [9], [20]). The Belyi function of degree 24 giving X(1,5,13,5) was discussed in [13] from a viewpoint of the present paper.

Finally, we shall remark that all X-,Y-shaped mono-nodal trees defined over $\mathbf{Q}$ are obtained from Table B by composition with Chebyshev polynomials (cf. Remark 3.3). Suppose that a pair $(p, q) \in \mathbf{Q}[x]^2$ with $p^2 - Dq^2 = \kappa^{-1} \in \mathbf{Q}^\times$ provides such a tree, i.e., satisfies the condition (I) or (II) of §4. Then, by the above discussion, we may assume $D(x)$ is equal to some $D_N(x)$ ($N \in \{2, \ldots, 10, 12\}$) with special moduli parameters t (or, a, b) given in Table B. Note first that, in either case of (I), (II), we have $q(x_0)^2 D(x_0) = 0$, so that $p(x_0)^2 = \kappa^{-1}$. Hence, writing $\kappa = \lambda^2$ ($\lambda \in \mathbf{Q}^\times$) and replacing p, q by λp, λq respectively, we may assume $p^2 - Dq^2 = 1$. This argument works also for the primitive (i.e., the smallest degree) solution $p_{\nu n_0 - 1}(x)^2 - q_{\nu n_0 - 1}(x)^2 D(x) = \kappa_0^{-1} = \lambda_0^{-2}$ to (I), (II) given in Table B with the same moduli parameter.

In each entry of Table B, the last column indicates the corresponding quartic $D(x)$ as well as the factorizations of $(p_{\nu n_0 - 1}(x) \pm \lambda_0^{-1}) \in \mathbf{Q}[x]$ that reflect properties of the Shabat polynomial $\lambda_0\, p_{\nu n_0 - 1}(x)$. Take the sign of λ_0 so that $(p_{\nu n_0 - 1}(x) - \lambda_0^{-1})$ has the unique factor of multiplicity 3 or 4.

Now, by Lemma 3.1 and Proposition 4.1, we see that there exist $\kappa \in \mathbf{N}$, $c \in \mathbf{Q}^\times$ and

Table B ($N \geq 4$)

N	t	x_0	[XY-type] $j(E)$	Elliptic Curve E(Cremona Symbol) : $y^2 = D(x)$ / Shabat polynomial $p(x)$
4	$-\frac{9}{16}$	$\frac{1}{4}$	Y{1,1,2} $j = \frac{2 \cdot 47^3}{3^8}$	$E(48a6)$: $y^2 = D(x) = x^4 - \frac{13}{8}x^2 + \frac{9}{4}x - \frac{119}{256}$ $p(x) - \frac{3}{2} = -\frac{1}{144}(16x^2 - 24x + 17)(4x + 5)^2$ $p(x) + \frac{3}{2} = -\frac{1}{144}(4x + 7)(4x - 1)^3$
4	$-\frac{3}{16}$	$\frac{1}{4}$	X(1,5,13,5) $j = \frac{2^3 \cdot 23^3}{3^4}$	$E(96a4)$: $y^2 = D(x) = x^4 - \frac{7}{8}x^2 + \frac{3}{4}x - \frac{47}{256}$ $p(x) - 1 = -\frac{1}{161243136}(4x+5)^2(4x+1)^2(4x-1)^4\lambda(x)^2 \cdot \mu(x)^2,$ where $\lambda(x) = 256x^4 + 256x^3 - 96x^2 - 16x + 77,$ $\mu(x) = 256x^4 + 256x^3 - 96x^2 - 16x + 41.$ $p(x) + 1 = -\frac{1}{161243136}(256x^4 - 224x^2 + 192x - 47)$ $\times(16x^2 + 16x + 1)^2\nu(x)^2$ where $\nu(x) = 65536x^8 + 131072x^7 + 16384x^6 - 57344x^5$ $+22016x^4 + 24064x^3 - 7616x^2 - 1312x + 1249.$
5	$\frac{4}{3}$	$\frac{1}{6}$	Y{1,2,2} $j = -\frac{269^3}{2^{10} \cdot 3^5}$	$E(150a3)$: $y^2 = D(x) = x^4 + \frac{47}{18}x^2 - \frac{16}{3}x + \frac{1057}{1296}$ $p(x) - 1 = -\frac{1}{27648}(36x^2 + 48x + 151)(6x - 1)^3$ $p(x) + 1 = -\frac{1}{27648}(6x - 7)(36x^2 + 36x + 89)^2$
5	$-\frac{1}{3}$	$\frac{1}{3}$	Y{1,1,3} $j = \frac{5 \cdot 2^{12}}{3^5}$	$E(75c1)$: $y^2 = D(x) = x^4 - \frac{14}{9}x^2 + \frac{4}{3}x - \frac{23}{81}$ $p(x) - 1 = -\frac{1}{54}(27x^3 + 9x^2 - 39x + 23)(3x + 2)^2$ $p(x) + 1 = -\frac{1}{54}(3x + 4)^2(3x - 1)^3$
5	8	$-\frac{1}{2}$	X(1,1,1,2) $j = \frac{5 \cdot 211^3}{2^{15}}$	$E(50b2)$: $y^2 = D(x) = x^4 - \frac{17}{2}x^2 - 32x - \frac{1503}{16}$ $p(x) - 1 = -\frac{1}{4096}(8x^3 + 36x^2 + 94x + 167)(2x - 7)^2$ $p(x) + 1 = -\frac{1}{4096}(2x - 9)(2x + 1)^4$
6	$-\frac{25}{9}$	$\frac{1}{3}$	X(1,1,2,2) $j = \frac{11^3 \cdot 1979^3}{3 \cdot 2^3 \cdot 5^{12}}$	$E(90c7)$: $y^2 = D(x) = x^4 + \frac{74}{27}x^2 - \frac{1600}{81}x + \frac{9523}{243}$ $p(x) - \frac{10}{3} = \frac{9}{160000}(9x^2 + 30x + 89)(3x - 1)^4$ $p(x) + \frac{10}{3} = \frac{1}{480000}(27x^2 - 90x + 107)(27x^2 + 72x + 173)^2$
7	5	$-\frac{1}{2}$	X(1,2,2,2) $j = \frac{7 \cdot 3^3 \cdot 2099^3}{2^{14} \cdot 5^7}$	$E(490k2)$: $y^2 = D(x) = x^4 + \frac{39}{2}x^2 - 400x - \frac{59279}{16}$ $p(x) - 1 = -\frac{1}{51200000}(2x + 11)(8x^3 - 36x^2 + 278x - 3051)^2$ $p(x) + 1 = -\frac{1}{51200000}(8x^3 - 44x^2 + 398x - 5389)(2x + 1)^4$

$\varepsilon = \pm 1$ such that $p = cp_{\kappa n_0 - 1}$, $q = c\varepsilon q_{\kappa n_0 - 1}$, and find that both of the above ν and κ belong to the set $M := \{k \in \mathbf{N} \mid q_{kn_0-1}(x_0)^2 D(x_0) = 0\}$. On the other hand, by virtue of §3 (10), it is not difficult to see that the set M is a principal submonoid of $\mathbf{N}^\times$ generated by ν. Indeed, the point to see this assertion is to show that if $k, l \in M$ and $k > l$ then $k - l \in M$. But (10) and the Pell properties of $p_{mn_0-1} + q_{mn_0-1}\sqrt{D}$ ($m = k, l$) imply that $p_{(k-l)n_0-1} + q_{(k-l)n_0-1}\sqrt{D}$ is a constant multiple of $(p_{kn_0-1} + q_{kn_0-1}\sqrt{D})(p_{ln_0-1} + q_{ln_0-1}\sqrt{D})$. Noting then that $\sqrt{D(x)} \notin \mathbf{Q}[x]$, we also see that $q_{(k-l)n_0-1}\sqrt{D}$ is a constant multiple of $(p_{kn_0-1}q_{ln_0-1} + p_{ln_0-1}q_{kn_0-1})\sqrt{D}$. Specializing $x = x_0$ leads to the assertion. Thus, one finds $d \in \mathbf{N}$ such that $\kappa = d\nu$. This together with Lemma 3.2 concludes that $p(x)$ is given by the Chebyshev composition $T_d(\lambda_0 p_{\nu n_0-1}(x))$. This also shows that the lengths of the arms of the mono-nodal tree of $p(x)$ are d times those of $p_{\nu n_0-1}(x)$.

EXAMPLE. Let $N=6$ so that $n_0=5$ and $n_1=10$, and consider the case $X(1,1,2,2)$. In this case, Table B gives the quartic

$$D(x)=D_6(x)=x^4+\frac{74}{27}x^2-\frac{1600}{81}x+\frac{9523}{243}=\frac{1}{243}\big(27x^2-90x+107\big)\big(9x^2+30x+89\big).$$

The continued fraction algorithm produces a first Pell solution $(p(x),q(x))=(p_4(x),q_4(x))$ with $p(x)^2-D(x)q(x)^2=\frac{100}{9}$. Putting $\lambda_0=u_0^{-1}=\frac{3}{10}$, we obtain a Belyi function $\beta(x,y)=p(x)+q(x)y$ on the elliptic curve $E: y^2=D(x)$ fitting into the Pakovich diagram (1). Figure B topologically illustrates the corresponding dessins, where solid lines indicate the segment $t\in[-1,1]$ and its inverse image for the dessin of Shabat polynomial $P(x)=\lambda_0 p(x)$, and dashed lines indicate the segment $u\in[0,u_0]$ and its inverse image for the dessin of the Belyi function β. The elliptic curve E is developed into the "wallpaper" uniformization where a period lattice is indicated by dotted lines.

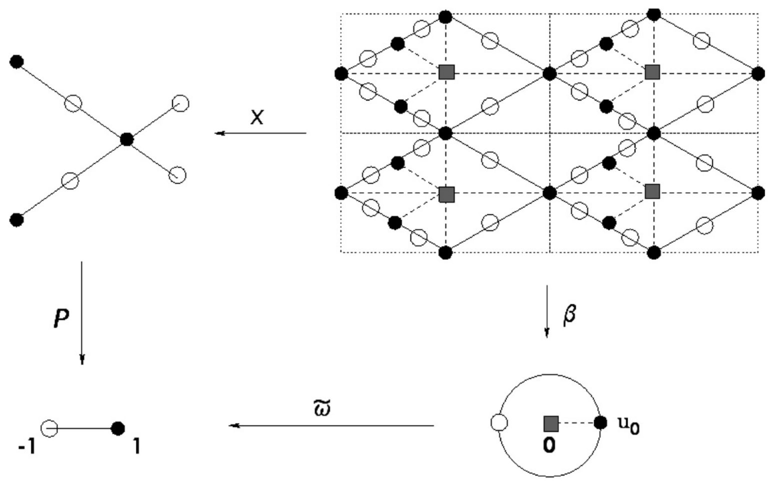

Figure B

ACKNOWLEDGMENT. The author thanks Hiroshi Tsunogai for valuable discussions and information on [9], [19], [20] which are closely related to the subject of this paper. He also would like to express his sincere gratitude to the referee for a number of remarks that were very useful to improve the presentation of this paper. This work was partially supported by KAKENHI (24654006).

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