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The Equivalence Theorem of Kinetic Solutions and Entropy Solutions for Stochastic Scalar Conservation Laws

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Abstract. In this paper, we prove the equivalence of kinetic solutions and entropy solutions for the initial-boundary value problem with a non-homogeneous boundary condition for a multi-dimensional scalar first-order conservation law with a multiplicative noise. We somewhat generalized the definitions of kinetic solutions and of entropy solutions given in Kobayasi and Noboriguchi [8] and Bauzet, Vallet and Wittobolt [1], respectively.

1. Introduction

In this paper we study the first order stochastic conservation law of the following type

$$du + \operatorname{div}(A(u))dt = \Phi(u)dW(t) \quad \text{in } \ \Omega \times Q\,, \tag{1}$$

with the initial condition

$$u(0,\cdot) = u_0(\cdot) \quad \text{in } \ \Omega \times D\,, \tag{2}$$

and the formal boundary condition

$$\text{"}u = u_b\text{"} \quad \text{on } \ \Omega \times \Sigma\,. \tag{3}$$

Here $D \subset \mathbb{R}^d$ is a bounded domain with a Lipschitz boundary ∂D, $T > 0$, $Q = (0,T) \times D$, $\Sigma = (0,T) \times \partial D$ and W is a cylindrical Wiener process defined on a stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t), P)$. More precisely, $(\mathscr{F}_t)$ is a complete right-continuous filtration and $W(t) = \sum_{k=1}^{\infty} \beta_k(t)e_k$ with $(\beta_k)_{k\geq 1}$ being mutually independent real-valued standard Wiener processes relative to $(\mathscr{F}_t)$ and $(e_k)_{k\geq 1}$ is a complete orthonormal system in a separable Hilbert space H (cf. [3] for example). Our purpose of this paper is to present the definitions of a kinetic solution and an entropy solution to the initial-boundary value problem (1)–(3) and to prove that such solutions are equivalent.

In the case of $\Phi = 0$, Eq.(1) becomes a deterministic scalar conservation law. In this case, a well known difficulty for the boundary condition (3) is that if (3) were assumed in the

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classical sense the problem (1)–(3) would be overdetermined. In the BV setting Bardos, Le Roux and Nédélec [2] first gave an interpretation of the boundary condition (3) as an "entropy" inequality on Σ. This condition is known as the BLN condition. However the BLN condition makes sense only if there exists a trace of solutions on ∂D. Otto [10] extended it to the L^∞ setting by introducing the notion of boundary entropy-flux pairs. Imbert and Vovelle [5] gave a kinetic formulation of weak entropy solutions of the initial-boundary value problem and proved equivalence between such a kinetic solution and weak entropy solution. Concerning deterministic degenerate parabolic equations, see [7] and [9].

To add a stochastic forcing $\Phi(u)dW(t)$ is natural for applications, which appears in wide variety of fields as physics, engineering and others. There are a few paper concerning the Dirichlet boundary value problem for stochastic conservation laws. See Kim [6], Vallet and Wittbold [11] and the references therein. Using the notion of entropy solutions, Bauzet, Vallet and Wittbold [1] studied the Dirichlet problem in the case of multiplicative noise under the restricted assumption that the flux function A is global Lipschitz. On the other hand, using the notion of kinetic solutions, Kobayasi and Noboriguchi [8] extended the result of Debussche and Vovelle [4] to the multidimensional Dirichlet problem with multiplicative noise without the assumption that A is global Lipschitz.

The aim of the present paper is to prove an equivalence theorem between kinetic solutions and entropy solutions to the problem (1)–(3). In the deterministic case the equivalence has been proved in [5]. Therefore our result is a counterpart of it in the stochastic case. In order to show that an entropy solution is a kinetic solution we introduce a notion of renormalized kinetic solutions in which both of the boundary defect measure $\bar{m}^\pm$ and the defect measure m are renormalized. We note that in [8] only the boundary defect measure is renormalized. We note that the uniqueness of such renormalized kinetic solution can be also proved in the same way as in [8]

We now give the precise assumptions under which the problem (1)–(3) is considered:

(H_1) The flux function $A\colon \mathbb{R} \to \mathbb{R}^d$ is of class C^2 and its derivatives have at most polynomial growth.

(H_2) For each $z \in L^2(D)$, $\Phi(z) : H \to L^2(D)$ is defined by $\Phi(z)e_k = g_k(\cdot, z(\cdot))$, where $g_k \in C(D \times \mathbb{R})$ satisfies the following conditions:

$$G^2(x,\xi) = \sum_{k=1}^{\infty} |g_k(x,\xi)|^2 \le L(1 + |\xi|^2)\,, \tag{4}$$

$$\sum_{k=1}^{\infty} |g_k(x,\xi) - g_k(y,\zeta)|^2 \le L\big(|x-y|^2 + |\xi - \zeta|\, r(|\xi - \zeta|)\big) \tag{5}$$

for every $x, y \in D$, $\xi, \zeta \in \mathbb{R}$. Here, L is a positive constant and r is a continuous nondecreasing function on $\mathbb{R}_+$ with $r(0) = 0$.

(H_3) $u_0 \in L^\infty(\Omega \times D)$ and u_0 is $\mathscr{F}_0 \otimes \mathscr{B}(D)$-measurable, where $\mathscr{B}(D)$ is the Borel σ-field on D. $u_b \in L^\infty(\Omega \times \Sigma)$ and $\{u_b(t)\}$ is predictable, in the following sense: For every $p \in [1, \infty)$, the $L^p(\partial D)$-valued process $\{u_b(t)\}$ is predictable with respect to the filtration $(\mathscr{F}_t)$.

Note that under the assumption (H_1) an important example of inviscid Burgers' equation (i.e. $A(\xi) = \xi^2/2$) can be included and that by (4) one has

$$\Phi : L^2(D) \to L_2(H; L^2(D)), \tag{6}$$

where $L_2(H; L^2(D))$ denotes the set of Hilbert–Schmidt operators from H to $L^2(D)$.

This paper is organized as follows. In Section 2, we introduce the notion of kinetic solutions and entropy solutions to (1)–(3) and state our result. In Section 3, we give a proof of it.

2. Statement of the result

We now give some notations and the definitions of kinetic solutions and entropy solutions in this section. We choose a finite open cover $\{U_{\lambda_i}\}_{i=0,\dots,M}$ of $\overline{D}$ and a partition of unity $\{\lambda_i\}_{i=0,\dots,M}$ on $\overline{D}$ subordinated to $\{U_{\lambda_i}\}$ such that $U_{\lambda_0} \cap \partial D = \emptyset$, for $i = 1, \dots, M$,

$$D^{\lambda_i} := D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d > h_{\lambda_i}(\overline{\mathcal{A}_i x})\} \quad \text{and}$$

$$\partial D^{\lambda_i} := \partial D \cap U_{\lambda_i} = \{x \in U_{\lambda_i}; (\mathcal{A}_i x)_d = h_{\lambda_i}(\overline{\mathcal{A}_i x})\},$$

with a Lipschitz function $h_{\lambda_i} : \mathbb{R}^{d-1} \to \mathbb{R}$, where $\mathcal{A}_i$ is an orthogonal matrix corresponding to a change of coordinates of $\mathbb{R}^d$ and $\bar{y}$ stands for $(y_1, \dots, y_{d-1})$ if $y \in \mathbb{R}^d$. For the sake of clarity, we will drop the index i of λ_i and we will suppose that the matrix $\mathcal{A}_i$ equals to the identity. We also set $Q^\lambda = (0, T) \times D^\lambda$, $\Sigma^\lambda = (0, T) \times \partial D^\lambda$ and $\Pi^\lambda = \{\bar{x}; x \in U_\lambda\}$.

To regularize functions that are defined on D^λ and $\mathbb{R}$, let us consider a standard mollifier ρ on $\mathbb{R}$, that is, ρ is a nonnegative and even function in $C_c^\infty((-1, 1))$ such that $\int_{\mathbb{R}} \rho = 1$. We set $\rho^\lambda(x) = \Pi_{i=1}^{d-1} \rho(x_i)\rho(x_d - (L_\lambda + 1))$ for $x = (x_1, \dots, x_d)$ with the Lipschitz constant L_λ of h_λ on Π^λ. Moreover we denote by ψ a standard mollifier on $\mathbb{R}_\xi$. For $\varepsilon, \delta > 0$ we set $\rho_\varepsilon^\lambda(x) = \frac{1}{\varepsilon^d}\rho^\lambda(\frac{x}{\varepsilon})$ and $\psi_\delta(\xi) = \frac{1}{\delta}\psi(\frac{\xi}{\delta})$.

We denote by $\mathcal{E}^+$ the set of non-negative convex functions η in $C^\infty(\mathbb{R})$ such that $\eta(x) = 0$ if $x \le 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$. We also denote by $\mathcal{E}^-$ the set $\{\eta \in C^\infty(\mathbb{R}); \eta(-\cdot) \in \mathcal{E}^+\}$. For convenience, define

$$\operatorname{sgn}^+(r) = \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r \le 0, \end{cases} \quad \text{and} \quad \operatorname{sgn}^-(r) = \begin{cases} -1 & \text{if } r < 0, \\ 0 & \text{if } r \ge 0, \end{cases}$$

$$\mathcal{F}^\pm(\xi, \kappa) = \operatorname{sgn}^\pm(\xi - \kappa)\,(A(\xi) - A(\kappa))$$

$$\text{and for any } \eta \in \mathcal{E}^+ \cup \mathcal{E}^-, \quad \mathcal{F}^\eta(\xi, \kappa) = \int_\kappa^\xi \eta'(\zeta - \kappa)a(\zeta)\,d\zeta,$$

where $a(\xi) = A'(\xi)$.

DEFINITION 2.1 (Kinetic measure). The set $\{m_N; N > 0\}$ of maps m_N from Ω to $\mathcal{M}_b^+([0,T) \times D \times (-N,N))$ is said to be a kinetic measure if

(i) for each $N > 0$, m_N are weak measurable,

(ii) if $A_N = [0,T) \times D \times \{\xi \in \mathbb{R}; N-1 \leq |\xi| \leq N\}$ then

$$\lim_{N\to\infty} \mathbb{E} m_N(A_N) = 0\,, \tag{7}$$

(iii) for all $\phi \in C_b(D \times (-N,N))$, the process

$$t \mapsto \int_{[0,t]\times D\times(-N,N)} \phi(x,\xi)\, dm_N(s,x,\xi) \tag{8}$$

is predictable,

where $\mathbb{E}X$ denotes the expectation of a random variable X, $C_b(D \times (-N,N))$ is the set of bounded continuous functions over $D \times (-N,N)$ and $\mathcal{M}_b^+([0,T) \times D \times (-N,N))$ is the set of non-negative finite measures over $[0,T) \times D \times (-N,N)$.

DEFINITION 2.2 (Kinetic solution). Let u_0 and u_b satisfy (H$_3$). A measurable function $u : \Omega \times Q \to \mathbb{R}$ is said to be a kinetic solution of (1)–(3) if the following conditions (i)–(iii) hold:

(i) $\{u(t)\}$ is predictable,

(ii) for all $p \in [1,\infty)$ there exists a constant $C_p \geq 0$ such that for a.e. $t \in [0,T]$,

$$\|u(t)\|_{L^p(\Omega\times D)} \leq C_p, \tag{9}$$

(iii) there exist kinetic measures $\{m_N^\pm; N > 0\}$ and for any $N > 0$ nonnegative functions $\bar{m}_N^\pm \in L^1(\Omega \times \Sigma \times (-N,N))$ such that $\{\bar{m}_N^\pm(t)\}$ is predictable, $\bar{m}_N^+(N-0) = \bar{m}_N^-(-N+0) = 0$ for sufficiently large $N > 0$ and $f_+ := \mathbf{1}_{u>\xi}$, $f_- := f_+ - 1 = -\mathbf{1}_{u\leq\xi}$ satisfy: for all $\varphi \in C_c^\infty([0,T) \times \overline{D} \times (-N,N))$,

$$\begin{aligned}
&\int_Q \int_{-N}^N f_\pm(\partial_t + a(\xi)\cdot\nabla)\varphi\, d\xi dx dt \\
&\quad + \int_D \int_{-N}^N f_\pm^0 \varphi(0)\, d\xi dx + M_N \int_\Sigma \int_{-N}^N f_\pm^b \varphi\, d\xi d\sigma dt \\
&= -\sum_{k=1}^\infty \int_0^T \int_D g_k(x,u)\, \varphi(t,x,u)\, dx d\beta_k(t) \\
&\quad - \frac{1}{2} \int_Q G^2(x,u)\, \partial_\xi \varphi(t,x,u)\, dx dt
\end{aligned}$$

$$+\int_{[0,T)\times D\times(-N,N)} \partial_\xi \varphi dm_N^\pm(t,x,\xi) + \int_\Sigma \int_{-N}^N \partial_\xi \varphi\, \bar{m}_N^\pm \, d\xi d\sigma dt \quad \text{a.s.} \quad (10)$$

where $M_N = \max_{-N\le\xi\le N} |a(\xi)|$. In (10), $f_+^0 = \mathbf{1}_{u_0>\xi}$, $f_+^b = \mathbf{1}_{u_b>\xi}$, $f_-^0 = f_+^0 - 1$ and $f_-^b = f_+^b - 1$.

REMARK 2.3. Even when we define kinetic solutions as above, we can get a unique existence theorem of kinetic solutions in a similar method as in [8].

REMARK 2.4. The main difficulty is to give an appropriate boundary condition. We treat this difficulty by using the kinetic trace $\bar{f}_\pm$ (see (27) below) of the kinetic solution $f_\pm$. Two boundary defect measures $\bar{m}_N^\pm$ are introduced and characterized by the formula

$$\partial_\xi \bar{m}_N^\pm = -M_N f_\pm^b \pm (-a\cdot n)\bar{f}_\pm \quad (11)$$

(see (30) and (31) below). The relation (11) can be understood as a kinetic analogue of the notion of boundary entropy-flux pairs introduced by Otto [10].

DEFINITION 2.5 (Entropy solution). Let u_0 and u_b satisfy (H$_3$). Let $u : \Omega\times Q \to \mathbb{R}$ be a measurable function.

(1) u is said to be an entropy sub-solution of (1)–(3) with data (u_0, u_b) if it satisfies:

(i) $\{u(t)\}$ is predictable,

(ii) for all $p\in[1,\infty)$ there exists a constant $C_p \ge 0$ such that for a.e. $t\in[0,T]$,

$$\|u(t)\|_{L^p(\Omega\times D)} \le C_p\,, \quad (12)$$

(iii) a.e. $N\in\mathbb{R}^+$, for any $\kappa\in(-N,N)$, $\eta\in\mathcal{E}^+$ and $\varphi\in C_c^\infty([0,T)\times\overline{D})$ with $\varphi\ge 0$,

$$\begin{aligned}\int_Q &\eta(u\wedge N-\kappa)\partial_t\varphi\, dxdt + \int_Q \mathcal{F}^\eta(u\wedge N,\kappa)\cdot\nabla\varphi\, dxdt \\ &+ \int_D \eta(u_0\wedge N-\kappa)\varphi(0)\,dx + M_N\int_\Sigma \eta(u_b\wedge N-\kappa)\varphi\, d\sigma dt \\ &+ \sum_{k=1}^\infty \int_0^T\int_D \mathbf{1}_{u\in(-N,N)}\eta'(u-\kappa)g_k(x,u)\varphi\, dxd\beta_k(t) \\ &+ \frac{1}{2}\int_Q \mathbf{1}_{u\in(-N,N)}\eta''(u-\kappa)G^2(x,u)\varphi\, dxdt \ge I_N^+(\varphi) \quad \text{a.s.}\,, \end{aligned} \quad (13)$$

where $I_N^+(\varphi)$ are non-positive and satisfy $\limsup_{N\to\infty} I_N^+(\varphi) = 0$.

(2) u is said to be an entropy super-solution of (1)–(3) with data (u_0, u_b) if definition of entropy sub-solution (iii) is replaced by

a.e. $N \in \mathbb{R}^+$, for any $\kappa \in (-N, N)$, $\eta \in \mathcal{E}^-$ and $\varphi \in C_c^\infty([0,T) \times \overline{D})$ with $\varphi \geq 0$,

$$\begin{aligned}
&\int_Q \eta(u \vee N - \kappa)\partial_t \varphi \, dxdt + \int_Q \mathcal{F}^\eta(u \vee N, \kappa) \cdot \nabla\varphi \, dxdt \\
&\quad + \int_D \eta(u_0 \vee N - \kappa)\varphi(0)\, dx + M_N \int_\Sigma \eta(u_b \vee N - \kappa)\varphi \, d\sigma dt \\
&\quad + \sum_{k=1}^\infty \int_0^T \int_D \mathbf{1}_{u \in (-N,N)} \eta'(u-\kappa) g_k(x,u)\varphi \, dxd\beta_k(t) \\
&\quad + \frac{1}{2}\int_Q \mathbf{1}_{u \in (-N,N)} \eta''(u-\kappa) G^2(x,u)\varphi \, dxdt \geq I_N^-(\varphi) \quad \text{a.s.}, \qquad (14)
\end{aligned}$$

where $I_N^-(\varphi)$ are non-positive and satisfy $\limsup_{N\to\infty} I_N^-(\varphi) = 0$.

(3) u is said to be an entropy solution of (1)–(3) with data (u_0, u_b) if it is both an entropy sub- and super-solution.

We are in a position to state our result.

THEOREM 2.6. *Let the assumptions* (H_1)–(H_3) *hold true. Then the following statements are equivalent each other:*

(i) *u is a kinetic solution of* (1)–(3) *with data* (u_0, u_b).
(ii) *u is an entropy solution of* (1)–(3) *with data* (u_0, u_b).

REMARK 2.7. If u is an entropy solution to (1)–(3), then u is a weak solution over $[0,T) \times D$ in the following sense; for any $\varphi \in C_c^\infty([0,T) \times D)$

$$\begin{aligned}
&\int_Q u(t,x)\partial_t\varphi \, dxdt + \int_D u_0(x)\varphi(0)\, dx + \int_Q A(u(t,x)) \cdot \nabla\varphi \, dxdt \\
&\quad + \sum_{k=1}^\infty \int_0^T \int_D g_k(x, u(t,x))\varphi \, dxd\beta_k(t) = 0 \quad \text{a.s.} \qquad (15)
\end{aligned}$$

Indeed, we take $\varphi \in C_c^\infty([0,T) \times D)$. Passing $N \to \infty$ in (13) and then passing $\kappa \to -\infty$, we obtain

$$\begin{aligned}
&\int_Q u(t,x)\partial_t\varphi \, dxdt + \int_D u_0(x)\varphi(0)\, dx + \int_Q A(u(t,x)) \cdot \nabla\varphi \, dxdt \\
&\quad + \sum_{k=1}^\infty \int_0^T \int_D g_k(x, u(t,x))\varphi \, dxd\beta_k(t) \geq 0 \quad \text{a.s.} \qquad (16)
\end{aligned}$$

In a similar manner, from (14) we obtain

$$\int_Q u(t,x)\partial_t\varphi\,dxdt + \int_D u_0(x)\varphi(0)\,dx + \int_Q A(u(t,x))\cdot\nabla\varphi\,dxdt$$
$$+\sum_{k=1}^{\infty}\int_0^T\int_D g_k(x,u(t,x))\varphi\,dxd\beta_k(t) \le 0 \quad \text{a.s.} \tag{17}$$

Combining (16) with (17) yields (15).

3. Proof

To prove that a kinetic solution is an entropy solution, we need the following two lemmas.

LEMMA 3.1. *We assume that for each* $N > 0$ (10) *holds true. Then for any* $N_2 > N_1 > 0$ *and* $\varphi \in C_c^\infty([0,T)\times\overline{D}\times(-N_1,N_1))$,

$$\int_{[0,T)\times D\times(-N_1,N_1)} \partial_\xi\varphi\,dm_{N_1}^{\pm}(t,x,\xi) = \int_{[0,T)\times D\times(-N_1,N_1)} \partial_\xi\varphi\,dm_{N_2}^{\pm}(t,x,\xi)\,. \tag{18}$$

PROOF. Let $\bar{f}_{\pm}^{\lambda}$ be any weak* limit of $\{f_{\pm}^{\lambda,\varepsilon}\}$ as $\varepsilon \to +0$ in $L^\infty(\Sigma^\lambda\times\mathbb{R})$ for any element λ of the partition of unity $\{\lambda_i\}$ on $\overline{D}$, where $f_{\pm}^{\lambda,\varepsilon}$ is denoted by

$$f_{\pm}^{\lambda,\varepsilon}(t,x,\xi) = \int_{D^\lambda} f_{\pm}(t,x,\xi)\rho_\varepsilon^\lambda(y-x)\,dy\,,$$

and let $\bar{f}_{\pm} = \sum_{i=0}^{M}\lambda_i\bar{f}_{\pm}^{\lambda_i}$. We take $\phi \in C_c^\infty([0,T)\times\overline{D}\times(-N,N))$. Put $\varphi = \bar{W}_\varepsilon\phi^\lambda$ in (10), where $\phi^\lambda = \phi\lambda$ and for any $\varepsilon > 0$

$$\bar{W}_\varepsilon = \int_0^{x_d-h_\lambda(\bar{x})} \rho_\varepsilon^\lambda(r-\varepsilon(L_\lambda+1))\,dr\,. \tag{19}$$

Letting $\varepsilon \to +0$ and summing over i, we obtain

$$\int_Q\int_{-N}^{N} f_{\pm}(\partial_t + a(\xi)\cdot\nabla)\phi\,d\xi dxdt + \int_D\int_{-N}^{N} f_{\pm}^0\phi(0)\,d\xi dx$$
$$+\int_\Sigma\int_{-N}^{N}(-a(\xi)\cdot\mathbf{n})\bar{f}_{\pm}\phi\,d\xi d\sigma dt$$
$$= -\sum_{k=1}^{\infty}\int_0^T\int_D g_k(x,u)\phi(t,x,u)\,dxd\beta_k(t)$$
$$-\frac{1}{2}\int_Q G^2(x,u)\partial_\xi\phi(t,x,u)\,dxdt + \int_{[0,T)\times D\times(-N,N)}\partial_\xi\phi\,dm_N^{\pm}(t,x,\xi) \quad \text{a.s.} \tag{20}$$

We take $\phi \in C_c^\infty([0,T)\times\overline{D}\times(-N_1,N_1))$. Subtracting the case $N = N_1$ from the case $N = N_2$ in (20), we can get the desired result. □

For a kinetic measure $\{m_N; N > 0\}$, from (7) there exists a set $\Omega_0 \subset \Omega$ of full measure and a sequence $\{N_k; k \in \mathbb{N}\} \subset \mathbb{N}$ such that for all $\omega \in \Omega_0$,

$$\begin{aligned}&\lim_{k\to\infty} m_{N_k}([0,T)\times D\times[N_k-1,N_k))\\ &\qquad = \lim_{k\to\infty} m_{N_k}([0,T)\times D\times(-N_k,-N_k+1]) = 0\,.\end{aligned}\tag{21}$$

For each $N \in \mathbb{N}$ and fixed $\omega \in \Omega_0$, we define the non-decreasing functions on $[-N, N]$ by

$$\mu_N(\xi) = m_N([0,T)\times D\times(-N,\xi))\,.\tag{22}$$

Let $\mathbb{D}_N$ be the sets of $\xi \in (N-1, N)$ such that μ_i, $i = N, N+1, \dots,$ are differentiable at ξ and $-\xi$. We also set $\mathbb{D} = \cup_{N=1}^{\infty}\mathbb{D}_N$. It is easy to see that $\mathbb{D}_N$ and $\mathbb{D}$ are full sets of $(N-1, N)$ and $\mathbb{R}$, respectively.

LEMMA 3.2. *It holds true*:

(i) *Let $N_0 \in \mathbb{N}$. If $a \in \mathbb{D}_{N_0}$, then for all $N \in \mathbb{N}$ with $N \geq N_0$, as $\delta \downarrow 0$*

$$\int_{-N}^{N} \psi_\delta(\xi \pm a)\,d\mu_N(\xi) \to \mu_N'(\mp a)\,.\tag{23}$$

(ii) *There exists a sequence $\{a_N; N \in \mathbb{N}\} \subset \mathbb{A}$ such that*

$$\liminf_{N\to\infty} \mu_N'(\pm N \mp a_N) = 0\,,\tag{24}$$

where $\mathbb{A} = \cap_{N=1}^{\infty}\{a \in (0,1); N - a \in \mathbb{D}_N\}$.

PROOF. Let $a \in \mathbb{D}_{N_0}$. Since $\mu_N(\zeta \mp a) = \mu_N(\mp a) + \mu_N'(\mp a)\zeta + o(\zeta)$ for each $N \in \mathbb{N}$ with $N \geq N_0$, it follows that

$$\int_{-N}^{N} \psi_\delta(\zeta \pm a)\,d\mu_N(\zeta) = -\int_{-\delta}^{\delta} \mu_N(\zeta \mp a)\psi_\delta'(\zeta)\,d\zeta = \mu_N'(\mp a) - \int_{-\delta}^{\delta} o(\zeta)\psi_\delta'(\zeta)\,d\zeta\,.$$

Besides, the last term of the right hand on the above equality tends to 0 as $\delta \to +0$. Indeed, since for any $\varepsilon > 0$ there exists $\delta_0 > 0$ such that if $|\zeta| < \delta_0$ then $|o(\zeta)| \leq \varepsilon\,|\zeta|$, if $0 < \delta < \delta_0$ then

$$\left|\int_{-\delta}^{\delta} o(\zeta)\psi_\delta'(\zeta)\,d\zeta\right| \leq \varepsilon \int_{-\delta}^{\delta} \left|\zeta\,\psi_\delta'(\zeta)\right| d\,\zeta \leq \varepsilon\,.$$

Thus we obtain the claim of (i).

Next, let us assume that there exists a number $k \in \mathbb{N}$ such that for any $N \geq k$ and $a \in \mathbb{A}$,

$$\mu_N'(N-a) > \frac{1}{k}\,.$$

Since the function $\xi \mapsto \mu_N(\xi)$ is non-decreasing, for all $N \in \mathbb{N}$ with $N \geq k$

$$\mu_N(N) - \mu_N(N-1) \geq \int_{N-1}^{N} \mu_N'(\xi)\,d\xi$$

$$= \int_0^1 \mu'_N(N-1+\xi)\,d\xi > \int_0^1 \frac{1}{k}\,d\xi = \frac{1}{k} > 0\,.$$

This contradicts the limit (21). Thus for each $k \in \mathbb{N}$, there exist a number $N_k \geq k$ and $a_k \in \mathbb{A}$ such that

$$\mu'_{N_k}(N_k - a_k) \leq \frac{1}{k}\,.$$

$\square$

PROOF OF THEOREM 2.6 . Consider a kinetic solution u of (1)–(3). We take $N \in \mathbb{D}$, $\varphi \in C_c^\infty([0,T)\times\overline{D})$ with $\varphi \geq 0$, $\kappa \in (-N, N)$ and $\eta \in \mathcal{E}^+$. We define the cutoff functions as follows. For each $\delta > 0$,

$$\Psi_\delta(\xi) = \int_{-\infty}^{\xi} \{\psi_\delta(\zeta + N - \delta) - \psi_\delta(\zeta - N + \delta)\}\,d\zeta\,.$$

Putting $(t,x,\xi) \mapsto \varphi(t,x)\eta'(\xi-\kappa)\Psi_\delta(\xi)$ in (10), we obtain

$$\begin{aligned}
&\int_Q \int_{-N}^{N} f_\pm(\partial_t + a(\xi)\cdot\nabla)\varphi\eta'(\xi-\kappa)\Psi_\delta(\xi)\,d\xi dx dt \\
&\quad + \int_D \int_{-N}^{N} f_\pm^0 \varphi(0)\eta'(\xi-\kappa)\Psi_\delta(\xi)\,d\xi dx \\
&\quad + M_N \int_\Sigma \int_{-N}^{N} f_\pm^b \varphi\eta'(\xi-\kappa)\Psi_\delta(\xi)\,d\xi d\sigma dt \\
&= -\sum_{k=1}^{\infty}\int_0^T \int_D g_k(x,u)\,\varphi(t,x)\eta'(u-\kappa)\Psi_\delta(u)\,dx d\beta_k(t) \\
&\quad - \frac{1}{2}\int_Q G^2(x,u)\,\partial_\xi\varphi(t,x)\eta'(u-\kappa)\Psi_\delta(u)\,dxdt \\
&\quad + \int_{[0,T)\times D\times(-N,N)} \varphi\{\eta''(\xi-\kappa)\Psi_\delta(\xi) + \eta'(\xi-\kappa)\Psi'_\delta(\xi)\}\,dm_N(t,x,\xi) \\
&\quad + \int_\Sigma \int_{-N}^{N} \varphi\bar{m}_N^\pm\{\eta''(\xi-\kappa)\Psi_\delta(\xi) + \eta'(\xi-\kappa)\Psi'_\delta(\xi)\}\,d\xi d\sigma dt \quad \text{a.s.}
\end{aligned} \tag{25}$$

Using Lemma 3.1 and Lemma 3.2 (i), we can compute the third term on the right hand of (25) as follows. For some $N_0 \in \mathbb{N}$ with $N_0 > N$,

$$\begin{aligned}
&\int_{[0,T)\times D\times(-N,N)} \varphi\{\eta''(\xi-\kappa)\Psi_\delta(\xi) + \eta'(\xi-\kappa)\Psi'_\delta(\xi)\}\,dm_N(t,x,\xi) \\
&\quad = \int_{[0,T)\times D\times(-N,N)} \varphi\{\eta''(\xi-\kappa)\Psi_\delta(\xi) + \eta'(\xi-\kappa)\Psi'_\delta(\xi)\}\,dm_{N_0}(t,x,\xi) \\
&\quad \geq \int_{[0,T)\times D\times(-N,N)} \varphi\,\eta'(\xi-\kappa)\Psi'_\delta(\xi)\,dm_{N_0}(t,x,\xi)
\end{aligned}$$

$$\geq -C\int_{[0,T)\times D\times(-N,N)} \psi_\delta(\xi - N + \varepsilon)\, dm_{N_0}(t,x,\xi)$$
$$\geq -C\int_{-N}^{N} \psi_\delta(\xi - N + \varepsilon)\, d\mu_{N_0}(\xi)$$
$$\to -C\mu'_{N_0}(N) =: I_N^+(\varphi), \quad \text{as } \delta \downarrow 0\,.$$

Since by Lemma 3.2 (ii) there exists a sequence $\{a_i\} \subset \mathbb{A}$ such that $\liminf_{i\to\infty} \mu'_i(\pm i \mp a_i) = 0$, if we set $N_i = i - a_i$, then $\limsup_{i\to\infty} I_{N_i}^+(\varphi) = 0$. Therefore using integration by parts it is easy to see that as $\delta \downarrow 0$ we get (13). In a similar manner, we can also obtain (14).

Next we prove that an entropy solution is a kinetic solution. Consider an entropy solution u of (1)–(3). Define linear forms $\nu_N^\pm$ on $C_c^\infty([0,T)\times\overline{D}\times(-N,N))$ by

$$\begin{aligned}
\nu_N^+(\varphi) :=& \int_Q\int_{-N}^{N} (u\wedge N - \xi)^+ \partial_t\varphi\, d\xi dx dt \\
&+\int_Q\int_{-N}^{N} \mathcal{F}^+(u\wedge N,\xi)\nabla\varphi\, d\xi dx dt \\
&+\int_D\int_{-N}^{N} (u_0\wedge N - \xi)^+\varphi(0)\, d\xi dx \\
&+M_N\int_\Sigma\int_{-N}^{N} (u_b\wedge N - \xi)^+\varphi\, d\xi d\sigma dt \\
&+\sum_{k=1}^{\infty}\int_0^T\int_D\int_{-N}^{N} \mathbf{1}_{u\in(-N,N)}\operatorname{sgn}_+(u-\xi) g_k(x,u)\varphi\, d\xi dx d\beta_k(t) \\
&+\frac{1}{2}\int_Q G^2(x,u)\varphi(t,x,u)\, dx dt - \int_{-N}^{N} I_N^+(\varphi)\, d\xi
\end{aligned}$$

and

$$\begin{aligned}
\nu_N^-(\varphi) :=& \int_Q\int_{-N}^{N} (u\vee N - \xi)^- \partial_t\varphi\, d\xi dx dt \\
&+\int_Q\int_{-N}^{N} \mathcal{F}^-(u\vee N,\xi)\nabla\varphi\, d\xi dx dt \\
&+\int_D\int_{-N}^{N} (u_0\vee N - \xi)^-\varphi(0)\, d\xi dx \\
&+M_N\int_\Sigma\int_{-N}^{N} (u_b\vee N - \xi)^-\varphi\, d\xi d\sigma dt \\
&+\sum_{k=1}^{\infty}\int_0^T\int_D\int_{-N}^{N} \mathbf{1}_{u\in(-N,N)}\operatorname{sgn}_-(u-\xi) g_k(x,u)\varphi\, d\xi dx d\beta_k(t)
\end{aligned}$$

$$+\frac{1}{2}\int_Q G^2(x,u)\varphi(t,x,u)\,dxdt - \int_{-N}^{N} I_N^-(\varphi)\,d\xi\,.$$

Since u is an entropy solution, we know that $\nu_N^{\pm}(\varphi)$ are nonnegative for a.e. $N \in \mathbb{R}^+$ and any $\varphi \geq 0$. We conclude that for a.e. $N \in \mathbb{R}^+$, $\nu_N^{\pm}$ are nonnegative finite measures on $[0,T) \times \overline{D} \times (-N,N)$ and satisfy the condition (7). Moreover, we have

$$\begin{aligned}
\nu_N^{\pm}(\partial_\xi\varphi) = &\int_Q \int_{-N}^{N} f_{\pm}(\partial_t + a(\xi)\cdot\nabla)\varphi\,d\xi dxdt \\
&+ \int_D \int_{-N}^{N} f_{\pm}^0\varphi(0)\,d\xi dx + M_N \int_\Sigma \int_{-N}^{N} f_{\pm}^b\varphi\,d\xi d\sigma dt \\
&+ \sum_{k=1}^{\infty}\int_0^T \int_D g_k(x,u)\varphi(t,x,u)\,dxd\beta_k(t) \\
&+ \frac{1}{2}\int_Q G^2(x,u)\partial_\xi\varphi(t,x,u)\,dxdt\,.
\end{aligned} \tag{26}$$

We now denote by $m_N^{\pm}$ the restriction of $\nu_N^{\pm}$ to $[0,T)\times D\times(-N,N)$ and by $\bar{\nu}_N^{\pm}$ the restriction of $\nu_N^{\pm}$ to $\Sigma\times(-N,N)$. Let $\varphi \in C_c^{\infty}([0,T)\times\overline{D}\times(-N,N))$. Take a test function $(t,x,\xi)\mapsto \bar{W}_\varepsilon(x)\varphi^{\lambda}(t,x,\xi)$ in (26), where $\varphi^{\lambda} = \varphi\lambda$ and

$$\bar{W}_\varepsilon(x) = \int_0^{x_d - h_\lambda(\bar{x})} \rho_\varepsilon^{\lambda}(r - \varepsilon(L_\lambda + 1))\,dr\,.$$

Let $\bar{f}_{\pm}^{\lambda}$ be any weak* limit of $\{f_{\pm}^{\lambda,\varepsilon}\}$ as $\varepsilon \to +0$ in $L^{\infty}(\Sigma^{\lambda}\times\mathbb{R})$ for any element λ of the partition of unity $\{\lambda_i\}$ on $\overline{D}$, where $f_{\pm}^{\lambda,\varepsilon}$ is denoted by

$$f_{\pm}^{\lambda,\varepsilon}(t,x,\xi) = \int_{D^{\lambda}} f_{\pm}(t,x,\xi)\rho_\varepsilon^{\lambda}(y-x)\,dy\,,$$

and let

$$\bar{f}_{\pm} = \sum_{i=0}^{M}\lambda_i \bar{f}_{\pm}^{\lambda_i}\,. \tag{27}$$

Then letting the limit $\varepsilon \downarrow 0$ and summing over i, we obtain

$$\begin{aligned}
m_N^{\pm}(\partial_\xi\varphi) = &\int_Q \int_{-N}^{N} f_{\pm}(\partial_t + a(\xi)\cdot\nabla)\varphi\,d\xi dxdt \\
&+ \int_D \int_{-N}^{N} f_{\pm}^0\varphi(0)\,d\xi dx + \int_\Sigma \int_{-N}^{N} (-a(\xi)\cdot n)\bar{f}_{\pm}\varphi\,d\xi d\sigma dt \\
&+ \sum_{k=1}^{\infty}\int_0^T \int_D g_k(x,u)\varphi(t,x,u)\,dxd\beta_k(t)
\end{aligned}$$

$$+\frac{1}{2}\int_Q G^2(x,u)\partial_\xi\varphi(t,x,u)\,dxdt\,. \tag{28}$$

It follows from (26) and (28) that for all $\varphi \in C_c^\infty([0,T)\times\overline{D}\times(-N,N))$,

$$\begin{aligned}&\int_\Sigma\int_{-N}^{N}(-a(\xi)\cdot n)\bar{f}_\pm\varphi\,d\xi d\sigma dt\\ &\qquad = M_N\int_\Sigma\int_{-N}^{N} f_\pm^b\varphi\,d\xi d\sigma dt-\int_\Sigma\int_{-N}^{N}\partial_\xi\varphi\,d\bar{\nu}_N^\pm(t,x,\xi).\end{aligned} \tag{29}$$

Defining the functions $\bar{m}_N^\pm$ by

$$\bar{m}_N^+(t,x,\xi) := M_N(u_b(t,x)-\xi)^+ - \int_\xi^N(-a(\xi)\cdot n(x))\bar{f}_+(t,x,\zeta)\,d\zeta\,, \tag{30}$$

$$\bar{m}_N^-(t,x,\xi) := M_N(u_b(t,x)-\xi)^- + \int_{-N}^\xi(-a(\xi)\cdot n(x))\bar{f}_-(t,x,\zeta)\,d\zeta\,, \tag{31}$$

we can replace $\int_\Sigma\int_{-N}^N\partial_\xi\varphi\,d\bar{\nu}_N^\pm(t,x,\xi)$ with $\int_\Sigma\int_{-N}^N\partial_\xi\varphi\bar{m}_N^\pm\,d\xi dt d\sigma$ in (29). Since obviously $\bar{m}_N^\pm(\pm N\mp 0)=0$ for sufficiently large $N\in\mathbb{R}$, we conclude the desired claim. □

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