A Cohomological Splitting Criterion for Rank 2 Vector Bundles on Hirzebruch Surfaces

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(Communicated by N. Suwa)

Abstract. In this note, we give a cohomological characterization of all rank 2 split vector bundles on Hirzebruch surfaces.

1. Introduction

Throughout this paper we work over an algebraically closed field $k$. A vector bundle on a smooth projective variety is called split if it is decomposed into a direct sum of line bundles. Recently, Fulger and Marchitan ([2]) obtained a cohomological characterization of some rank 2 split vector bundles on Hirzebruch surfaces over the complex number field, by using Buchdahl’s Beilinson type spectral sequence ([1]). However, it seems difficult to apply their argument to general cases. The purpose of this paper is to give a simple characterization of all rank 2 split vector bundles on Hirzebruch surfaces by cohomological informations in arbitrary characteristic.

THEOREM 1. Let $\mathcal{E}$ be a rank 2 vector bundle on a Hirzebruch surface $F_n = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. Let $\mathcal{F}$ be a split rank 2 vector bundle on $F_n$. If $\dim_k H^i(\mathcal{E} \otimes \mathcal{L}) = \dim_k H^i(\mathcal{F} \otimes \mathcal{L})$ for any $0 \leq i \leq 2$ and any line bundle $\mathcal{L}$ on $F_n$, then $\mathcal{E}$ is isomorphic to $\mathcal{F}$.

It seems that our assumption of Theorem 1 is stronger than the one of Theorem in [2]. However, if we know the Chern classes of $\mathcal{E}$, we need only a few assumptions (see Lemma 2). In the cases treated in [2], the assumption of Lemma 2 is essentially equivalent to the one of the Theorem of [2].

Notation

For a smooth projective variety $X$ and a vector bundle $\mathcal{E}$ on $X$, let $P_X(\mathcal{E})$ be the projectivization of $\mathcal{E}$ in the sense of Grothendieck. We denote $\dim_k H^i(\mathcal{E})$ by $h^i(\mathcal{E})$. We put $CH_0(X)$
the Chow group of 0-cycles of \( X \). We denote the Hirzebruch surface \( \mathbb{P}^1(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)) \) by \( F_n \), the natural projection \( F_n \to \mathbb{P}^1 \) by \( \pi \), the minimal section on \( F_n \) by \( \sigma \) (i.e., \( \sigma \cong \mathbb{P}^1 \), \( \sigma^2 = -n \)) and a fiber of \( \pi \) by \( f \). It is well-known that \( \text{Pic}(F_n) = \mathbb{Z}\sigma \oplus \mathbb{Z}f \).

2. Proof of Theorem 1

To give the proof of Theorem 1, we show the following lemma.

**Lemma 2.** Let \( \mathcal{E} \) be a rank 2 vector bundle on \( F_n \). Assume that \( h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{E}(-f)) = 0 \), \( h^0(\mathcal{E}) \geq 1 \) and \( c_2(\mathcal{E}) = 0 \).

1. Assume that \( c_1(\mathcal{E}) = -a\sigma - bf \), where \( a \) and \( b \) are nonnegative integers, and that \( h^0(\mathcal{E}(a\sigma + bf)) \geq 1 + h^0(\mathcal{O}_{F_n}(a\sigma + bf)) \). Then \( \mathcal{E} \) is isomorphic to \( \mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(-a\sigma - bf) \).

2. Assume that \( c_1(\mathcal{E}) = a\sigma - bf \), where \( a \) and \( b \) are integers such that \( ab > 0 \), and that \( h^0(\mathcal{E}(-a\sigma + bf)) \geq 1 + h^0(\mathcal{O}_{F_n}(-a\sigma + bf)) \) Then \( \mathcal{E} \) is isomorphic to \( \mathcal{O}_{F_n} \oplus \mathcal{O}_{F_n}(a\sigma - bf) \).

**Proof.** Put \( \mathcal{L} = \mathcal{O}_{F_n}(-a\sigma - bf) \) in Case 1 and \( \mathcal{L} = \mathcal{O}_{F_n}(a\sigma - bf) \) in Case 2. Since \( h^0(\mathcal{E}) \neq 0 \), we can take a nonzero section \( 0 \neq s \in H^0(\mathcal{E}) \). Put \( Z := (s = 0) \). If \( s \) takes zero on some nonzero effective divisor \( D > 0 \), we have a nonzero section of \( H^0(\mathcal{E}(-D)) \). This is a contradiction because \( h^0(\mathcal{E}(-D)) \leq \max\{h^0(\mathcal{E}(-\sigma)), h^0(\mathcal{E}(-f))\} = 0 \). Therefore, \( Z \) is of codimension 2. Since \( c_2(\mathcal{E}) = 0 \), we have \( Z = \emptyset \). Hence we obtain an exact sequence,

\[
0 \to \mathcal{O}_{F_n} \xrightarrow{s} \mathcal{E} \to \mathcal{L} \to 0,
\]

since \( c_1(\mathcal{E}) = -a\sigma - bf \) in Case 1 and \( c_1(\mathcal{E}) = a\sigma - bf \) in Case 2. We show that the above exact sequence splits. Consider the long exact sequence:

\[
0 \to \text{Hom}(\mathcal{L}, \mathcal{O}_{F_n}) \to \text{Hom}(\mathcal{L}, \mathcal{E}) \to \text{Hom}(\mathcal{L}, \mathcal{L}) \to \text{Ext}^1(\mathcal{L}, \mathcal{O}_{F_n}) \to \ldots
\]

By the assumption, we have

\[
\dim_k \text{Hom}(\mathcal{L}, \mathcal{E}) = h^0(\mathcal{E} \otimes \mathcal{L}^\vee) \geq h^0(\mathcal{L}^\vee) + h^0(\mathcal{O}_{F_n}) = \dim_k \text{Hom}(\mathcal{L}, \mathcal{O}_{F_n}) + \dim_k \text{Hom}(\mathcal{L}, \mathcal{L}).
\]

Therefore, the homomorphism \( \text{Hom}(\mathcal{L}, \mathcal{E}) \to \text{Hom}(\mathcal{L}, \mathcal{L}) \) is surjective. Hence \( \mathcal{E} \) is isomorphic to \( \mathcal{O}_{F_n} \oplus \mathcal{L} \).

**Remark 3.** Let \( \mathcal{E} \) be a split rank 2 vector bundle on \( F_n \). It is readily seen that there exists a line bundle \( \mathcal{L} \) on \( F_n \) such that \( c_1(\mathcal{L} \otimes \mathcal{L}) = a\sigma + bf \) with \( \alpha, \beta \leq 0 \) or \( \alpha \beta < 0 \) and that \( c_2(\mathcal{L} \otimes \mathcal{L}) = 0 \). Therefore, by Lemma 2, we can characterize all split rank 2 vector bundles on \( F_n \).

From now on, we give a proof of Theorem 1. We will begin with a proof of the following Claim.
Claim 4. Under the assumptions of Theorem 1, we have \( \det(\mathcal{E}) = \det(\mathcal{F}) \) in \( \text{Pic}(F_n) \) and \( c_2(\mathcal{E}) = c_2(\mathcal{F}) \) in \( \text{CH}_0(F_n) \).

Proof. Take arbitrarily a very ample divisor \( D \) on \( F_n \). We may assume that \( D \) is smooth. By the assumptions, we have \( \chi(\mathcal{E}) = \chi(\mathcal{F}) \) and \( \chi(\mathcal{E}(-D)) = \chi(\mathcal{F}(-D)) \). Therefore, we obtain \( \chi(\mathcal{E}|_D) = \chi(\mathcal{F}|_D) \). By Riemann-Roch theorem on the smooth curve \( D \), we have \( c_1(\mathcal{E}) \cdot D = c_1(\mathcal{F}) \cdot D \) (cf. [3], Example 15.2.1.). Hence \( c_1(\mathcal{E}) \) is numerically equivalent to \( c_1(\mathcal{F}) \). Because \( F_n \) is rational, we get \( \det(\mathcal{E}) = \det(\mathcal{F}) \). We also have \( c_2(\mathcal{E}) = c_2(\mathcal{F}) \) from the Riemann-Roch theorem on \( F_n \) (cf. [3], Example 15.2.2.).

Now we conclude the proof of Theorem 1. By Remark 3, we may assume that \( \mathcal{F} \) is isomorphic to \( O \oplus \mathcal{L} \), where \( \mathcal{L} = O_{F_n}(-a\sigma - bf) \) with nonnegative integers \( a, b \geq 0 \) or \( \mathcal{L} = O_{F_n}(a\sigma - bf) \) with positive integers \( a, b > 0 \).

By Claim 4, we have \( c_1(\mathcal{E}) = c_1(\mathcal{F}) = c_1(\mathcal{L}) \) and \( c_2(\mathcal{E}) = 0 \). By the assumptions of Theorem 1, we also have \( h^0(\mathcal{E}(-\sigma)) = h^0(\mathcal{F}(-\sigma)) = 0 \), \( h^0(\mathcal{E}(-f)) = h^0(\mathcal{F}(-f)) = 0 \), \( h^0(\mathcal{E}) = h^0(\mathcal{F}) \geq 1 \) and \( h^0(\mathcal{E} \otimes \mathcal{L}^e) = h^0(\mathcal{F} \otimes \mathcal{L}^e) = 1 + h^0(\mathcal{L}^e) \). Then, by Lemma 2, we obtain the result of Theorem 1.

Similar arguments of the proof of Theorem 1 imply the following theorem.

Theorem 5. Let \( S \) be a smooth projective surface with the Picard group \( \text{Pic}(S) \cong \mathbb{Z} \). Let \( \mathcal{E} \) be a rank 2 vector bundle on \( S \). Let \( \mathcal{F} \) be a split rank 2 vector bundle on \( S \). If \( h^i(\mathcal{E} \otimes \mathcal{L}) = h^i(\mathcal{F} \otimes \mathcal{L}) \) for any \( 0 \leq i \leq 2 \) and any line bundle \( \mathcal{L} \) on \( S \), then \( \mathcal{E} \) is isomorphic to \( \mathcal{F} \).

Proof. We may assume that \( \mathcal{F} \cong O_S \oplus \mathcal{M} \) where \( \mathcal{M} \) is a line bundle on \( S \) such that \( \text{deg} \mathcal{M} \leq 0 \). In the same manner as in Claim 4, we can verify that \( \det(\mathcal{E}) = \det(\mathcal{F}) \) in \( \text{Pic}(S) \) since \( \text{Pic}(S) \) is isomorphic to \( \mathbb{Z} \). Moreover we also have \( c_2(\mathcal{E}) = c_2(\mathcal{F}) \) in \( \text{CH}_0(S) \). Therefore, we have an exact sequence

\[
0 \to O_S \to \mathcal{E} \to \mathcal{M} \to 0.
\]

Moreover, we can show that \( \mathcal{E} \cong O_S \oplus \mathcal{M} \) in the same way as in the proof of Theorem 1.

Remark 6. There are many surfaces having the Picard group \( \text{Pic}(S) \cong \mathbb{Z} \). In fact, on the complex number field, it is known that a very general surface \( S \subseteq \mathbb{P}^3 \) of degree \( d \geq 4 \) has the Picard group \( \text{Pic}(S) \cong \mathbb{Z} \) (cf. [4], Theorem).

Acknowledgment. The author would like to express his gratitude to Professors Hajime Kaji, Yasunari Nagai, Yasuyuki Nagatomo, Eiichi Sato and Noriyuki Suwa for useful comments. The author also thanks the referee for careful reading the manuscript and for giving him useful comments.

References


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