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The Structure Theorem for the Cut Locus of a Certain Class of Cylinders of Revolution II

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Abstract. In the previous paper [C], the structure of the cut locus was determined for a class of surfaces of revolution homeomorphic to a cylinder. In this paper, we prove the structure theorem of the cut locus for a wider class of surfaces of revolution homeomorphic to a cylinder.

1. Introduction

The following structure theorem was proved in [C] for a class of surfaces of revolution homeomorphic to a cylinder.

THEOREM. *Let (M, ds^2) be a complete Riemannian manifold $R^1 \times S^1$ with a warped product metric $ds^2 = dt^2 + m(t)^2 d\theta^2$ of the real line (R^1, dt^2) and the unit circle $(S^1, d\theta^2)$. Suppose that the warping function m is a positive-valued even function and the Gaussian curvature of M is decreasing along the half meridian $t^{-1}[0, \infty] \cap \theta^{-1}(0)$. If the Gaussian curvature of M is positive on $t = 0$, then the structure of the cut locus C_q of a point $q \in \theta^{-1}(0)$ in M is given as follows:*

1. *The cut locus C_q is the union of a subarc of the parallel $t = -t(q)$ opposite to q and the meridian opposite to q if $|t(q)| < t_0 := \sup\{t > 0 | m'(t) < 0\}$ and $\varphi(m(t(q))) < \pi$. More precisely,*

$$C_q = \theta^{-1}(\pi) \cup (t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]) .$$

2. *The cut locus C_q is the meridian $\theta^{-1}(\pi)$ opposite to q if $\varphi(m(t(q))) \geq \pi$ or if $|t(q)| \geq t_0$.*

Here, the half period function $\varphi(\nu)$ on $(\inf m, m(0))$ is defined as

$$\varphi(\nu) := 2\int_{-\xi(\nu)}^{0} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt = 2\int_{0}^{\xi(\nu)} \frac{\nu}{m\sqrt{m^2 - \nu^2}} dt , \tag{1.1}$$

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where $\xi(\nu) := \min\{t > 0 | m(t) = \nu\}$. *Notice that the point* q *is an arbitrarily given point if the coordinates* (t, θ) *are chosen so as to satisfy* $\theta(q) = 0$.

Crucial properties of the manifold (M, ds^2) in the theorem above are

1. M has a reflective symmetry with respect to a parallel.
2. The Gaussian curvature is decreasing along each upper half meridian.

In this paper, the second property is replaced by the following property:

The cut locus of a point on $\tilde{t} = 0$ *is a nonempty subset of* $\tilde{t} = 0$, for the universal covering space $(\tilde{M}, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ of a cylinder of revolution $(M, dt^2 + m(t)^2 d\theta^2)$ with a reflective symmetry with respect to the parallel $t = 0$.

We will prove the following structure theorem of the cut locus for a cylinder of revolution satisfying the property above.

MAIN THEOREM. *Let* (M, ds^2) *denote a complete Riemannian manifold* $R^1 \times S^1$ *with a warped product metric* $ds^2 = dt^2 + m(t)^2 d\theta^2$ *of the real line* (R^1, dt^2) *and the unit circle* $(S^1, d\theta^2)$, *and by* $(\widetilde{M}, d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2)$ *we denote the universal covering space of* (M, ds^2). *Suppose that* m *is an even positive-valued function. If the cut locus of a point on* $\tilde{t}^{-1}(0)$ *is a nonempty subset of* $\tilde{t}^{-1}(0)$, *then the cut locus* C_q *of a point* q *of* M *with* $|t(q)| < t_0 := \sup\{t > 0 \mid m'(t) < 0\}$ *equals the union of a subarc of the parallel* $t = -t(q)$ *opposite to* q *and the meridian opposite to* q. *More precisely, there exists a number* $t_\pi \in [0, t_0)$ *such that for any point* q *with* $|t(q)| < t_\pi$,

$$C_q = \theta^{-1}(\pi) \cup \Big(t^{-1}(-t(q)) \cap \theta^{-1}[\varphi(m(t(q))), 2\pi - \varphi(m(t(q)))]\Big)$$

and for any point q *with* $t_\pi \leq |t(q)| < t_0$, $C_q = \theta^{-1}(\pi)$. *Moreover, if* t_0 *is finite, then* $C_q = \theta^{-1}(\pi)$ *for any point* q *with* $|t(q)| = t_0$.

Here the coordinates (t, θ) *are chosen so as to satisfy* $\theta(q) = 0$. *Notice that the domain of the half period function* $\varphi(\nu)$ *is* $(m(t_0), m(0))$ (*respectively* $(\inf m, m(0))$) *if* t_0 *is finite* (*respectively infinite*).

REMARK 1.1. If the Gaussian curvature of the manifold M in the Main Theorem is nonpositive on $\tilde{t}^{-1}(t_0, \infty)$, then the cut locus C_q of any point q with $|t(q)| > t_0$ is equal to $\theta^{-1}(\pi)$, the meridian opposite to q.

We refer to [C], [SST] and [ST] for some fundamental properties of geodesics on a surface of revolution and the structure theorem of the cut locus on a surface.

2. A necessary and sufficient condition for $\varphi(\nu)$ to be decreasing

A complete Riemannian manifold (M, ds^2) homeomorphic to $R^1 \times S^1$ is called a *cylinder of revolution* if $ds^2 = dt^2 + m(t)^2 d\theta^2$ is a warped product metric of the real line (R^1, dt^2)

and the unit circle $(S^1, d\theta^2)$.

Throughout this paper, we assume that the warping function m of a cylinder of revolution M is an even function. Hence M has a reflective symmetry with respect to $t = 0$, which is called the *equator.* Let $(\widetilde{M}, d\tilde{s}^2)$ denote the universal covering space of (M, ds^2). Thus $d\tilde{s}^2 = d\tilde{t}^2 + m(\tilde{t})^2 d\tilde{\theta}^2$. Since $m'(0) = 0$, it follows from Lemma 7.1.4 in [SST] that the equator $t = 0$ and $\tilde{t} = 0$ are geodesics in M and $\widetilde{M}$ respectively.

The following lemma is a corresponding one to Lemma 3.2 in [BCST] in the case of a two-sphere of revolution.

LEMMA 2.1. *If the cut locus of a point in $\tilde{t}^{-1}(0)$ is a nonempty subset of $\tilde{t}^{-1}(0)$, then the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$ and for any $t > 0$ satisfying $m'|_{(0,t)} < 0$, the function $\varphi(\nu)$ is decreasing on $(m(t), m(0))$.*

PROOF. Let q be an end point of the cut locus of a point $p \in \tilde{t}^{-1}(0)$. Since the end point q is conjugate to p along the subarc of $\tilde{t}^{-1}(0)$, the Gaussian curvature on $\tilde{t}^{-1}(0)$ is positive. We omit the proof of the second claim, since the proof of Proposition 4.6 in [C] is applicable. □

LEMMA 2.2. *Suppose that the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$. Let $t > 0$ be any number satisfying $m'|_{(0,t)} < 0$. If $\varphi(\nu)$ is decreasing on $(m(t), m(0))$ then for any point $\tilde{p} \in \tilde{t}^{-1}(0)$, $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is a nonempty subset of $\tilde{t}^{-1}(0)$. Here $C_{\tilde{p}}$ denotes the cut locus of $\tilde{p}$.*

PROOF. Choose an arbitrary point $\tilde{p} \in \tilde{t}^{-1}(0)$ and fix it. Since the Gaussian curvature is positive constant on $\tilde{t} = 0$, there exists a conjugate point of $\tilde{p}$ along the subarc of $\tilde{t} = 0$. Thus, $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is nonempty. We omit the proof of the claim that $C_{\tilde{p}} \cap \tilde{t}^{-1}(-t, t)$ is a subset of $\tilde{t}^{-1}(0)$, since the proof of Lemma 3.3 in [BCST] is still valid in our case. □

Combining Lemmas 2.1 and 2.2 we get

PROPOSITION 2.3. *Suppose that $m' \neq 0$ on $(0, \infty)$. Then the cut locus of a point on $\tilde{t}^{-1}(0)$ is a nonempty subset of $\tilde{t}^{-1}(0)$ if and only if the Gaussian curvature of $\widetilde{M}$ is positive on $\tilde{t}^{-1}(0)$ and the half period function $\varphi(\nu)$ defined by (1.1) is decreasing on $(\inf m, m(0))$.*

3. Preliminaries

From now on, we assume that *the cut locus of a point on $\tilde{t} = 0$ is a nonempty subset of $\tilde{t} = 0$*. Hence, from Lemma 2.1, the function $\varphi(\nu)$ is decreasing on $(m(t_0), m(0))$, where $t_0 := \sup\{ t > 0 \,|\, m'(t) < 0\}$ and $m(t_0)$ means $\inf m$ when $t_0 = \infty$. For each $\nu \in [0, m(0))$ let $\gamma_\nu : [0, \infty) \to \widetilde{M}$ denote a unit speed geodesic emanating from the point $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ on $\tilde{t}^{-1}(0)$ with Clairaut constant ν. It is known (see [C], for example) that γ_ν intersects $\tilde{t}^{-1}(0)$ again at the point $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$ if ν is greater than $m(t_0)$. Notice that γ_ν is a submeridian of $\tilde{\theta} = 0$, when $\nu = 0$.

LEMMA 3.1. *If* $0 \leq \nu \leq m(t_0)$, *then* γ_ν *is not tangent to any parallel arc* $\tilde{t} = c$. *In particular, the geodesic does not intersect* $\tilde{t} = 0$ *again.*

PROOF. Since there does not exist a cut point of $\tilde{p}$ in $\tilde{t} \neq 0$, the subarc $\gamma_\nu|_{[0,l(\nu)]}$ of γ_ν is minimal for each $\nu \in (m(t_0), m(0))$. Here $l(\nu)$ denotes the length of the subarc of γ_ν having end points $\tilde{p}$ and $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$. Therefore, the limit geodesic $\gamma_{m(t_0)} = \lim_{\nu \searrow m(t_0)} \gamma_\nu|_{[0,l(\nu)]}$ is a ray emanating from $\tilde{p}$ and in particular, $\gamma_{m(t_0)}$ is not tangent to any parallel arc and does not intersect $\tilde{t} = 0$ again. We will prove that for any $\nu \in [0, m(t_0))$, γ_ν is not tangent to any parallel arc. Suppose that for some $\nu_0 \in (0, m(t_0))$, γ_{ν_0} is tangent to a parallel arc. Since $\widetilde{M}$ has a reflection symmetry with respect to $\tilde{t} = 0$, we may assume that $(\tilde{t} \circ \gamma_{\nu_0})'(0) < 0$ and $(\tilde{t} \circ \gamma_{m(t_0)})'(0) < 0$. By applying the Clairaut relation at the point $\tilde{p}$, $\gamma_{\nu_0}|_{(0,t)}$ lies in the domain D cut off by $\gamma_{m(t_0)}$ and the submeridian γ_0 of $\tilde{\theta} = 0$ for some positive t. Since there does not exist a cut point of $\tilde{p}$ in $\tilde{t}^{-1}(-\infty, 0)$, the geodesic γ_{ν_0} does not intersect $\gamma_{m(t_0)}$ again. Hence $\gamma_{\nu_0}|_{(0,\infty)}$ lies in the domain D. Since γ_{ν_0} is tangent to a parallel arc, the geodesic intersects $\tilde{t} = 0$ again, which is a contradiction. □

LEMMA 3.2. *Let* $\tilde{\gamma}_\nu : R \to \widetilde{M}$ *denote a unit speed geodesic with Clairaut constant* $\nu \in (0, m(t_0)]$. *If* $\tilde{\gamma}_\nu$ *passes through a point of* $\tilde{t}^{-1}(-t_0, t_0)$, *then* $\tilde{\gamma}_\nu$ *is not tangent to any parallel arc* $\tilde{t} = c$.

PROOF. First, we will prove that $\tilde{\gamma}_\nu$ intersects $\tilde{t} = 0$ for any $\nu \in [0, m(t_0)]$. Supposing that $\tilde{\gamma}_\nu$ does not intersect $\tilde{t} = 0$ for some $\nu \in [0, m(t_0)]$, we will get a contradiction. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $(\tilde{t} \circ \tilde{\gamma}_\nu)(s) < 0$ for any real number s. By the Clairaut relation, $(\tilde{t} \circ \tilde{\gamma}_\nu)'(s) \neq 0$ for any s satisfying $-t_0 < \tilde{t} \circ \tilde{\gamma}_\nu(s) < 0 < t_0$. From the assumptions, we may assume that $\tilde{t} \circ \tilde{\gamma}_\nu(0) \in (-t_0, 0)$. If $(\tilde{t} \circ \tilde{\gamma}_\nu)'(0) > 0$ (respectively $(\tilde{t} \circ \tilde{\gamma}_\nu)'(0) < 0$) then $\tilde{t} \circ \tilde{\gamma}_\nu(s)$ is increasing (respectively decreasing) and bounded above by 0. Thus, there exists a unique limit $-t_0 < \tilde{t}_1 := \lim_{s\to\infty} \tilde{t} \circ \tilde{\gamma}_\nu(s) \leq 0$ (respectively $-t_0 < \tilde{t}_1 := \lim_{s\to-\infty} \tilde{t} \circ \tilde{\gamma}_\nu(s) \leq 0$). It follows from Lemma 7.1.7 in [SST] that $m'(\tilde{t}_1) = 0$ and $m(\tilde{t}_1) = \nu$. This is a contradiction, since $\nu \in [0, m(t_0)]$ and $-t_0 < \tilde{t}_1 \leq 0$. Therefore, $\tilde{\gamma}_\nu$ intersects $\tilde{t} = 0$ for any $\nu \in [0, m(t_0)]$, and hence by Lemma 3.1, the geodesic is not tangent to any parallel arc. □

LEMMA 3.3. *If* $t_0 = \sup\{ t > 0 \mid m'(t) < 0\}$ *is finite, then any subarc of the parallel arc* $\tilde{t} = -t_0$ *is minimal, i.e., the parallel arc is a straight line. Hence,* $\tilde{t} = t_0$ *is also a straight line.*

PROOF. Since $m'(t_0) = 0$, the parallel arc $\tilde{t} = -t_0$ is a geodesic by Lemma 7.1.4 in [SST]. Let c be a geodesic emanating from a point on $\tilde{t} = -t_0$ which is not tangent to $\tilde{t} = -t_0$. By Lemma 3.2, c is not tangent to any parallel arc. In particular, c does not intersect $\tilde{t} = -t_0$ again. This implies that $\tilde{t} = -t_0$ is a straight line. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, $\tilde{t} = t_0$ is also a straight line. □

4. The cut locus of a point in $\widetilde{M}$

Choose any point q in $\widetilde{M}$ with $-t_0 < \tilde{t}(q) < 0$. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$. For each $\nu \in [0, m(0))$ let $\gamma_\nu : [0, \infty) \to \widetilde{M}$ denote a geodesic emanating from the point $\tilde{p} := (\tilde{t}, \tilde{\theta})^{-1}(0, 0)$ on $\tilde{t}^{-1}(0)$ with Clairaut constant ν. The geodesic γ_ν intersects $\tilde{t} = 0$ again at the point $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$, if $\nu > m(t_0)$.

We consider two geodesics α_ν and β_ν emanating from the point $q = \alpha_\nu(0) = \beta_\nu(0)$ with Clairaut constant $\nu > 0$. Here we assume that the angle $\angle((\partial/\partial\tilde{t})_q, \alpha'_\nu(0))$ made by the tangent vectors $(\partial/\partial\tilde{t})_q$ and $\alpha'_{\nu(0)}$ is greater than the angle $\angle((\partial/\partial\tilde{t})_q, \beta'_\nu(0))$ by $(\partial/\partial\tilde{t})_q$ and $\beta'_\nu(0)$, if $\nu < m(t(q))$. Notice that $\alpha_\nu = \beta_\nu$ if $\nu = m(t(q))$.

It follows from Lemma 5.1 in [C] that the geodesics α_ν and β_ν intersect again at the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$, where $u := -\tilde{t}(q)$, if $\nu \in (m(t_0), m(u))$. The subarcs of α_ν and β_ν having end points q and $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$ have the same length and its length equals $l(\nu)$, where $l(\nu)$ denotes the length the subarc of γ_ν having end points $\tilde{p}$ and $(\tilde{t}, \tilde{\theta})^{-1}(0, \varphi(\nu))$.

LEMMA 4.1. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| \in (0, t_0)$. Then, for any $\nu \in (m(t_0), m(u)]$, where $u = -\tilde{t}(q)$, $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \tilde{\theta}(q) + \varphi(\nu))$, and in particular, $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq \varphi(m(u)) + \tilde{\theta}(q)\}$ is a subset of the cut locus of the point q.*

PROOF. Without loss of generality, we may assume that $\tilde{\theta}(q) = 0$ and $-t_0 < \tilde{t}(q) < 0$. We will prove that $\alpha_\nu|_{[0,l(\nu)]}$ is a minimal geodesic segment joining q to the point $\alpha_\nu(l(\nu)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu))$. Suppose that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is not minimal for some $\nu_0 \in (m(t_0), m(u)]$. Here we assume that ν_0 is the minimum solution $\nu = \nu_0$ of $\varphi(\nu) = \varphi(\nu_0)$.

Let $\alpha : [0, d(q, x)] \to M$ be a unit speed minimal geodesic segment joining q to $x := \alpha_{\nu_0}(l(\nu_0)) = (\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu_0))$. Hence, $\varphi(\nu_1) = \varphi(\nu_0) = \tilde{\theta}(x)$ and α equals $\alpha_{\nu_1}|_{[0,l(\nu_1)]}$ or $\beta_{\nu_1}|_{[0,l(\nu_1)]}$, where $\nu_1 \in (m(t_0), m(u))$ denotes the Clairaut constant of α. By Lemma 2.1, $\varphi(\nu) = \varphi(\nu_0)$ for any $\nu \in [\nu_0, \nu_1]$. Hence, by Lemma 3.2 in [C] we get, $l(\nu_1) = l(\nu_0)$. This implies that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is minimal, which is a contradiction, since we assumed that $\alpha_{\nu_0}|_{[0,l(\nu_0)]}$ is not minimal. Therefore, for any $\nu \in (m(t_0), m(u)]$, the geodesic segments $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal geodesic segments joining q to the point $(\tilde{t}, \tilde{\theta})^{-1}(u, \varphi(\nu)) = \alpha_\nu(l(\nu))$. In particular, the point $\alpha_\nu(l(\nu)) = \beta_\nu(l(\nu))$ is a cut point of q. □

PROPOSITION 4.2. *The cut locus of any point q with $|\tilde{t}(q)| < t_0$ equals the set*

$$\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}.$$

Here the coordinates $(\tilde{t}, \tilde{\theta})$ are chosen so as to satisfy $\tilde{\theta}(q) = 0$.

PROOF. Without of loss of generality, we may assume that $-t_0 < \tilde{t}(q) < 0$. By Lemma 4.1, the geodesic segments $\alpha_\nu|_{[0,l(\nu)]}$ and $\beta_\nu|_{[0,l(\nu)]}$ are minimal for any $\nu \in (m(t_0), m(u)]$. Hence their limit geodesics $\alpha^- := \alpha_{m(t_0)}$ and $\beta^+ := \beta_{m(t_0)}$ are rays, that is, any their subarcs are minimal. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{\theta} = 0$, it is trivial from Lemma 4.1 that the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$ is a subset of the cut locus of q. Suppose that there exists a cut point $y \notin \{(\tilde{t}, \tilde{\theta}) \mid \tilde{t} = u, \tilde{\theta} \geq |\varphi(m(u))|\}$. Without loss of generality, we may assume that $\tilde{\theta}(y) > 0 = \tilde{\theta}(q)$. Since the cut locus of q has a tree structure, there exists an end point x of the cut locus in the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{\theta} > 0\} \setminus D(\beta^+, \alpha^-)$, where $D(\beta^+, \alpha^-)$ denotes the closure of the unbounded domain cut off by β^+ and α^-. Hence, x is conjugate to q along any minimal geodesic segments γ joining q to x. Since such a minimal geodesic γ runs in the set $\{(\tilde{t}, \tilde{\theta}) \mid \tilde{\theta} > 0\} \setminus D(\beta^+, \alpha^-)$, by applying the Clairaut relation at the point q, we get that the Clairaut constant of γ is positive and less than $m(t_0)$. Notice that the geodesics β^+ and α^- have the same Clairaut constant $m(t_0)$. It follows from Lemma 3.1 that the geodesic γ cannot be tangent to any parallel arc. From Corollary 7.2.1 in [SST], γ has no conjugate point of q, which is a contradiction. □

LEMMA 4.3. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| = t_0$. Then, the cut locus of q is empty.*

PROOF. We may assume that $\tilde{t}(q) = -t_0$, since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$. Supposing that there exists a cut point x of q, we will get a contradiction. Since $\widetilde{M}$ is simply connected, the cut locus has an end point. Hence, we may assume that the cut point x is an end point of C_q. Let γ be a minimal geodesic segment joining q to x. Then, x is a conjugate point of q along γ, since x is an end point of the cut locus. From Lemma 3.3, $\tilde{t}(x) \neq -t_0$. By applying the Clairaut relation, we obtain that the Clairaut constant of γ is smaller than $m(t_0)$, and hence γ is not tangent to any parallel arc by Lemma 3.2. Therefore, by Corollary 7.2.1 in [SST], there does not exist a conjugate point of q along γ, which is a contradiction. □

LEMMA 4.4. *Let q be a point in $\widetilde{M}$ with $|\tilde{t}(q)| > t_0$. If the Gaussian curvature of $\tilde{M}$ is nonpositive on $\tilde{t}^{-1}(-\infty, -t_0) \cup \tilde{t}^{-1}(t_0, \infty)$, then the cut locus of the point q is empty.*

PROOF. Suppose that the cut locus of a point q with $|\tilde{t}(q)| > t_0$ is nonempty. Since $\widetilde{M}$ has a reflective symmetry with respect to $\tilde{t} = 0$, we may assume that $\tilde{t}(q) < -t_0$. Supposing the existence of a cut point of q, we will get a contradiction. We may assume that there exists an end point x of C_q, since $\widetilde{M}$ is simply connected. Let $\gamma : [0, d(q, x)] \to \widetilde{M}$ be a unit speed minimal geodesic joining q to x. If $\tilde{t}(\gamma(s)) \leq -t_0$ for any $s \in (0, d(q, x)]$, then γ has no conjugate point of q, since the Gaussian curvature is nonpositive on $\tilde{t}^{-1}(-\infty, -t_0) \cup \tilde{t}^{-1}(t_0, \infty)$. This contradicts the fact that x is an end point of C_q. Thus we may assume that $\tilde{t}(\gamma(s)) > -t_0$ for some $s \in (0, d(q, x)]$. This implies that γ passes through a point of $\tilde{t}^{-1}(-t_0, t_0)$. It follows from the Clairaut relation that the Clairaut constant of γ is smaller than $m(t_0)$. Hence, from Corollary 7.2.1 in [SST] and Lemma 3.2, there does not exist a conjugate point of q along γ, which is a contradiction. □

PROOF OF MAIN THEOREM. Since the functions m and φ are decreasing on $[0, t_0)$ and $(m(t_0), m(0))$ respectively, the composite function $\varphi \circ m$ is increasing on $(0, t_0)$. It is clear to see that $\lim_{t \nearrow t_0} \varphi \circ m(t) = \infty$, since the minimal geodesic segment $\gamma_\nu|_{[0,l(\nu)]}$ converges to the ray $\gamma_{m(t_0)}$ as $\nu \searrow m(t_0)$. Let $t = t_\pi \in [0, t_0)$ be a solution of $\varphi \circ m(t) = \pi$. Define $t_\pi = 0$ if there is no solution. Hence, $\varphi \circ m(t) \geq \pi$ on $[t_\pi, t_0)$ and $\varphi \circ m(t) \leq \pi$ on $(0, t_\pi)$. Now the Main Theorem is clear from Proposition 4.2 and Lemma 4.3.

5. A family of cylinders of revolution

An example of a cylinder of revolution satisfying the two properties 1 and 2 in the introduction was given by Tamura [T]. The Riemannian metric ds^2 is defined by $ds^2 = dt^2 + e^{-t^2} d\theta^2$. It is easy to see that $m' = -2t \cdot m < 0$ on $(0, \infty)$, and the Gaussian curvature $G(q)$ at a point q is $-4t^2(q) + 2$. This implies that the Gaussian curvature is decreasing on each upper half meridian of the surface. By Lemma 4.5 in [C], the cut locus of a point on $\tilde{t} = 0$ on the universal covering space of the surface is a nonempty subset of $\tilde{t} = 0$. Hence, this surface satisfies the assumptions of the Main Theorem. The following family of cylinders of revolution shows that the converse is not true, i.e., under the assumptions of the Main Theorem, the decline of the Gaussian curvature does not always hold.

In this section we give a family of cylinders of revolution $\{M_\lambda\}_\lambda := \{(R^1 \times S^1, dt^2 + m_\lambda(t)^2 d\theta^2)\}_\lambda$ satisfying the assumptions in the Main Theorem, where $\lambda > 1$ denotes a parameter and

$$m_\lambda(t) := \frac{\cosh t}{\sqrt{1 + \lambda \sinh^2 t}} . \tag{5.1}$$

LEMMA 5.1. *The Gaussian curvature $G(q)$ at a point $q \in M_\lambda$ is given by*

$$G(q) = (\lambda - 1)\left(\frac{3}{h^2(t(q))} - \frac{2}{h(t(q))}\right), \tag{5.2}$$

where $h(t) = 1 + \lambda \sinh^2 t$. In particular, the Gaussian curvature G is not monotonic along the upper half meridian $\theta^{-1}(0) \cap t^{-1}(0, \infty)$.

PROOF. From (5.1), we get

$$m'_\lambda(t) = (1 - \lambda) m_\lambda(t) \tanh t / h(t) , \tag{5.3}$$

and

$$m''_\lambda(t) = ((1 - \lambda)\tanh t / h(t))^2 \, m_\lambda(t) + (1 - \lambda) m_\lambda(t)(h(t)/\cosh^2 t - h'(t) \tanh t)/h(t)^2 .$$

Thus, we obtain

$$m''_\lambda = (1 - \lambda) m_\lambda(t)((1 - \lambda) \tanh^2 t + h(t)/\cosh^2 t - h'(t) \tanh t)/h^2.$$

Since $(1 - \lambda) \tanh^2 t + h(t)/\cosh^2 t = 1$ holds, we have

$$-m''_\lambda(t)/m_\lambda(t) = (\lambda - 1) \left(3/h^2(t) - 2/h(t)\right) .$$

Since $G(q) = -m''_\lambda(t(q))/m_\lambda(t(q))$, we obtain (5.2). By (5.3), it is trivial to see that the Gaussian curvature is not monotonic along the upper half meridian. □

LEMMA 5.2. *Let $a, b \in (0, 1)$ be numbers with $a < b$. Then,*

$$\int_b^1 \frac{dx}{x(x-a)\sqrt{(x-b)(1-x)}} = \frac{\pi}{a(1-a)}\left(\frac{a-1}{\sqrt{b}} + \frac{1}{c}\right) \tag{5.4}$$

holds, where $c = \sqrt{(b-a)/(1-a)}$.

PROOF. From a direct computation, we obtain

$$\frac{d}{du}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}} + \frac{1}{c}\arctan\frac{u}{c}\right) = \frac{a-1}{u^2+b} + \frac{1}{u^2+c^2} \tag{5.5}$$

and

$$\frac{du}{dx} = \frac{(1-b)u}{2(x-b)(1-x)}\,, \tag{5.6}$$

where $u = \sqrt{(x-b)/(1-x)}$. Since $c^2 = (b-a)/(1-a)$, we get

$$\frac{a-1}{u^2+b} + \frac{1}{u^2+c^2} = \frac{a(1-a)(1-x)}{(1-b)x(x-a)}\,. \tag{5.7}$$

By (5.5), (5.6) and (5.7), we have

$$\frac{d}{dx}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}} + \frac{1}{c}\arctan\frac{u}{c}\right) = \frac{a(1-a)}{2}\frac{1}{x(x-a)\sqrt{(x-b)(1-x)}}\,.$$

This implies that

$$\int \frac{dx}{x(x-a)\sqrt{(x-b)(1-x)}} = \frac{2}{a(1-a)}\left(\frac{a-1}{\sqrt{b}}\arctan\frac{u}{\sqrt{b}} + \frac{1}{c}\arctan\frac{u}{c}\right)$$

holds. Hence, we obtain (5.4). □

By (5.1) and (5.3), we get $\inf m_\lambda = 1/\sqrt{\lambda}$ and $m'_\lambda(t) < 0$ for any $t > 0$. Hence the half period function $\varphi(\nu)$ for M_λ is defined on $(1/\sqrt{\lambda}, 1)$.

LEMMA 5.3. *The half period function $\varphi(\nu)$ is given by*

$$\varphi(\nu) = \pi\left(-\sqrt{\lambda-1} + \frac{\lambda\nu}{\sqrt{\lambda\nu^2-1}}\right)$$

on $(\frac{1}{\sqrt{\lambda}}, 1)$. In particular φ is decreasing on $(\frac{1}{\sqrt{\lambda}}, 1)$ and the surface M_λ satisfies the assumptions of the Main Theorem.

PROOF. By putting $x := m_\lambda^2(t)$, we get, by (5.3),

$$dt = \frac{h(t)}{2(1-\lambda)x\tanh t}dx\,. \tag{5.8}$$

Since $x = (1+\sinh t^2)/h(t)$,

$$\sinh^2 t = \frac{1-x}{\lambda x - 1}, \quad \cosh^2 t = \frac{(\lambda-1)x}{\lambda x - 1}, \quad \text{and } h(t) = \frac{(\lambda-1)}{\lambda x - 1}\,. \tag{5.9}$$

By combining (5.8) and (5.9), we obtain,

$$dt = \frac{-\sqrt{\lambda-1}}{2(\lambda x-1)\sqrt{x(1-x)}}dx\,. \tag{5.10}$$

Therefore, by (1.1),

$$\varphi(\nu) = \nu\sqrt{\lambda-1}\int_{\nu^2}^{1}\frac{dx}{x(\lambda x-1)\sqrt{(x-\nu^2)(1-x)}}$$

for $\nu \in (1/\sqrt{\lambda}, 1)$. It follows from Lemma 5.2 that $\varphi(\nu) = \pi\left(-\sqrt{\lambda-1} + \lambda\nu/\sqrt{\lambda\nu^2-1}\right)$. It is easy to check that $\varphi'(\nu) = -\pi\left(1/(2\sqrt{\lambda-1}) + \lambda/\sqrt{\lambda\nu^2-1}^3\right) < 0$ on $(1/\sqrt{\lambda}, 1)$. Therefore, by Proposition 2.3, the surface M_λ satisfies the assumptions of the Main Theorem. □

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