TOKYO J. MATH.
VOL. 38, NO. 1, 2015

The $*$-transforms of Acyclic Complexes

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(Communicated by H. Nawa)

Abstract. Let R be an n-dimensional Cohen-Macaulay local ring and Q a parameter ideal of R. Suppose that an acyclic complex $(F_\bullet, \varphi_\bullet)$ of length n of finitely generated free R-modules is given. We put $M = \operatorname{Im}\varphi_1$, which is an R-submodule of F_0. Then $F_\bullet$ is an R-free resolution of F_0/M. In this paper, we describe a concrete procedure to get an acyclic complex ${}^*F_\bullet$ of length n that resolves $F_0/(M :_{F_0} Q)$.

1. Introduction

Let I and J be ideals of a commutative ring R. The ideal quotient

$$I :_R J = \{a \in R \mid aJ \subseteq I\}$$

is an important notion in the theory of commutative algebra. For example, if $(R, \mathfrak{m})$ is a Noetherian local ring and I is an $\mathfrak{m}$-primary ideal of R, the Gorenstein property of R/I is characterized by the socle $\operatorname{Soc}(R/I) = (I :_R \mathfrak{m})/I$. The $*$-transform of an acyclic complex of length 3 is introduced in [1] for the purpose of composing an R-free resolution of the ideal quotient of a certain ideal I whose R-free resolution is given. Here, let us recall its outline.

Let $(R, \mathfrak{m})$ be a 3-dimensional Cohen-Macaulay local ring and Q a parameter ideal of R. Suppose that an acyclic complex

$$F_\bullet \;\;:\;\; 0 \longrightarrow F_3 \xrightarrow{\varphi_3} F_2 \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = R$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_3 \subseteq QF_2$ is given. Then, taking the $*$-transform of $F_\bullet$, we get an acyclic complex

$${}^*F_\bullet \;\;:\;\; 0 \longrightarrow {}^*F_3 \xrightarrow{{}^*\varphi_3} {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = R$$

of finitely generated free R-modules such that $\operatorname{Im}{}^*\varphi_1 = \operatorname{Im}\varphi_1 :_R Q$ and $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\cdot{}^*F_2$. If R is regular, for any ideal I of R, we can take $\mathfrak{m}$ and the minimal R-free resolution of R/I as Q and $F_\bullet$, respectively, and then ${}^*F_\bullet$ gives an R-free resolution of $R/(I :_R \mathfrak{m})$. Here, let us notice that we can take the $*$-transform of ${}^*F_\bullet$ again since $\operatorname{Im}{}^*\varphi_3 \subseteq \mathfrak{m}\cdot{}^*F_2$, and an R-free resolution of $R/(I :_R \mathfrak{m}^2)$ is induced. Repeating this procedure, we get an R-free resolution

Received December 17, 2013; revised March 25, 2014

of $R/(I :_R \mathfrak{m}^k)$ for any $k > 0$, and it contains complete information about the 0-th local cohomology module of R/I with respect to $\mathfrak{m}$. This method is very useful for computing the symbolic powers of the ideal generated by the maximal minors of a certain 2×3 matrix as is described in [1].

Thus, in [1], the theory of $*$-transform is developed for only acyclic complexes of length 3 on a 3-dimensional Cohen-Macaulay local ring. The purpose of this paper is to generalize the machinery of $*$-transform so that we can apply it to acyclic complexes of length n as follows. Let $(R, \mathfrak{m})$ be an n-dimensional Cohen-Macaulay local ring, where $2 \leq n \in \mathbb{Z}$, and let Q be a parameter ideal of R. Suppose that an acyclic complex

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$ is given. We aim to give a concrete procedure to get an acyclic complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

of finitely generated free R-modules such that $\operatorname{Im}{}^*\varphi_1 = \operatorname{Im}\varphi_1 :_{F_0} Q$ and $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$. Let us notice that we do not need any restriction on the rank of F_0, so there may be some application to the study of $M :_F Q$, where F is a finitely generated free R-module and M is an R-submodule of F. Moreover, as the generalized $*$-transform works for acyclic complexes of length $n \geq 2$, we can apply it to the study of some ideal quotients in n-dimensional Cohen-Macaulay local rings. In fact, in the subsequent paper [2], setting I to be the m-th power of the ideal generated by the maximal minors of the matrix

$$\begin{pmatrix} x_1^{\alpha_{1,1}} & x_2^{\alpha_{1,2}} & x_3^{\alpha_{1,3}} & \cdots & x_m^{\alpha_{1,m}} & x_{m+1}^{\alpha_{1,m+1}} \\ x_2^{\alpha_{2,1}} & x_3^{\alpha_{2,2}} & x_4^{\alpha_{2,3}} & \cdots & x_{m+1}^{\alpha_{2,m}} & x_1^{\alpha_{2,m+1}} \\ x_3^{\alpha_{3,1}} & x_4^{\alpha_{3,2}} & x_5^{\alpha_{3,3}} & \cdots & x_1^{\alpha_{3,m}} & x_2^{\alpha_{3,m+1}} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ x_m^{\alpha_{m,1}} & x_{m+1}^{\alpha_{m,2}} & x_1^{\alpha_{m,3}} & \cdots & x_{m-2}^{\alpha_{m,m}} & x_{m-1}^{\alpha_{m,m+1}} \end{pmatrix}$$

and setting $Q = (x_1, x_2, x_3, \ldots, x_m, x_{m+1})R$, where $x_1, x_2, x_3, \ldots, x_m, x_{m+1}$ is an sop for an $(m+1)$-dimensional Cohen-Macaulay local ring R and $\{\alpha_{i,j}\}_{1 \leq i \leq m, 1 \leq j \leq m+1}$ is a family of positive integers, the ideal quotient $I :_R Q$ is computed, and it is proved that $I :_R Q$ coincides with the saturation of I, that is, the depth of $R/(I :_R Q)$ is positive.

Throughout this paper, R is a commutative ring, and in the last section, we assume that R is an n-dimensional Cohen-Macaulay local ring. For R-modules G and H, the elements of $G \oplus H$ are denoted by

$$(g, h) \quad (g \in G, h \in H).$$

In particular, the elements of the forms

$$(g, 0) \quad \text{and} \quad (0, h)$$

are denoted by $[g]$ and $\langle h \rangle$, respectively. Moreover, if V is a subset of G, then the family $\{[v]\}_{v \in V}$ is denoted by $[V]$. Similarly $\langle W \rangle$ is defined for a subset W of H. If T is a subset of an R-module, we denote by $R \cdot T$ the R-submodule generated by T. If S is a finite set, $\sharp\, S$ denotes the number of elements of S.

2. Preliminaries

In this section, we summarize preliminary results. Let R be a commutative ring.

LEMMA 2.1. *Let $G_\bullet$ and $F_\bullet$ be acyclic complexes, whose boundary maps are denoted by $\partial_\bullet$ and $\varphi_\bullet$, respectively. Suppose that a chain map $\sigma_\bullet : G_\bullet \longrightarrow F_\bullet$ is given and $\sigma_0^{-1}(\operatorname{Im}\varphi_1) = \operatorname{Im}\partial_1$ holds. Then the mapping cone* $\operatorname{Cone}(\sigma_\bullet)$:

$$\cdots \longrightarrow G_{p-1} \oplus F_p \xrightarrow{\psi_p} G_{p-2} \oplus F_{p-1} \longrightarrow \cdots \longrightarrow G_1 \oplus F_2 \xrightarrow{\psi_2} G_0 \oplus F_1 \xrightarrow{\psi_1} F_0 \longrightarrow 0$$

is acyclic, where

$$\psi_p = \begin{pmatrix} \partial_{p-1} & (-1)^{p-1} \cdot \sigma_{p-1} \\ 0 & \varphi_p \end{pmatrix} \quad \textit{for all } p \geq 2 \textit{ and } \psi_1 = \begin{pmatrix} \sigma_0 \\ \varphi_1 \end{pmatrix} .$$

Hence, if $G_\bullet$ and $F_\bullet$ are complexes of finitely generated free R-modules, then $\operatorname{Cone}(\sigma_\bullet)$ *gives an R-free resolution of $F_0/(\operatorname{Im}\varphi_1 + \operatorname{Im}\sigma_0)$.*

PROOF. See [1, 2.1]. □

LEMMA 2.2. *Let $2 \leq n \in \mathbb{Z}$ and $C_{\bullet\bullet}$ be a double complex such that $C_{p,q} = 0$ unless $0 \leq p, q \leq n$. For any $p, q \in \mathbb{Z}$, we denote the boundary maps $C_{p,q} \longrightarrow C_{p-1,q}$ and $C_{p,q} \longrightarrow C_{p,q-1}$ by $d'_{p,q}$ and $d''_{p,q}$, respectively. We assume that $C_{p\bullet}$ and $C_{\bullet q}$ are acyclic for $0 \leq p, q \leq n$. Let $T_\bullet$ be the total complex of $C_{\bullet\bullet}$ and let $d_\bullet$ be its boundary map, that is, if $\xi \in C_{p,q} \subseteq T_r$ $(p + q = r)$, then*

$$d_r(\xi) = (-1)^p \cdot d''_{p,q}(\xi) + d'_{p,q}(\xi) \in C_{p,q-1} \oplus C_{p-1,q} \subseteq T_{r-1} .$$

Then the following assertions hold.

(1) *Suppose that $\xi_n \in C_{n,0}$ and $\xi_{n-1} \in C_{n-1,1}$ such that $d'_{n,0}(\xi_n) = (-1)^n \cdot d''_{n-1,1}(\xi_{n-1})$ are given. Then there exist elements $\xi_p \in C_{p,n-p}$ for all $p = 0, 1, \ldots, n-2$ such that*

$$\begin{aligned} \xi_n + \xi_{n-1} + \xi_{n-2} + \cdots + \xi_0 &\in \operatorname{Ker} d_n \\ &\subseteq T_n = C_{n,0} \oplus C_{n-1,1} \oplus C_{n-2,2} \oplus \cdots \oplus C_{0,n} . \end{aligned}$$

(2) *Suppose that* $\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Ker} d_n \subseteq T_n = C_{n,0} \oplus C_{n-1,1} \oplus \cdots \oplus C_{1,n-1} \oplus C_{0,n}$ *and* $\xi_0 \in \operatorname{Im} d'_{1,n}$. *Then*

$$\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Im} d_{n+1}\,.$$

In particular, we have $\xi_n \in \operatorname{Im} d''_{n,1}$.

PROOF. (1) It is enough to show that if $1 \le p \le n-1$ and two elements $\xi_{p+1} \in C_{p+1,n-p-1}$, $\xi_p \in C_{p,n-p}$ such that

$$d'_{p+1,n-p-1}(\xi_{p+1}) = (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p)$$

are given, then we can take $\xi_{p-1} \in C_{p-1,n-p+1}$ so that

$$d'_{p,n-p}(\xi_p) = (-1)^p \cdot d''_{p-1,n-p+1}(\xi_{p-1})\,.$$

In fact, if the assumption of the claim stated above is satisfied, we have

$$\begin{aligned} d''_{p-1,n-p}(d'_{p,n-p}(\xi_p)) &= d'_{p,n-p-1}(d''_{p,n-p}(\xi_p)) \\ &= d'_{p,n-p-1}((-1)^{p+1} \cdot d'_{p+1,n-p-1}(\xi_{p+1})) \\ &= 0\,, \end{aligned}$$

and so

$$d'_{p,n-p}(\xi_p) \in \operatorname{Ker} d''_{p-1,n-p} = \operatorname{Im} d''_{p-1,n-p+1}\,,$$

which means the existence of the required element ξ_{p-1}.

(2) We set $\eta_0 = 0$. By the assumption, there exists $\eta_1 \in C_{1,n}$ such that

$$\xi_0 = d'_{1,n}(\eta_1) = d'_{1,n}(\eta_1) + d''_{0,n+1}(\eta_0)\,.$$

Here we assume $0 \le p \le n-1$ and two elements $\eta_p \in C_{p,n-p+1}$, $\eta_{p+1} \in C_{p+1,n-p}$ such that

$$\xi_p = d'_{p+1,n-p}(\eta_{p+1}) + (-1)^p \cdot d''_{p,n-p+1}(\eta_p)$$

are fixed. We would like to find $\eta_{p+2} \in C_{p+2,n-p-1}$ such that

$$\xi_{p+1} = d'_{p+2,n-p-1}(\eta_{p+2}) + (-1)^{p+1} \cdot d''_{p+1,n-p}(\eta_{p+1})\,.$$

Now $d'_{p+1,n-p-1}(\xi_{p+1}) = (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p)$ holds, since $\xi_n + \xi_{n-1} + \cdots + \xi_1 + \xi_0 \in \operatorname{Ker} d_n$. Hence, we have

$$\begin{aligned} &d'_{p+1,n-p-1}(\xi_{p+1} + (-1)^p \cdot d''_{p+1,n-p}(\eta_{p+1})) \\ &\quad = d'_{p+1,n-p-1}(\xi_{p+1}) + (-1)^p \cdot d'_{p+1,n-p-1}(d''_{p+1,n-p}(\eta_{p+1})) \\ &\quad = (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p) + (-1)^p \cdot d''_{p,n-p}(d'_{p+1,n-p}(\eta_{p+1})) \end{aligned}$$

$$
\begin{aligned}
&= (-1)^{p+1} \cdot d''_{p,n-p}(\xi_p - d'_{p+1,n-p}(\eta_{p+1})) \\
&= (-1)^{p+1} \cdot d''_{p,n-p}((-1)^p \cdot d''_{p,n-p+1}(\eta_p)) \\
&= 0\,,
\end{aligned}
$$

and it follows that

$$\xi_{p+1} + (-1)^p \cdot d''_{p+1,n-p}(\eta_{p+1}) \in \operatorname{Ker} d'_{p+1,n-p-1} = \operatorname{Im} d'_{p+2,n-p-1}\,.$$

Thus we see the existence of the required element η_{p+2}. □

LEMMA 2.3. *Suppose that*

$$0 \longrightarrow F \xrightarrow{\varphi} G \xrightarrow{\psi} H \xrightarrow{\rho} L$$

is an exact sequence of R-modules. Then the following assertions hold.

(1) *If there exists a homomorphism $\phi : G \longrightarrow F$ of R-modules such that $\phi \circ \varphi = \mathrm{id}_F$, then*

$$0 \longrightarrow {}^*G \xrightarrow{{}^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${}^*G = \operatorname{Ker}\phi$ and ${}^*\psi$ is the restriction of ψ to ${}^*G$.

(2) *If $F = {}'F \oplus {}^*F$, $G = {}'G \oplus {}^*G$, $\varphi({}'F) = {}'G$ and $\varphi({}^*F) \subseteq {}^*G$, then*

$$0 \longrightarrow {}^*F \xrightarrow{{}^*\varphi} {}^*G \xrightarrow{{}^*\psi} H \xrightarrow{\rho} L$$

is exact, where ${}^*\varphi$ and ${}^*\psi$ are the restrictions of φ and ψ to ${}^*F$ and ${}^*G$, respectively.

PROOF. See [1, 2.3]. □

3. ∗-transform

Let $2 \le n \in \mathbb{Z}$ and let R be an n-dimensional Cohen-Macaulay local ring with the maximal ideal $\mathfrak{m}$. Suppose that an acyclic complex

$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

of finitely generated free R-modules such that $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$ is given, where $Q = (x_1, x_2, \ldots, x_n)R$ is a parameter ideal of R. We put $M = \operatorname{Im}\varphi_1$, which is an R-submodule of F_0. In this section, transforming $F_\bullet$ suitably, we aim to construct an acyclic complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0$$

of finitely generated free R-modules such that $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$ and $\operatorname{Im}{}^*\varphi_1 = M :_{F_0} Q$. Let us call ${}^*F_\bullet$ the ∗-transform of $F_\bullet$ with respect to $x_1, x_2, \ldots, x_n$.

In this operation, we use the Koszul complex $K_\bullet = K_\bullet(x_1, x_2, \ldots, x_n)$. We denote the boundary map of $K_\bullet$ by $\partial_\bullet$. Let $e_1, e_2, \ldots, e_n$ be an R-free basis of K_1 such that $\partial_1(e_i) = x_i$ for all $i = 1, 2, \ldots, n$. Moreover, we use the following notation:

- $N := \{1, 2, \ldots, n\}$.
- $N_p := \{I \subseteq N \mid \sharp I = p\}$ for $1 \leq p \leq n$ and $N_0 := \{\emptyset\}$.
- If $1 \leq p \leq n$ and $I = \{i_1, i_2, \ldots, i_p\} \in N_p$, where $1 \leq i_1 < i_2 < \cdots < i_p \leq n$, we set
$$e_I = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_p} \in K_p\,.$$
In particular, for $1 \leq i \leq n$, $\check{e}_i := e_{N\setminus\{i\}}$. Furthermore, $e_\emptyset$ denotes the identity element 1_R of $R = K_0$.
- If $1 \leq p \leq n$, $I \in N_p$ and $i \in I$, we set
$$s(i, I) = \sharp\{j \in I \mid j < i\}\,.$$
We define $\sharp\,\emptyset = 0$, so $s(i, I) = 0$ if $i = \min I$.

Then, for any $p = 0, 1, \ldots, n$, $\{e_I\}_{I\in N_p}$ is an R-free basis of K_p and
$$\partial_p(e_I) = \sum_{i\in I}(-1)^{s(i,I)} \cdot x_i \cdot e_{I\setminus\{i\}}\,.$$

THEOREM 3.1. $(M :_{F_0} Q)/M \cong F_n/QF_n$.

PROOF. We put $L_0 = F_0/M$. Moreover, for $1 \leq p \leq n-1$, we put $L_p = \operatorname{Im}\varphi_p \subseteq F_{p-1}$ and consider the exact sequence
$$0 \longrightarrow L_p \longrightarrow F_{p-1} \xrightarrow{\varphi_{p-1}} L_{p-1} \longrightarrow 0\,,$$
where $\varphi_0 : F_0 \longrightarrow L_0$ is the canonical surjection. Because
$$\operatorname{Ext}_R^{p-1}(R/Q, F_{p-1}) = \operatorname{Ext}_R^{p}(R/Q, F_{p-1}) = 0\,,$$
we get
$$\operatorname{Ext}_R^{p}(R/Q, L_p) \cong \operatorname{Ext}_R^{p-1}(R/Q, L_{p-1})\,.$$
Therefore $\operatorname{Ext}_R^{n-1}(R/Q, L_{n-1}) \cong \operatorname{Hom}_R(R/Q, F_0/M) \cong (M :_{F_0} Q)/M$. Now, we see that
$$\operatorname{Ext}_R^{n}(R/Q, F_n) \cong \operatorname{Hom}_R(R/Q, F_n/QF_n) \cong F_n/QF_n$$
and
$$\operatorname{Ext}_R^{n}(R/Q, F_{n-1}) \cong \operatorname{Hom}_R(R/Q, F_{n-1}/QF_{n-1}) \cong F_{n-1}/QF_{n-1}$$
hold, because $x_1, x_2, \ldots, x_n$ is an R-regular sequence. Furthermore, we look at the exact sequence
$$0 \longrightarrow F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} L_{n-1} \longrightarrow 0\,.$$

Then, we get the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}_R^{n-1}(R/Q, L_{n-1}) & \longrightarrow & \mathrm{Ext}_R^n(R/Q, F_n) & \xrightarrow{\widetilde{\varphi_n}} & \mathrm{Ext}_R^n(R/Q, F_{n-1}) \quad \text{(ex)} \\ & & & & \downarrow\cong & & \downarrow\cong \\ & & & & F_n/QF_n & \xrightarrow{\overline{\varphi_n}} & F_{n-1}/QF_{n-1}\,, \end{array}$$

where $\widetilde{\varphi_n}$ and $\overline{\varphi_n}$ denote the maps induced from φ_n. Let us notice $\overline{\varphi_n} = 0$ as $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$. Hence

$$\mathrm{Ext}_R^{n-1}(R/Q, L_{n-1}) \cong F_n/QF_n\,,$$

and so the required isomorphism follows. □

Let us fix an R-free basis of F_n, say $\{v_\lambda\}_{\lambda\in\Lambda}$. We set $\widetilde{\Lambda} = \Lambda \times N$ and take a family $\{v_{(\lambda,i)}\}_{(\lambda,i)\in\widetilde{\Lambda}}$ of elements in F_{n-1} so that

$$\varphi_n(v_\lambda) = \sum_{i\in N} x_i \cdot v_{(\lambda,i)}$$

for all $\lambda \in \Lambda$. This is possible as $\operatorname{Im}\varphi_n \subseteq QF_{n-1}$. The next result is the essential part of the process to get ${}^*F_\bullet$.

THEOREM 3.2. *There exists a chain map $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$*

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & F_n\otimes_R K_n & \xrightarrow{F_n\otimes\partial_n} & F_n\otimes_R K_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_n\otimes_R K_1 & \xrightarrow{F_n\otimes\partial_1} & F_n\otimes_R K_0 \\ & & \downarrow\sigma_n & & \downarrow\sigma_{n-1} & & & & \downarrow\sigma_1 & & \downarrow\sigma_0 \\ 0 & \longrightarrow & F_n & \xrightarrow{\varphi_n} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{\varphi_1} & F_0 \end{array}$$

satisfying the following conditions.

(1) $\sigma_0^{-1}(\operatorname{Im}\varphi_1) = \operatorname{Im}(F_n\otimes\partial_1)$.
(2) $\operatorname{Im}\sigma_0 + \operatorname{Im}\varphi_1 = M :_{F_0} Q$.
(3) $\sigma_{n-1}(v_\lambda\otimes\check{e}_i) = (-1)^{n+i-1}\cdot v_{(\lambda,i)}$ *for all* $(\lambda,i)\in\widetilde{\Lambda}$.
(4) $\sigma_n(v_\lambda\otimes e_N) = (-1)^n\cdot v_\lambda$ *for all* $\lambda\in\Lambda$.

PROOF. Let us notice that, for any $p = 0, 1, \ldots, n$, $\{v_\lambda\otimes e_I\}_{(\lambda,I)\in\Lambda\times N_p}$ is an R-free basis of $F_n\otimes_R K_p$, so $\sigma_p : F_n\otimes_R K_p \longrightarrow F_p$ can be defined by choosing suitable element $w_{(\lambda,I)}\in F_p$ that corresponds to $v_\lambda\otimes e_I$ for $(\lambda,I)\in\Lambda\times N_p$. We set $w_{(\lambda,N)} = (-1)^n\cdot v_\lambda$ for $\lambda\in\Lambda$ and $w_{(\lambda,N\setminus\{i\})} = (-1)^{n+i-1}\cdot v_{(\lambda,i)}$ for $(\lambda,i)\in\widetilde{\Lambda}$. Then

$$\begin{aligned} \varphi_n(w_{(\lambda,N)}) &= (-1)^n\cdot\varphi_n(v_\lambda) \\ &= (-1)^n\cdot\sum_{i\in N} x_i\cdot v_{(\lambda,i)} \\ &= \sum_{i\in N}(-1)^{s(i,N)}\cdot x_i\cdot w_{(\lambda,N\setminus\{i\})}\,. \end{aligned}$$

Moreover, we can take families $\{w_{(\lambda,I)}\}_{(\lambda,I)\in\Lambda\times N_p}$ of elements in F_p for any $p = 0, 1, \ldots, n-2$ so that

$$\varphi_p(w_{(\lambda,I)}) = \sum_{i\in I}(-1)^{s(i,I)} \cdot x_i \cdot w_{(\lambda,I\setminus\{i\})} \tag{$\sharp$}$$

for all $p = 1, 2, \ldots, n$ and $(\lambda, I) \in \Lambda \times N_p$. If this is true, an R-linear map $\sigma_p : F_n \otimes_R K_p \longrightarrow F_p$ is defined by setting $\sigma_p(v_\lambda \otimes e_I) = w_{(\lambda,I)}$ for $(\lambda, I) \in \Lambda \times N_p$ and $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$ becomes a chain map satisfying (3) and (4).

In order to see the existence of $\{w_{(\lambda,I)}\}_{(\lambda,I)\in\Lambda\times N_p}$, let us consider the double complex $F_\bullet \otimes_R K_\bullet$.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & F_p \otimes_R K_q \arrow[r, "\varphi_p\otimes K_q"] \arrow[d, "F_p\otimes\partial_q"] & F_{p-1} \otimes_R K_q \arrow[r] \arrow[d, "F_{p-1}\otimes\partial_q"] & \cdots \\
\cdots \arrow[r] & F_p \otimes_R K_{q-1} \arrow[r, "\varphi_p\otimes K_{q-1}"] \arrow[d] & F_{p-1} \otimes_R K_{q-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

We can take it as $C_{\bullet\bullet}$ of 2.2. Let $T_\bullet$ be the total complex and $d_\bullet$ be its boundary map. In particular, we have

$$T_n = (F_n \otimes_R K_0) \oplus (F_{n-1} \otimes_R K_1) \oplus \cdots \oplus (F_1 \otimes_R K_{n-1}) \oplus (F_0 \otimes_R K_n)\,.$$

For $I \subseteq N$, we define

$$t(I) = \begin{cases} \sum_{i\in I}(i-1) & \text{if } I \neq \emptyset\,, \\ 0 & \text{if } I = \emptyset\,. \end{cases}$$

For a while, we fix $\lambda \in \Lambda$ and set

$$\xi_n(\lambda) = (-1)^{\frac{n(n+1)}{2}} \cdot (-1)^{t(N)} \cdot w_{(\lambda,N)} \otimes e_\emptyset \in F_n \otimes_R K_0\,,$$
$$\xi_{n-1}(\lambda) = (-1)^{\frac{(n-1)n}{2}} \cdot \sum_{i\in N}(-1)^{t(N\setminus\{i\})} \cdot w_{(\lambda,N\setminus\{i\})} \otimes e_i \in F_{n-1} \otimes_R K_1\,.$$

It is easy to see that

$$\xi_n(\lambda) = v_\lambda \otimes e_\emptyset$$

since $t(N) = (n-1)n/2$ and $n^2 + n \equiv 0 \pmod 2$. Moreover, we have

$$\xi_{n-1}(\lambda) = (-1)^n \cdot \sum_{i\in N} v_{(\lambda,i)} \otimes e_i$$

since $t(N \setminus \{i\}) = (n-1)n/2 - (i-1)$. Then

$$\begin{aligned}(\varphi_n \otimes K_0)(\xi_n(\lambda)) &= \varphi_n(v_\lambda) \otimes e_\emptyset \\ &= \Big(\sum_{i\in N} x_i \cdot v_{(\lambda,i)}\Big) \otimes e_\emptyset \\ &= \sum_{i\in N} v_{(\lambda,i)} \otimes x_i \\ &= (F_{n-1} \otimes \partial_1)\Big(\sum_{i\in N} v_{(\lambda,i)} \otimes e_i\Big) \\ &= (-1)^n \cdot (F_{n-1} \otimes \partial_1)(\xi_{n-1}(\lambda)) .\end{aligned}$$

Hence, by (1) of 2.2 there exist elements $\xi_p(\lambda) \in F_p \otimes K_{n-p}$ for all $p = 0, 1, \dots, n-2$ such that

$$\xi_n(\lambda) + \xi_{n-1}(\lambda) + \xi_{n-2}(\lambda) + \cdots + \xi_0(\lambda) \in \operatorname{Ker} d_n \subseteq T_n ,$$

which means

$$(\varphi_p \otimes K_{n-p})(\xi_p(\lambda)) = (-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda))$$

for any $p = 1, 2, \dots, n$. Let us denote $N \setminus I$ by I^{c} for $I \subseteq N$. Because $\{e_{I^{\mathrm{c}}}\}_{I\in N_p}$ is an R-free basis of K_{n-p}, it is possible to write

$$\xi_p(\lambda) = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I\in N_p} (-1)^{t(I)} \cdot w_{(\lambda,I)} \otimes e_{I^{\mathrm{c}}}$$

for any $p = 0, 1, \dots, n-2$ (Notice that $\xi_n(\lambda)$ and $\xi_{n-1}(\lambda)$ are defined so that they satisfy the same equalities), where $w_{(\lambda,I)} \in F_p$. Then we have

$$(\varphi_p \otimes K_{n-p})(\xi_p(\lambda)) = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I\in N_p} (-1)^{t(I)} \cdot \varphi_p(w_{(\lambda,I)}) \otimes e_{I^{\mathrm{c}}} .$$

On the other hand,

$$\begin{aligned}&(-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda)) \\ &\quad = (-1)^p \cdot (-1)^{\frac{(p-1)p}{2}} \cdot \sum_{J\in N_{p-1}} \left\{(-1)^{t(J)} \cdot w_{(\lambda,J)} \otimes \Big(\sum_{i\in J^{\mathrm{c}}} (-1)^{s(i,J^{\mathrm{c}})} \cdot x_i \cdot e_{J^{\mathrm{c}}\setminus\{i\}}\Big)\right\} .\end{aligned}$$

Here we notice that if $I \in N_p$, $J \in N_{p-1}$ and $i \in N$, then

$$I^{\mathrm{c}} = J^{\mathrm{c}} \setminus \{i\} \iff I = J \cup \{i\} .$$

Hence we get

$$
\begin{aligned}
&(-1)^p \cdot (F_{p-1} \otimes \partial_{n-p+1})(\xi_{p-1}(\lambda)) \\
&\quad = (-1)^{\frac{p(p+1)}{2}} \cdot \sum_{I \in N_p} \left\{ \left(\sum_{i \in I} (-1)^{t(I\setminus\{i\})+s(i, I^{\mathrm{c}}\cup\{i\})} \cdot x_i \cdot w_{(\lambda, I\setminus\{i\})} \right) \otimes e_{I^{\mathrm{c}}} \right\} .
\end{aligned}
$$

For $I \in N_p$ and $i \in I$, we have

$$
\begin{gathered}
t(I \setminus \{i\}) = t(I) - (i-1) , \\
s(i, I) + s(i, I^{\mathrm{c}} \cup \{i\}) = s(i, N) = i - 1 ,
\end{gathered}
$$

and so

$$
\begin{aligned}
t(I \setminus \{i\}) + s(i, I^{\mathrm{c}} \cup \{i\}) &= t(I) - s(i, I) \\
&\equiv t(I) + s(i, I) \pmod 2 .
\end{aligned}
$$

Therefore we see that the required equality ($\sharp$) holds for all $I \in N_p$.

Let us prove (1). We have to show $\sigma_0^{-1}(\operatorname{Im}\varphi_1) \subseteq \operatorname{Im}(F_n \otimes \partial_1)$. Take any $\eta_n \in F_n \otimes_R K_0$ such that $\sigma_0(\eta_n) \in \operatorname{Im}\varphi_1$. As $\{\xi_n(\lambda)\}_{\lambda\in\Lambda}$ is an R-free basis of $F_n \otimes_R K_0$, we can express

$$
\eta_n = \sum_{\lambda\in\Lambda} a_\lambda \cdot \xi_n(\lambda) = \sum_{\lambda\in\Lambda} a_\lambda \cdot (v_\lambda \otimes e_\emptyset) ,
$$

where $a_\lambda \in R$ for $\lambda \in \Lambda$. Then we have

$$
\sum_{\lambda\in\Lambda} a_\lambda \cdot w_{(\lambda,\emptyset)} = \sum_{\lambda\in\Lambda} a_\lambda \cdot \sigma_0(v_\lambda \otimes e_\emptyset) = \sigma_0(\eta_n) \in \operatorname{Im}\varphi_1 .
$$

Now we set

$$
\eta_p = \sum_{\lambda\in\Lambda} a_\lambda \cdot \xi_p(\lambda) \in F_p \otimes_R K_{n-p}
$$

for $0 \le p \le n-1$. Then

$$
\begin{aligned}
\eta_n + \eta_{n-1} + \cdots + \eta_1 + \eta_0 &= \sum_{\lambda\in\Lambda} a_\lambda \cdot (\xi_n(\lambda) + \xi_{n-1}(\lambda) + \cdots + \xi_1(\lambda) + \xi_0(\lambda)) \\
&\in \operatorname{Ker} d_n \subseteq T_n .
\end{aligned}
$$

Because

$$
\begin{aligned}
\eta_0 &= \sum_{\lambda\in\Lambda} a_\lambda \cdot \xi_0(\lambda) \\
&= \sum_{\lambda\in\Lambda} a_\lambda \cdot (w_{(\lambda,\emptyset)} \otimes e_N) \\
&= \left(\sum_{\lambda\in\Lambda} a_\lambda \cdot w_{(\lambda,\emptyset)} \right) \otimes e_N
\end{aligned}
$$

$$\in \operatorname{Im}(\varphi_1 \otimes K_n)\,,$$

we get $\eta_n \in \operatorname{Im}(F_n \otimes \partial_1)$ by (2) of 2.2.

Finally we prove (2). Let us consider the following commutative diagram

$$\begin{array}{ccccccc} F_n \otimes_R K_1 & \xrightarrow{F_n \otimes \partial_1} & F_n \otimes_R K_0 & \longrightarrow & F_n/QF_n & \longrightarrow & 0 \quad (\text{ex}) \\ \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \sigma_0} & & \downarrow{\scriptstyle \overline{\sigma_0}} & & \\ F_1 & \xrightarrow{\varphi_1} & F_0 & \longrightarrow & F_0/M & \longrightarrow & 0 \quad (\text{ex})\,, \end{array}$$

where $\overline{\sigma_0}$ is the map induced from σ_0. For all $\lambda \in \Lambda$ and $i \in N$, we have

$$x_i \cdot w_{(\lambda,\emptyset)} = \varphi_1(w_{(\lambda,\{i\})}) \in M\,,$$

which means $w_{(\lambda,\emptyset)} \in M :_{F_0} Q$. Hence $\operatorname{Im}\sigma_0 \subseteq M :_{F_0} Q$, and so $\operatorname{Im}\overline{\sigma_0} \subseteq (M :_{F_0} Q)/M$. On the other hand, as $\sigma_0^{-1}(\operatorname{Im}\varphi_1) = \operatorname{Im}(F_n \otimes \partial_1)$, we see that $\overline{\sigma_0}$ is injective. Therefore we get $\operatorname{Im}\overline{\sigma_0} = (M :_{F_0} Q)/M$ since $(M :_{F_0} Q)/M \cong F_n/QF_n$ by 3.1 and F_n/QF_n has a finite length. Thus the assertion (2) follows and the proof is complete. □

In the rest, $\sigma_\bullet : F_n \otimes_R K_\bullet \longrightarrow F_\bullet$ is the chain map constructed in 3.2. Then, by 2.1 the mapping cone $\operatorname{Cone}(\sigma_\bullet)$ gives an R-free resolution of $F_0/(M :_{F_0} Q)$, that is,

$$\begin{aligned} 0 \longrightarrow F_n \otimes_R K_n &\xrightarrow{\psi_{n+1}} (F_n \otimes_R K_{n-1}) \oplus F_n \xrightarrow{\psi_n} (F_n \otimes_R K_{n-2}) \oplus F_{n-1} \\ &\xrightarrow{{}'\varphi_{n-1}} (F_n \otimes_R K_{n-3}) \oplus F_{n-2} \xrightarrow{{}^*\varphi_{n-2}} (F_n \otimes_R K_{n-4}) \oplus F_{n-3} \longrightarrow \cdots \\ &\longrightarrow (F_n \otimes_R K_1) \oplus F_2 \xrightarrow{{}^*\varphi_2} (F_n \otimes_R K_0) \oplus F_1 \xrightarrow{{}^*\varphi_1} F_0 \end{aligned}$$

is acyclic and $\operatorname{Im}{}^*\varphi_1 = M :_{F_0} Q$, where

$$\psi_{n+1} = \begin{pmatrix} F_n \otimes \partial_n & (-1)^n \cdot \sigma_n \end{pmatrix}\,, \quad \psi_n = \begin{pmatrix} F_n \otimes \partial_{n-1} & (-1)^{n-1} \cdot \sigma_{n-1} \\ 0 & \varphi_n \end{pmatrix}\,,$$

$${}'\varphi_{n-1} = \begin{pmatrix} F_n \otimes \partial_{n-2} & (-1)^{n-2} \cdot \sigma_{n-2} \\ 0 & \varphi_{n-1} \end{pmatrix}\,,$$

$${}^*\varphi_p = \begin{pmatrix} F_n \otimes \partial_{p-1} & (-1)^{p-1} \cdot \sigma_{p-1} \\ 0 & \varphi_p \end{pmatrix} \quad \text{for } 2 \le p \le n-2 \text{ and } {}^*\varphi_1 = \begin{pmatrix} \sigma_0 \\ \varphi_1 \end{pmatrix}\,.$$

Because $\sigma_n : F_n \otimes_R K_n \longrightarrow F_n$ is an isomorphism by (4) of 3.2, we can define

$$\phi = \begin{pmatrix} 0 \\ (-1)^n \cdot \sigma_n^{-1} \end{pmatrix} \quad : \quad (F_n \otimes_R K_{n-1}) \oplus F_n \longrightarrow F_n \otimes_R K_n\,.$$

Then $\phi \circ \psi_{n+1} = \mathrm{id}_{F_n \otimes_R K_n}$ and $\operatorname{Ker}\phi = F_n \otimes_R K_{n-1}$. Hence, by (1) of 2.3, we get the acyclic complex

$$0 \longrightarrow {}'F_n \xrightarrow{{}'\varphi_n} {}'F_{n-1} \xrightarrow{{}'\varphi_{n-1}} {}^*F_{n-2} \xrightarrow{{}^*\varphi_{n-2}} {}^*F_{n-3} \longrightarrow \cdots \longrightarrow {}^*F_2 \xrightarrow{{}^*\varphi_2} {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0\,,$$

where

$$'F_n = F_n \otimes_R K_{n-1}\,, \quad 'F_{n-1} = (F_n \otimes_R K_{n-2}) \oplus F_{n-1}\,,$$

$$^*F_p = (F_n \otimes_R K_{p-1}) \oplus F_p \quad \text{for } 1 \le p \le n-2 \text{ and } '\varphi_n = \big(F_n \otimes \partial_{n-1}\ (-1)^{n-1} \cdot \sigma_{n-1}\big)\,.$$

Although $\operatorname{Im}{'\varphi_n}$ may not be contained in $\mathfrak{m} \cdot {'F_{n-1}}$, removing non-minimal components from $'F_n$ and $'F_{n-1}$, we get free R-modules $^*F_n$ and $^*F_{n-1}$ such that

$$0 \longrightarrow {^*F_n} \xrightarrow{^*\varphi_n} {^*F_{n-1}} \xrightarrow{^*\varphi_{n-1}} {^*F_{n-2}} \longrightarrow \cdots \longrightarrow {^*F_1} \xrightarrow{^*\varphi_1} {^*F_0} = F_0$$

is acyclic and $\operatorname{Im}{^*\varphi_n} \subseteq \mathfrak{m} \cdot {^*F_{n-1}}$, where $^*\varphi_n$ and $^*\varphi_{n-1}$ are the restrictions of $'\varphi_n$ and $'\varphi_{n-1}$, respectively. In the rest of this section, we describe a concrete procedure to get $^*F_n$ and $^*F_{n-1}$. For that purpose, we use the following notation. As described in Introduction, for any $\xi \in F_n \otimes_R K_{n-2}$ and $\eta \in F_{n-1}$,

$$[\xi] := (\xi, 0) \in {'F_{n-1}} \quad \text{and} \quad \langle \eta \rangle := (0, \eta) \in {'F_{n-1}}\,.$$

In particular, for any $(\lambda, I) \in \Lambda \times N_{n-2}$, we denote $[v_\lambda \otimes e_I]$ by $[\lambda, I]$. Moreover, for a subset U of F_{n-1}, $\langle U \rangle := \{\langle u \rangle\}_{u \in U}$.

Now, let us choose a subset $'\Lambda$ of $\widetilde{\Lambda}$ and a subset U of F_{n-1} so that

$$\{v_{(\lambda,i)}\}_{(\lambda,i) \in {'\Lambda}} \cup U$$

is an R-free basis of F_{n-1}. We would like to choose $'\Lambda$ as big as possible. The following almost obvious fact is useful to find $'\Lambda$ and U.

LEMMA 3.3. *Let V be an R-free basis of F_{n-1}. If a subset $'\Lambda$ of $\widetilde{\Lambda}$ and a subset U of V satisfy*

(i) $\sharp{'\Lambda} + \sharp U \le \sharp V$, *and*

(ii) $V \subseteq R \cdot \{v_{(\lambda,i)}\}_{(\lambda,i) \in {'\Lambda}} + R \cdot U + \mathfrak{m} F_{n-1}$,

then $\{v_{(\lambda,i)}\}_{(\lambda,i) \in {'\Lambda}} \cup U$ is an R-free basis of F_{n-1}.

Let us notice that

$$\{[\lambda, I]\}_{(\lambda,I) \in \Lambda \times N_{n-2}} \cup \{\langle v_{(\lambda,i)} \rangle\}_{(\lambda,i) \in {'\Lambda}} \cup \langle U \rangle$$

is an R-free basis of $'F_{n-1}$. We define $^*F_{n-1}$ to be the direct summand of $'F_{n-1}$ generated by

$$\{[\lambda, I]\}_{(\lambda,I) \in \Lambda \times N_{n-2}} \cup \langle U \rangle\,.$$

Let $^*\varphi_{n-1}$ be the restriction of $'\varphi_{n-1}$ to $^*F_{n-1}$.

THEOREM 3.4. *If we can take $\widetilde{\Lambda}$ itself as $'\Lambda$, then*

$$0 \longrightarrow {^*F_{n-1}} \xrightarrow{^*\varphi_{n-1}} {^*F_{n-2}} \longrightarrow \cdots \longrightarrow {^*F_1} \xrightarrow{^*\varphi_1} {^*F_0} = F_0$$

is acyclic. Hence we have $\operatorname{depth}_R F_0/(M :_{F_0} Q) > 0$.

PROOF. If ${}'\Lambda = \widetilde{\Lambda}$, there exists a homomorphism $\phi : {}'F_{n-1} \longrightarrow {}'F_n$ such that

$$\phi([\lambda, I]) = 0 \quad \text{for any } (\lambda, I) \in \Lambda \times N_{n-2} ,$$

$$\phi(\langle v_{(\lambda,i)} \rangle) = (-1)^i \cdot v_\lambda \otimes \check{e}_i \quad \text{for any } (\lambda, i) \in \widetilde{\Lambda} ,$$

$$\phi(\langle u \rangle) = 0 \quad \text{for any } u \in U .$$

Then $\phi \circ {}'\varphi_n = \mathrm{id}_{{}'F_n}$ and $\operatorname{Ker} \phi = {}^*F_{n-1}$. Hence, by (1) of 2.3 we get the required assertion. □

In the rest of this section, we assume ${}'\Lambda \subsetneq \widetilde{\Lambda}$ and put ${}^*\Lambda = \widetilde{\Lambda} \setminus {}'\Lambda$. Then, for any $(\mu, j) \in {}^*\Lambda$, it is possible to write

$$v_{(\mu,j)} = \sum_{(\lambda,i) \in {}'\Lambda} a_{(\lambda,i)}^{(\mu,j)} \cdot v_{(\lambda,i)} + \sum_{u \in U} b_u^{(\mu,j)} \cdot u ,$$

where $a_{(\lambda,i)}^{(\mu,j)}, b_u^{(\mu,j)} \in R$. Here, if ${}'\Lambda$ is big enough, we can choose every $b_u^{(\mu,j)}$ from $\mathfrak{m}$. In fact, if $b_u^{(\mu,j)} \notin \mathfrak{m}$ for some $u \in U$, then we can replace ${}'\Lambda$ and U by ${}'\Lambda \cup \{(\mu, j)\}$ and $U \setminus \{u\}$, respectively. Furthermore, because of a practical reason, let us allow that some terms of $v_{(\lambda,i)}$ for $(\lambda, i) \in {}^*\Lambda$ with non-unit coefficients appear in the right hand side, that is, for any $(\mu, j) \in {}^*\Lambda$, we write

$$v_{(\mu,j)} = \sum_{(\lambda,i) \in \widetilde{\Lambda}} a_{(\lambda,i)}^{(\mu,j)} \cdot v_{(\lambda,i)} + \sum_{u \in U} b_u^{(\mu,j)} \cdot u ,$$

where

$$a_{(\lambda,i)}^{(\mu,j)} \in \begin{cases} R & \text{if } (\lambda, i) \in {}'\Lambda , \\ \mathfrak{m} & \text{if } (\lambda, i) \in {}^*\Lambda \end{cases} \quad \text{and } b_u^{(\mu,j)} \in \mathfrak{m} .$$

Using this expression, for any $(\mu, j) \in {}^*\Lambda$, the following element in ${}'F_n$ can be defined.

$${}^*v_{(\mu,j)} := (-1)^j \cdot v_\mu \otimes \check{e}_j + \sum_{(\lambda,i) \in \widetilde{\Lambda}} (-1)^{i-1} \cdot a_{(\lambda,i)}^{(\mu,j)} \cdot v_\lambda \otimes \check{e}_i .$$

LEMMA 3.5. *For any* $(\mu, j) \in {}^*\Lambda$, *we have*

$${}'\varphi_n({}^*v_{(\mu,j)}) = (-1)^j \cdot [v_\mu \otimes \partial_{n-1}(\check{e}_j)] + \sum_{(\lambda,i) \in \widetilde{\Lambda}} (-1)^{i-1} \cdot a_{(\lambda,i)}^{(\mu,j)} \cdot [v_\lambda \otimes \partial_{n-1}(\check{e}_i)]$$
$$+ \sum_{u \in U} b_u^{(\mu,j)} \cdot \langle u \rangle .$$

As a consequence, we have ${}'\varphi_n({}^*v_{(\mu,j)}) \in \mathfrak{m} \cdot {}^*F_{n-1}$ *for any* $(\mu, j) \in {}^*\Lambda$.

PROOF. By the definition of ${}'\varphi_n$, for any $(\mu, j) \in {}^*\Lambda$, we have

$${}'\varphi_n({}^*v_{(\mu,j)}) = [(F_n \otimes \partial_{n-1})({}^*v_{(\mu,j)})] + \langle (-1)^{n-1} \cdot \sigma_{n-1}({}^*v_{(\mu,j)}) \rangle .$$

Because

$$(F_n \otimes \partial_{n-1})({}^*v_{(\mu,j)}) = (-1)^j \cdot v_\mu \otimes \partial_{n-1}(\check{e}_j) + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot v_\lambda \otimes \partial_{n-1}(\check{e}_i)$$

and

$$\begin{aligned}
\sigma_{n-1}({}^*v_{(\mu,j)}) &= (-1)^j \cdot \sigma_{n-1}(v_\mu \otimes \check{e}_j) + \sum_{(\lambda,i)\in\widetilde{\Lambda}} (-1)^{i-1} \cdot a^{(\mu,j)}_{(\lambda,i)} \cdot \sigma_{n-1}(v_\lambda \otimes \check{e}_i) \\
&= (-1)^{n-1} \cdot v_{(\mu,j)} + (-1)^n \cdot \sum_{(\lambda,i)\in\widetilde{\Lambda}} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)} \\
&= (-1)^{n-1} \cdot (v_{(\mu,j)} - \sum_{(\lambda,i)\in\widetilde{\Lambda}} a^{(\mu,j)}_{(\lambda,i)} \cdot v_{(\lambda,i)}) \\
&= (-1)^{n-1} \cdot \sum_{u\in U} b^{(\mu,j)}_u \cdot u\,,
\end{aligned}$$

we get the required equality. □

Let ${}^*F_n$ be the R-submodule of ${}'F_n$ generated by $\{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda}$ and let ${}^*\varphi_n$ be the restriction of ${}'\varphi_n$ to ${}^*F_n$. By 3.5 we have $\operatorname{Im}{}^*\varphi_n \subseteq {}^*F_{n-1}$. Thus we get a complex

$$0 \longrightarrow {}^*F_n \xrightarrow{{}^*\varphi_n} {}^*F_{n-1} \longrightarrow \cdots \longrightarrow {}^*F_1 \xrightarrow{{}^*\varphi_1} {}^*F_0 = F_0\,.$$

This is the complex we desire. In fact, the following result holds.

THEOREM 3.6. *$({}^*F_\bullet, {}^*\varphi_\bullet)$ is an acyclic complex of finitely generated free R-modules with the following properties.*

(1) $\operatorname{Im}{}^*\varphi_1 = M :_{F_0} Q$ *and* $\operatorname{Im}{}^*\varphi_n \subseteq \mathfrak{m} \cdot {}^*F_{n-1}$.
(2) $\{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda}$ *is an R-free basis of* ${}^*F_n$.
(3) $\{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \langle U\rangle$ *is an R-free basis of* ${}^*F_{n-1}$.

PROOF. First, let us notice that $\{v_\lambda \otimes \check{e}_i\}_{(\lambda,i)\in\widetilde{\Lambda}}$ is an R-free basis of ${}'F_n$ and

$$v_\mu \otimes \check{e}_j \in R \cdot {}^*v_{(\mu,j)} + R \cdot \{v_\lambda \otimes \check{e}_i\}_{(\lambda,i)\in{}'\Lambda} + \mathfrak{m} \cdot {}'F_n$$

for any $(\mu, j) \in {}^*\Lambda$. Hence, by Nakayama's lemma it follows that ${}'F_n$ is generated by

$$\{v_\lambda \otimes \check{e}_i\}_{(\lambda,i)\in{}'\Lambda} \cup \{{}^*v_{(\mu,j)}\}_{(\mu,j)\in{}^*\Lambda}\,,$$

which must be an R-free basis since $\operatorname{rank}_R {}'F_n = \sharp\,\widetilde{\Lambda} = \sharp\,{}'\Lambda + \sharp\,{}^*\Lambda$. Let ${}''F_n$ be the R-submodule of ${}'F_n$ generated by $\{v_\lambda \otimes \check{e}_i\}_{(\lambda,i)\in{}'\Lambda}$. Then ${}'F_n = {}''F_n \oplus {}^*F_n$.

Next, let us recall that

$$\{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \{\langle v_{(\lambda,i)}\rangle\}_{(\lambda,i)\in{}'\Lambda} \cup \langle U\rangle$$

is an R-free basis of ${}'F_{n-1}$. Because

$$ {}'\varphi_n(v_\lambda \otimes \check{e}_i) = [v_\lambda \otimes \partial_{n-1}(\check{e}_i)] + (-1)^i \cdot \langle v_{(\lambda,i)} \rangle \, , $$

we see that

$$ \{[\lambda, I]\}_{(\lambda,I)\in\Lambda\times N_{n-2}} \cup \{{}'\varphi_n(v_\lambda \otimes \check{e}_i)\}_{(\lambda,i)\in{}'\Lambda} \cup \langle U \rangle $$

is also an R-free basis of ${}'F_{n-1}$. Let ${}''F_{n-1} = R \cdot \{{}'\varphi_n(v_\lambda \otimes \check{e}_i)\}_{(\lambda,i)\in{}'\Lambda}$. Then ${}'F_{n-1} = {}''F_{n-1} \oplus {}^*F_{n-1}$.

It is obvious that ${}'\varphi_n({}''F_n) = {}''F_{n-1}$. Moreover, by 3.5 we get ${}'\varphi_n({}^*F_n) \subseteq {}^*F_{n-1}$. Therefore, by (2) of 2.3, it follows that ${}^*F_\bullet$ is acyclic. We have already seen (3) and the first assertion of (1). The second assertion of (1) follows from 3.5. Moreover, the assertion (2) is now obvious. $\square$

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