

Product Hardy Operators on Hardy Spaces

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Abstract. We study the product Hausdorff operator H_Φ on the product Hardy spaces, and prove that, for a nonnegative valued function Φ , H_Φ is bounded on the product Hardy space $H^1(\mathbb{R} \times \mathbb{R})$ if and only if Φ is a Lebesgue integrable function on $(0, \infty) \times (0, \infty)$. As an application, we know that the product Hardy operator \mathcal{H} is not bounded on $H^1(\mathbb{R} \times \mathbb{R})$. On the other hand, we prove that $\|\mathcal{H}f\|_{H^1(\mathbb{R} \times \mathbb{R})} \leq \|f\|_{H^1(\mathbb{R} \times \mathbb{R})}$ if f is an even function. Furthermore, using the $H^p(\mathbb{R} \times \mathbb{R})$ boundedness criterion of Fefferman, we prove that the k -th order Hardy operator is bounded on $H^p(\mathbb{R} \times \mathbb{R})$ whenever $k > 1/p - 1$.

1. Introduction

Let Φ be a suitable function. The classical one-parameter Hausdorff operator with the kernel function Φ is defined in the dilation-integral form by

$$h_\Phi(f)(x) = \int_0^\infty \frac{\Phi(t)}{t} f\left(\frac{x}{t}\right) dt,$$

where, for the sake of simplicity, we initially assume that the operator acts on the class of Schwartz functions f . The Hausdorff operator h_Φ is an important operator on analysis and its boundedness on various function or distribution spaces was extensively studied in recent years. The reader can see [1, 2, 9–16] to find a lot of results and historical developments about this operator. As an analog, the product Hausdorff operator

$$H_\Phi(f)(x, y) = \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} f\left(\frac{x}{s}, \frac{y}{t}\right) ds dt$$

is also received some attention (see [19, 20]). Particularly, if one chooses

$$\Phi(s, t) = \frac{\chi_{(1, \infty) \times (1, \infty)}(s, t)}{st},$$

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then the operator H_Φ becomes the product Hardy operator

$$\mathcal{H}f(x, y) = \frac{1}{xy} \int_0^x \int_0^y f(t, s) dt ds,$$

that is also an interesting operator in harmonic analysis [8, 17, 18, 20].

When $p > 1$, by the Minkowski integral inequality and changing variables on the $L^p(\mathbb{R} \times \mathbb{R})$ integral, we obtain

$$\begin{aligned} \|H_\Phi(f)\|_{L^p(\mathbb{R} \times \mathbb{R})} &\leq \int_0^\infty \int_0^\infty \frac{|\Phi(s, t)|}{st} \left\| f\left(\frac{\cdot}{s}, \frac{\cdot}{t}\right) \right\|_{L^p(\mathbb{R} \times \mathbb{R})} ds dt \\ &= \|f\|_{L^p(\mathbb{R} \times \mathbb{R})} \int_0^\infty \int_0^\infty \frac{|\Phi(s, t)|}{st} s^{1/p} t^{1/p} ds dt. \end{aligned}$$

It indicates that $H_\Phi(f)$ is bounded on the Lebesgue space $L^p(\mathbb{R} \times \mathbb{R})$, if

$$\int_0^\infty \int_0^\infty \frac{|\Phi(s, t)|}{st} s^{1/p} t^{1/p} ds dt < \infty.$$

When $0 < p \leq 1$, a good substitution of L^p is the Hardy space H^p . The space H^1 is a subspace of L^1 and it is still a norm space. Thus, the Minkowski integral inequality gives

$$\|H_\Phi(f)\|_{H^1(\mathbb{R} \times \mathbb{R})} \leq \int_0^\infty \int_0^\infty \frac{|\Phi(s, t)|}{st} \left\| f\left(\frac{\cdot}{s}, \frac{\cdot}{t}\right) \right\|_{H^1(\mathbb{R} \times \mathbb{R})} ds dt.$$

Now an easy scaling argument shows that

$$\left\| f\left(\frac{\cdot}{s}, \frac{\cdot}{t}\right) \right\|_{H^1(\mathbb{R} \times \mathbb{R})} = st \|f\|_{H^1(\mathbb{R} \times \mathbb{R})}.$$

Hence, we know that H_Φ is bounded on $H^1(\mathbb{R} \times \mathbb{R})$, provided

$$\int_0^\infty \int_0^\infty |\Phi(s, t)| ds dt < \infty.$$

Furthermore, in the following theorem, we will show that the above condition on Φ is also a necessity condition, if Φ is nonnegative valued.

Theorem 1. *Assume that Φ is a nonnegative valued function. Then H_Φ is bounded on $H^1(\mathbb{R} \times \mathbb{R})$ if and only if $\Phi(s, t) \in L^1(\mathbb{R}_+ \times \mathbb{R}_+)$.*

Note that the kernel function $\Phi(s, t) = \frac{\chi_{(1,\infty) \times (1,\infty)}(s, t)}{st}$ of the product Hardy operator is not Lebesgue integrable. Thus by Theorem 1, we know that the product Hardy operator \mathcal{H} is not bounded on $H^1(\mathbb{R} \times \mathbb{R})$. But if $f \in H^1(\mathbb{R} \times \mathbb{R})$ is even, then we have the following result.

Theorem 2. *If $f(x, y) \in H^1(\mathbb{R} \times \mathbb{R})$ is even on both x and y variables, then we have*

$$\|\mathcal{H}f\|_{H^1(\mathbb{R} \times \mathbb{R})} \leq \|f\|_{H^1(\mathbb{R} \times \mathbb{R})}.$$

Additionally, recent papers [2, 14, 20] addressed the boundedness of H_Φ on the Hardy space $H^1(\mathbb{R} \times \mathbb{R})$, Besov space $B_0^{1,0}(\mathbb{R} \times \mathbb{R})$ and the local Hardy space $h^1(\mathbb{R} \times \mathbb{R})$, as well as the $H^1(\mathbb{R} \times \mathbb{R}) \rightarrow L^1(\mathbb{R} \times \mathbb{R})$ boundedness of H_Φ . However, we are not able to find any paper in the literature to study the boundedness of H_Φ on the product Hardy spaces $H^p(\mathbb{R} \times \mathbb{R})$ for $0 < p < 1$. This problem seems interesting and involved. First, $H^p(\mathbb{R} \times \mathbb{R})$ is merely a quasi-norm space. The argument involving the Minkowski integral inequality to treat the H^1 case does not work in this case. More significantly, the dual space and atomic structure of the product Hardy space $H^p(\mathbb{R} \times \mathbb{R})$ is much more complicated, compared to those of the one parameter Hardy space $H^p(\mathbb{R})$. Thus, all methods used to study the $H^p(\mathbb{R})$ boundedness of h_Φ (see [9, 11, 15]) fail in studying $H^p(\mathbb{R} \times \mathbb{R})$ boundedness of H_Φ . Motivated by above observations, in order to study the $H^p(\mathbb{R} \times \mathbb{R})$ boundedness of H_Φ , in this paper we will work on a special case of H_Φ , the k -th product Hardy operator

$$\mathcal{H}_{(k)}f(x, y) = \frac{1}{x^{k+1}y^{k+1}} \int_0^x \int_0^y t^k s^k f(t, s) dt ds .$$

By a changing of variables, we can write

$$\mathcal{H}_{(k)}f(x, y) = \int_1^\infty \int_1^\infty \frac{1}{t^{k+2}s^{k+2}} f(x/t, y/s) dt ds .$$

Hence, the k -th order product Hardy operator is also a special case of the product Hausdorff operator if we set

$$\Phi(s, t) = \frac{\chi_{(1,\infty) \times (1,\infty)}(s, t)}{s^{k+1}t^{k+1}} .$$

So, if $k = 0$ then $\mathcal{H}_{(0)}$ is just the product Hardy operator. We will establish the following result.

Theorem 3. $\mathcal{H}_{(k)}$ is bounded on $H^p(\mathbb{R} \times \mathbb{R})$ if $k > 1/p - 1$.

We will adopt some basic ideas used in one parameter case [5] to prove Theorem 1 and Theorem 2. However, the argument here is more technically difficult to execute in order to treat the product case. Furthermore, to prove Theorem 3, we will invoke a famous $H^p(\mathbb{R} \times \mathbb{R})$ boundedness criterion established by R. Fefferman. This criterion and some preliminary knowledge will be recalled in Section 2. Finally we prove these theorems in Section 3.

Throughout this paper, we use the notation $A \simeq B$ if there exist positive constants C and c , independent of all essential variables such that

$$cB \leq A \leq CB .$$

Also, we use the notation $A \leq B$ to mean that there is a constant $C > 0$ independent of all essential variables such that $A \leq CB$.

2. Some Preliminary Knowledge

We start with briefly describing the definition of the product Hardy spaces. Let $\phi \in S(\mathbb{R})$ and $\varphi \in S(\mathbb{R})$ satisfy

$$\int_{\mathbb{R}} \phi(x)dx \int_{\mathbb{R}} \varphi(y)dy \neq 0,$$

and denote

$$\phi_t(x) = \frac{1}{t}\phi\left(\frac{x}{t}\right), \quad \varphi_s(y) = \frac{1}{s}\varphi\left(\frac{y}{s}\right), \quad s, t > 0.$$

The product Hardy space $H^p(\mathbb{R} \times \mathbb{R})$ is the space of all distributions f satisfying

$$\|f\|_{H^p(\mathbb{R} \times \mathbb{R})} := \left\| \sup_{0 < s, t < \infty} |(\phi_t \otimes \varphi_s) * f| \right\|_{L^p(\mathbb{R} \times \mathbb{R})} < \infty,$$

where

$$(\phi_t \otimes \varphi_s)(x, y) = \phi_t(x)\varphi_s(y).$$

It is known that the definition of H^p is flexible on the choice of the functions ϕ and φ .

Compared to the one parameter atomic Hardy space (see [3]), the atomic decomposition of the product Hardy space is quite complicated. The following content can be found in [4].

Let R denote a rectangle and γR the rectangle with the same center of R and its side length is γ -times of the side length of R . For an open set $\Omega \subset \mathbb{R}^2$ of finite measure, let $m(\Omega)$ denote the set of all maximal dyadic rectangles in Ω . Assume $0 < p \leq 1$, and s is any integer larger than or equal to $[1/p - 1]$, the integer part of $1/p - 1$. A function $A(x, y)$ is called a (p, s) -atom if A satisfies the following three conditions:

- (1) $\text{supp}(A) \subset \Omega$, where $\Omega \subset \mathbb{R}^2$ is an open set of finite measure;
- (2) A can be written as

$$A = \sum_{R \in m(\Omega)} a_R,$$

where each a_R satisfies

- (2-i) $\text{supp}(a_R) \subset 3R$,
- (2-ii)

$$\int_{\mathbb{R}} x^\alpha a_R(x, y)dx = 0, \quad \text{all } y \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} a_R(x, y)y^\alpha dy = 0, \quad \text{all } x \in \mathbb{R},$$

for $\alpha = 0, 1, 2, \dots, s$;
 (3)

$$\|A\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq |\Omega|^{1/2-1/p}, \quad \sum_{R \in m(\Omega)} \|a_R\|_{L^2(\mathbb{R} \times \mathbb{R})}^2 \leq |\Omega|^{1-2/p}.$$

It is known that for $f \in H^p(\mathbb{R} \times \mathbb{R})$, f can be written as

$$f = \sum \lambda_j A_j$$

where each A_j is a (p, s) -atom and

$$\sum |\lambda_j|^p \simeq \|f\|_{H^p(\mathbb{R} \times \mathbb{R})}^p.$$

Also, to study the Hausdorff operator on $H^p(\mathbb{R} \times \mathbb{R})$, we will employ the Fefferman's criterion for $H^p(\mathbb{R} \times \mathbb{R})$ boundedness of linear operators with the rectangular atoms. Here, a function $a(x)$ is called a rectangular $(p, 2, s)$ -atom if it satisfies the following three conditions:

- (a) $\text{supp}(a) \subset R = I \times J$, where I and J are intervals in \mathbb{R} ;
- (b)

$$\int_{\mathbb{R}} x^\alpha a(x, y)dx = 0 \quad \text{for all } y \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} a(x, y)y^\alpha dy = 0 \quad \text{for all } x \in \mathbb{R}$$

for $\alpha = 0, 1, 2, \dots, s$, where s is a fixed integer larger than or equal to $[1/p - 1]$;
 (c)

$$\|a\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq |R|^{1/2-1/p}.$$

For a rectangular $(p, 2, s)$ -atom, using Hölder's inequality, we have

$$\int_{\mathbb{R} \times \mathbb{R}} |a(x, y)|^p dx dy \leq \|a\|_{L^2}^p |R|^{\frac{1-p}{2}} \leq 1.$$

The following theorem is well known.

Theorem A (R. Fefferman [6, 7]). *Let T be a linear operator bounded on $L^2(\mathbb{R} \times \mathbb{R})$. Suppose there exist positive numbers C and δ , for any rectangular $(p, 2, s)$ -atom a with*

support in R ,

$$\int_{(\gamma R)^c} |T(a)(x, y)|^p dx dy \leq C\gamma^{-\delta}$$

holds for all $\gamma \geq 2$. Then the operator T is bounded from $H^p(\mathbb{R} \times \mathbb{R})$ to $L^p(\mathbb{R} \times \mathbb{R})$.

The space $H^p(\mathbb{R} \times \mathbb{R})$ also can be characterized by using the Hilbert transforms. We define three Hilbert transforms as follows.

$$\mathfrak{R}_{1,1}(f)(x, y) = p.v. \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{1}{uv} f(x - u, y - v) dudv ,$$

$$\mathfrak{R}_1(f)(x, y) = p.v. \int_{\mathbb{R}} \frac{1}{u} f(x - u, y) du ,$$

$$\mathfrak{R}_2(f)(x, y) = p.v. \int_{\mathbb{R}} \frac{1}{v} f(x, y - v) dv .$$

Also, define \mathfrak{R}_0 as the identity map, namely

$$\mathfrak{R}_0(f)(x, y) = f(x, y) .$$

It is known (see [14, 18]) that for all $f \in H^p(\mathbb{R} \times \mathbb{R}) \cap L^2(\mathbb{R} \times \mathbb{R})$,

$$\|f\|_{H^p(\mathbb{R} \times \mathbb{R})} \simeq \|\mathfrak{R}_{1,1}(f)\|_{L^p(\mathbb{R} \times \mathbb{R})} + \sum_{m=0}^2 \|\mathfrak{R}_m(f)\|_{L^p(\mathbb{R} \times \mathbb{R})} .$$

3. Proof of Theorems

3.1. Proof of Theorem 1. By the Minkowski integral inequality and a scaling argument, we have

$$\begin{aligned} \|H_{\Phi}(f)\|_{H^1(\mathbb{R} \times \mathbb{R})} &\leq \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} \left\| f\left(\frac{\cdot}{s}, \frac{\cdot}{t}\right) \right\|_{H^1(\mathbb{R} \times \mathbb{R})} ds dt \\ &= \|f\|_{H^1(\mathbb{R} \times \mathbb{R})} \int_0^\infty \int_0^\infty \Phi(s, t) ds dt . \end{aligned}$$

This shows the sufficiency part.

Let a be a function with support on $[-1, 1]^2$, and be odd for both two variables and satisfies

- (i) $a(x, y) = 1/2$ for $(x, y) \in [1/4, 1/2]^2$,
- (ii) $1/2 \geq a(x_1, x_2) \geq 0$ on $[0, 1]^2$.

We can view such a as a rectangular $(1, 2, 0)$ -atom, so that by an easy computation, we know

$$\|a\|_{H^1(\mathbb{R} \times \mathbb{R})} \leq 1.$$

Suppose

$$\int_0^\infty \int_0^\infty \Phi(s, t) ds dt = \infty,$$

but H_Φ were bounded on H^1 , we have

$$\|H_\Phi(a)\|_{H^1} \leq \|a\|_{H^1} \leq 1.$$

On the other hand, by the Hilbert transform characterization of H^p , we have

$$\begin{aligned} \|H_\Phi(a)\|_{H^1} &\geq \|H_\Phi(a)\|_{L^1} \\ &\geq \int_0^1 \int_0^1 \left| \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} a\left(\frac{x}{s}, \frac{y}{t}\right) ds dt \right| dx_1 dx_2 \\ &= \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} \left(\int_0^1 \int_0^1 a\left(\frac{x}{s}, \frac{y}{t}\right) dx dy \right) ds dt \\ &\geq \int_0^\infty \int_0^\infty \Phi(s, t) ds dt = \infty. \end{aligned}$$

This leads to a contradiction.

3.2. Proof of Theorem 2. Let $f(x, y) \in H^1(\mathbb{R} \times \mathbb{R})$ be even on both x and y . Since H^1 is a subspace of L^1 , we have

$$f(x, y) = \sum \lambda_j A_j(x, y)$$

for almost all $(x, y) \in \mathbb{R} \times \mathbb{R}$, where each A_j is a $(1, 0)$ -atom and

$$\sum |\lambda_j| \simeq \|f\|_{H^1(\mathbb{R} \times \mathbb{R})}.$$

For a function g , we use the following notation.

$$\begin{aligned} g^{+,+} &= g, & g^{+,-}(x, y) &= g(x, -y), \\ g^{-,+}(x, y) &= g(-x, y), & g^{-,-}(x, y) &= g(-x, -y). \end{aligned}$$

Similarly, for an open set Λ in $\mathbb{R} \times \mathbb{R}$, we define

$$\begin{aligned} \Lambda^{+,+} &= \Lambda, & \Lambda^{+,-} &= \{(x, -y) : (x, y) \in \Lambda\}, \\ \Lambda^{-,+} &= \{(-x, y) : (x, y) \in \Lambda\}, & \Lambda^{-,-} &= \{(-x, -y) : (x, y) \in \Lambda\}. \end{aligned}$$

Since f is even on both x and y variables, we can write

$$f = \frac{1}{4} (f^{+,+} + f^{+,-} + f^{-,+} + f^{-,-}) .$$

Hence,

$$f(x, y) = \sum \lambda_j B_j(x, y) ,$$

where

$$B_j = \frac{1}{4} (A_j^{+,+} + A_j^{+,-} + A_j^{-,+} + A_j^{-,-}) .$$

Furthermore, we have

$$\mathcal{H}(f) = \sum \lambda_j \mathcal{H}(B_j) .$$

Now, by the atomic characterization of $H^1(\mathbb{R} \times \mathbb{R})$, it suffices to show that each $\mathcal{H}(B_j)$ is again a $(1, 0)$ -atom.

For simplicity in the notation, we write $A = A_j$ and $B = B_j$. Let Ω be in the definition of $(1, 0)$ -atom for which $\text{supp}(A) \subset \Omega$ and

$$\|A\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq |\Omega|^{-1/2} .$$

The set

$$\Omega^\sharp = \Omega^{+,+} \cup \Omega^{+,-} \cup \Omega^{-,+} \cup \Omega^{-,-}$$

is an open set in $\mathbb{R} \times \mathbb{R}$ so that it is a union of disjoint open rectangles

$$\Omega^\sharp = \cup_\alpha R_\alpha .$$

We denote

$$\Omega^* = \cup_\alpha 3R_\alpha .$$

Then Ω^* is also an open set in $\mathbb{R} \times \mathbb{R}$, $|\Omega^*| \simeq |\Omega|$, and $\text{supp}(B) \subset \Omega^*$.

Using the Minkowski integral inequality, we have

$$\begin{aligned} & \|\mathcal{H}(B)\|_{L^2(\mathbb{R} \times \mathbb{R})} \\ &= \left\| \int_1^\infty \int_1^\infty \frac{B(\frac{\cdot}{t}, \frac{\cdot}{s})}{t^2 s^2} dt ds \right\|_{L^2(\mathbb{R} \times \mathbb{R})} \leq \|B\|_{L^2(\mathbb{R} \times \mathbb{R})} \\ &\leq \frac{1}{4} \left\{ \|A^{+,+}\|_{L^2(\mathbb{R} \times \mathbb{R})} + \|A^{+,-}\|_{L^2(\mathbb{R} \times \mathbb{R})} + \|A^{-,+}\|_{L^2(\mathbb{R} \times \mathbb{R})} + \|A^{-,-}\|_{L^2(\mathbb{R} \times \mathbb{R})} \right\} \end{aligned}$$

$$\leq |\Omega^*|^{-1/2}.$$

We know that

$$A^{+,+}(x, y) = \sum_{R \in m(\Omega)} a_R^{+,+}(x, y),$$

where each a_R satisfies $\text{supp}(a_R) \subset R$, and $R = I \times J$. We denote

$$\begin{aligned} I^+ &= I, \quad I^- = \{-x, \quad x \in I\}, \\ J^+ &= J, \quad J^- = \{-y, \quad y \in J\}. \end{aligned}$$

Then

$$\begin{aligned} R^{+,+} &= R, \quad R^{+,-} = I^+ \times J^-, \\ R^{-,+} &= I^- \times J^+, \quad R^{-,-} = I^- \times J^-. \end{aligned}$$

With these definitions, we can write

$$\begin{aligned} &4\mathcal{H}(B)(x, y) \\ &= \sum_{R \in m(\Omega)} \left\{ \mathcal{H}\left(a_R^{+,+}\right)(x, y) + \mathcal{H}\left(a_R^{+,-}\right)(x, y) + \mathcal{H}\left(a_R^{-,+}\right)(x, y) + \mathcal{H}\left(a_R^{-,-}\right)(x, y) \right\}, \end{aligned}$$

where $a_R^{+,+}, a_R^{+,-}, a_R^{-,+}, a_R^{-,-}$ are rectangular $(1, 2, 0)$ -atoms and they satisfy

$$\begin{aligned} \text{supp}\left(a_R^{+,+}\right) &\subset R^{+,+} \in m(\Omega^{+,+}), \quad \text{supp}\left(a_R^{+,-}\right) \subset R^{+,-} \in m(\Omega^{+,-}), \\ \text{supp}\left(a_R^{-,+}\right) &\subset R^{-,+} \in m(\Omega^{-,+}), \quad \text{supp}\left(a_R^{-,-}\right) \subset R^{-,-} \in m(\Omega^{-,-}). \end{aligned}$$

Now, write $R = R^{+,+} = I \times J = (\alpha, \beta) \times (c, d)$. We consider the following four cases:

Case 1: $0 \notin I, 0 \notin J$;

Case 2: $0 \notin I, 0 \in J$;

Case 3: $0 \in I, 0 \notin J$;

Case 4: $0 \in I, 0 \in J$.

In Case 1, without loss of generality, we assume $I = [\alpha, \beta] \subset (0, \infty)$ and $J = [c, d] \subset (-\infty, 0)$. For any $x \notin [\alpha, \beta]$, if $x < \alpha$, clearly $\mathcal{H}(a_R)(x, y) = 0$. If $x > \beta$, by the cancellation condition of a_R we also see

$$\mathcal{H}(a_R)(x, y) = \frac{1}{xy} \int_0^x \int_0^y a_R(t, s) dt ds = \frac{1}{xy} \int_\alpha^\beta \int_0^y a_R(t, s) dt ds = 0.$$

This indicates that for any $(x, y) \notin R = I \times J$, $\mathcal{H}(a_R)(x, y) = 0$. Similarly, we can show

$$\mathcal{H}\left(a_R^{+,-}\right)(x, y) = 0, \quad \text{if } (x, y) \notin R^{+,-},$$

$$\begin{aligned}\mathcal{H}\left(a_R^{-,+}\right)(x, y) &= 0, \quad \text{if } (x, y) \notin R^{-,+}, \\ \mathcal{H}\left(a_R^{-,-}\right)(x, y) &= 0, \quad \text{if } (x, y) \notin R^{-,-}.\end{aligned}$$

Also, using the support condition and the Fubini theorem, it is easy to check that

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{H}\left(a_R^{+,+}\right)(x, y) dx &= \int_{\alpha}^{\beta} \int_1^{\infty} \int_1^{\infty} \frac{1}{s^2 t^2} a\left(\frac{x}{t}, \frac{y}{s}\right) dt ds dx = 0, \\ \int_{\mathbb{R}} \mathcal{H}\left(a_R^{+,-}\right)(x, y) dy &= \int_c^d \int_1^{\infty} \int_1^{\infty} \frac{1}{s^2 t^2} a\left(\frac{x}{t}, \frac{y}{s}\right) dt ds dy = 0.\end{aligned}$$

This shows that $\mathcal{H}\left(a_R^{+,+}\right)$ satisfies (2-i) and (2-ii).

Similarly, we can show that $\mathcal{H}\left(a_R^{+,-}\right), \mathcal{H}\left(a_R^{-,+}\right), \mathcal{H}\left(a_R^{-,-}\right)$ satisfy (2-i) and (2-ii). Since it is shown

$$\begin{aligned}\text{supp}(\mathcal{H}\left(a_R^{+,+}\right)) &\subset 3R^{+,+} \\ \text{supp}(\mathcal{H}\left(a_R^{+,-}\right)) &\subset 3R^{+,-} \\ \text{supp}(\mathcal{H}\left(a_R^{-,+}\right)) &\subset 3R^{-,+} \\ \text{supp}(\mathcal{H}\left(a_R^{-,-}\right)) &\subset 3R^{-,-},\end{aligned}$$

we conclude that in Case 1, all $\mathcal{H}\left(a_R^{+,+}\right), \mathcal{H}\left(a_R^{+,-}\right), \mathcal{H}\left(a_R^{-,+}\right), \mathcal{H}\left(a_R^{-,-}\right)$ are rectangular $(1, 2, 0)$ -atoms.

In Case 2, without loss of generality, we assume $I = [\alpha, \beta] \subset (0, \infty)$ and $J = [-c, c]$. We now write

$$\begin{aligned}&\left\{ \mathcal{H}\left(a_R^{+,+}\right)(x, y) + \mathcal{H}\left(a_R^{+,-}\right)(x, y) + \mathcal{H}\left(a_R^{-,+}\right)(x, y) + \mathcal{H}\left(a_R^{-,-}\right)(x, y) \right\} \\ &= \mathcal{H}\left(\left(a_R^{+,+} + a_R^{+,-}\right) \chi_{\mathbb{R} \times [-c, 0)}\right)(x, y) + \mathcal{H}\left(\left(a_R^{+,+} + a_R^{+,-}\right) \chi_{\mathbb{R} \times (0, c]}\right)(x, y) \\ &\quad + \mathcal{H}\left(\left(a_R^{-,+} + a_R^{-,-}\right) \chi_{\mathbb{R} \times [-c, 0]}\right)(x, y) + \mathcal{H}\left(\left(a_R^{-,+} + a_R^{-,-}\right) \chi_{\mathbb{R} \times [0, c]}\right)(x, y).\end{aligned}$$

Since both $a_R^{+,+} + a_R^{+,-}$ and $a_R^{-,+} + a_R^{-,-}$ are even on y , it is easy to check

$$\begin{aligned}\int_{\mathbb{R}} \left(a_R^{+,+} + a_R^{+,-}\right)(x, y) \chi_{\mathbb{R} \times [-c, 0]}(x, y) dx &= 0 = \int_{\mathbb{R}} \left(a_R^{+,+} + a_R^{+,-}\right)(x, y) \chi_{\mathbb{R} \times [0, c]}(x, y) dy, \\ \int_{\mathbb{R}} \left(a_R^{-,+} + a_R^{-,-}\right)(x, y) \chi_{\mathbb{R} \times [-c, 0]}(x, y) dx &= 0 = \int_{\mathbb{R}} \left(a_R^{-,+} + a_R^{-,-}\right)(x, y) \chi_{\mathbb{R} \times [0, c]}(x, y) dy.\end{aligned}$$

The known result for Case 1 implies that $\mathcal{H}\left(\left(a_R^{+,+}+a_R^{+,-}\right)\chi_{\mathbb{R}\times[-c,0]}\right)$, $\mathcal{H}\left(\left(a_R^{+,+}+a_R^{+,-}\right)\chi_{\mathbb{R}\times[0,c]}\right)$, $\mathcal{H}\left(\left(a_R^{-,+}+a_R^{-,-}\right)\chi_{\mathbb{R}\times[-c,0]}\right)$, $\mathcal{H}\left(\left(a_R^{-,+}+a_R^{-,-}\right)\chi_{\mathbb{R}\times[0,c]}\right)$ satisfy (2-i) and (2-ii). Thus, they are all rectangular $(1, 2, 0)$ -atoms.

In Case 3, without loss of generality, we assume $I = (-\alpha, \alpha)$ and $J = (c, d) \subset (0, \infty)$. Then both $a_R^{+,+} + a_R^{-,+}$ and $a_R^{+,-} + a_R^{-,-}$ are even in x . Similar to the discussion in Case 3, we know that $\mathcal{H}\left(\left(a_R^{+,+}+a_R^{-,+}\right)\chi_{(-\alpha,0)\times\mathbb{R}}\right)$, $\mathcal{H}\left(\left(a_R^{+,+}+a_R^{-,+}\right)\chi_{(0,\alpha)\times\mathbb{R}}\right)$, $\mathcal{H}\left(\left(a_R^{+,-}+a_R^{-,-}\right)\chi_{(-\alpha,0)\times\mathbb{R}}\right)$, $\mathcal{H}\left(\left(a_R^{+,-}+a_R^{-,-}\right)\chi_{(0,\alpha)\times\mathbb{R}}\right)$ are rectangular $(1, 2, 0)$ -atoms.

In Case 4, without loss of generality, we assume $I = (-\alpha, \alpha)$ and $J = [-c, c]$. Then, we know that the function

$$\mathfrak{a}_R = a_R^{+,+} + a_R^{+,-} + a_R^{-,+} + a_R^{-,-}$$

is even on both x and y variables. With the same reason as that of Case 2, it is easy to see that all $\mathcal{H}\left(\mathfrak{a}_R\chi_{[-\alpha,0]\times[-c,0]}\right)$, $\mathcal{H}\left(\mathfrak{a}_R\chi_{[-\alpha,0]\times[0,c]}\right)$, $\mathcal{H}\left(\mathfrak{a}_R\chi_{[0,\alpha]\times[0,c]}\right)$, $\mathcal{H}\left(\mathfrak{a}_R\chi_{[0,\alpha]\times[-c,0]}\right)$ are rectangular $(1, 2, 0)$ -atoms.

We now write

$$\begin{aligned} a_R^{+,+} &= a_{R^{+,+}}, & a_R^{+,-} &= a_{R^{+,-}}, \\ a_R^{-,+} &= a_{R^{-,+}}, & a_R^{-,-} &= a_{R^{-,-}}, \end{aligned}$$

and denote

$$\begin{aligned} \widetilde{\mathfrak{R}}_1 &= \{R^{+,+}, R^{+,-}, R^{-,+}, R^{-,-} : \text{if } R^{+,+} \text{ is in Case 1}\}, \\ \widetilde{\mathfrak{R}}_2 &= \{R^{+,+}, R^{+,-}, R^{-,+}, R^{-,-} : \text{if } R^{+,+} \text{ is in Case 2}\}, \\ \widetilde{\mathfrak{R}}_3 &= \{R^{+,+}, R^{+,-}, R^{-,+}, R^{-,-} : \text{if } R^{+,+} \text{ is in Case 3}\}, \\ \widetilde{\mathfrak{R}}_4 &= \{R^{+,+}, R^{+,-}, R^{-,+}, R^{-,-} : \text{if } R^{+,+} \text{ is in Case 4}\}, \end{aligned}$$

and use the notation $\widetilde{\mathfrak{R}}_j \in m(\Omega^*)$, $j = 1, 2, 3, 4$ to mean that $R^{+,+}$ is in Case j , and

$$R^{+,+}, R^{+,-}, R^{-,+}, R^{-,-} \in m(\Omega^*).$$

By this notation and the above discussion, we have $\text{supp}(\mathcal{H}(B)) \subset \Omega^*$, and

$$\begin{aligned} &4\mathcal{H}(B)(x, y) \\ &= \sum_{\widetilde{\mathfrak{R}}_1 \in m(\Omega^*)} \{\mathcal{H}(a_{R^{+,+}}) + \mathcal{H}(a_{R^{+,-}}) + \mathcal{H}(a_{R^{-,+}}) + \mathcal{H}(a_{R^{-,-}})\} \\ &+ \sum_{\widetilde{\mathfrak{R}}_2 \in m(\Omega^*)} \{\mathcal{H}((a_{R^{+,+}} + a_{R^{+,-}})\chi_{\mathbb{R}\times[-c,0]}) + \mathcal{H}((a_{R^{+,+}} + a_{R^{+,-}})\chi_{\mathbb{R}\times(0,c]})\} \end{aligned}$$

$$\begin{aligned}
& + \mathcal{H}((a_{R^-,+} + a_{R^-, -}) \chi_{\mathbb{R} \times [-c, 0)}) + \mathcal{H}((a_{R^-,+} + a_{R^-, -}) \chi_{\mathbb{R} \times (0, c]}) \\
& + \sum_{\widetilde{\mathfrak{R}}_3 \in m(\Omega^*)} \left\{ \mathcal{H}((a_{R^-,+} + a_{R^{+,+}}) \chi_{[-\alpha, 0) \times \mathbb{R}}) + \mathcal{H}((a_{R^-,+} + a_{R^{+,+}}) \chi_{(0, \alpha) \times \mathbb{R}}) \right\} \\
& + \mathcal{H}((a_{R^-, -} + a_{R^{+, -}}) \chi_{[-\alpha, 0) \times \mathbb{R}}) + \mathcal{H}((a_{R^-, -} + a_{R^{+, -}}) \chi_{(0, \alpha) \times \mathbb{R}}) \\
& + \sum_{\widetilde{\mathfrak{R}}_4 \in m(\Omega^*)} \left\{ \mathcal{H}(\mathfrak{a}_R \chi_{[-a, 0) \times [-c, 0)}) + \mathcal{H}(\mathfrak{a}_R \chi_{[-\alpha, 0) \times (0, c]}) \right\} \\
& + \mathcal{H}(\mathfrak{a}_R \chi_{(0, \alpha) \times (0, c]}) + \mathcal{H}(\mathfrak{a}_R \chi_{(0, \alpha) \times [-c, 0)}) \}.
\end{aligned}$$

Finally, since \mathcal{H} is bounded on $L^2(\mathbb{R} \times \mathbb{R})$ ([17]), we easily obtain

$$\begin{aligned}
& \sum_{\widetilde{\mathfrak{R}}_1 \in m(\Omega^*)} \left\{ \|\mathcal{H}(a_{R^{+,+}})\|_{L^2}^2 + \|\mathcal{H}(a_{R^{+,-}})\|_{L^2}^2 + \|\mathcal{H}(a_{R^-,+})\|_{L^2}^2 + \|\mathcal{H}(a_{R^-, -})\|_{L^2}^2 \right\} \\
& + \sum_{\widetilde{\mathfrak{R}}_2 \in m(\Omega^*)} \left\{ \|\mathcal{H}((a_{R^{+,+}} + a_{R^{+,-}}) \chi_{\mathbb{R} \times [-c, 0)})\|_{L^2}^2 + \|\mathcal{H}((a_{R^{+,+}} + a_{R^{+,-}}) \chi_{\mathbb{R} \times (0, c]})\|_{L^2}^2 \right\} \\
& + \|\mathcal{H}((a_{R^-,+} + a_{R^-, -}) \chi_{\mathbb{R} \times [-c, 0)})\|_{L^2}^2 + \|\mathcal{H}((a_{R^-,+} + a_{R^-, -}) \chi_{\mathbb{R} \times (0, c]})\|_{L^2}^2 \\
& + \sum_{\widetilde{\mathfrak{R}}_3 \in m(\Omega^*)} \left\{ \|\mathcal{H}((a_{R^-,+} + a_{R^{+,+}}) \chi_{[-\alpha, 0) \times \mathbb{R}})\|_{L^2}^2 + \|\mathcal{H}((a_{R^-,+} + a_{R^{+,+}}) \chi_{(0, \alpha) \times \mathbb{R}})\|_{L^2}^2 \right\} \\
& + \|\mathcal{H}((a_{R^-, -} + a_{R^{+, -}}) \chi_{[-\alpha, 0) \times \mathbb{R}})\|_{L^2}^2 + \|\mathcal{H}((a_{R^-, -} + a_{R^{+, -}}) \chi_{(0, \alpha) \times \mathbb{R}})\|_{L^2}^2 \\
& + \sum_{\widetilde{\mathfrak{R}}_4 \in m(\Omega^*)} \left\{ \|\mathcal{H}(\mathfrak{a}_R \chi_{[-a, 0) \times [-c, 0)})\|_{L^2}^2 + \|\mathcal{H}(\mathfrak{a}_R \chi_{[-\alpha, 0) \times (0, c]})\|_{L^2}^2 \right\} \\
& + \|\mathcal{H}(\mathfrak{a}_R \chi_{(0, \alpha) \times (0, c]})\|_{L^2}^2 + \|\mathcal{H}(\mathfrak{a}_R \chi_{(0, \alpha) \times [-c, 0)})\|_{L^2}^2 \} \leq |\Omega^*|^{-1}.
\end{aligned}$$

Combining all estimates, we know that $\mathcal{H}(B)$ is a $(1, 0)$ -atom. This completes the proof.

3.3. Proof of Theorem 3.

First, we note

$$\begin{aligned}
\mathcal{H}_{(k)} f(x, y) &= \frac{1}{x^{k+1} y^{k+1}} \int_0^x \int_0^y t^k s^k f(t, s) dt ds \\
&= \int_0^1 \int_0^1 t^k s^k f(xt, ys) dt ds = \int_1^\infty \int_1^\infty \frac{1}{t^{k+2} s^{k+2}} f(x/t, y/s) dt ds \\
&= \int_1^\infty \frac{1}{s^{k+2}} \left(\int_0^1 t^k f(xt, y/s) dt \right) ds.
\end{aligned}$$

By the Minkowski integral inequality and a scaling argument, for any $f \in L^2$, we have

$$\begin{aligned} \|\mathcal{H}_{(k)}(f)\|_{L^2(\mathbb{R} \times \mathbb{R})} &\leq \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} \left\| f\left(\frac{\cdot}{s}, \frac{\cdot}{t}\right) \right\|_{L^2(\mathbb{R} \times \mathbb{R})} ds dt \\ &= \|f\|_{L^2(\mathbb{R} \times \mathbb{R})} \int_0^\infty \int_0^\infty \frac{\Phi(s, t)}{st} s^{1/2} t^{1/2} ds dt, \end{aligned}$$

where

$$\Phi(s, t) = \chi_{(1, \infty) \times (1, \infty)}(s, t) s^{-1-k} t^{-1-k}.$$

Hence, the operator $\mathcal{H}_{(k)}$ is bounded on L^2 . We first show that $\mathcal{H}_{(k)}$ is bounded from $H^p(\mathbb{R} \times \mathbb{R})$ to $L^p(\mathbb{R} \times \mathbb{R})$. To this end, we only need to check that $\mathcal{H}_{(k)}$ satisfies the condition in Theorem A.

Let a be a rectangular $(p, 2, k)$ -atom with support in $R = I \times J$, where $I = [\alpha, \beta]$ and $J = [c, d]$. To verify that $\mathcal{H}_{(k)}(a)$ satisfies the condition in Theorem A, it suffices to consider the following three cases:

- Case 1: $0 \notin I, 0 \notin J$;
- Case 2: $0 \notin I, 0 \in J$;
- Case 3: $0 \in I, 0 \in J$.

In Case 1, without loss of generality, we assume $I = [\alpha, \beta] \subset (0, \infty)$ and $J = [c, d] \subset (-\infty, 0)$. For any $(x, y) \notin I \times J$, as argument in the proof of Theorem 2, it is easy to see that

$$\mathcal{H}_{(k)}(a)(x, y) = \frac{1}{x^{k+1}y^{k+1}} \int_0^x \int_0^y t^k s^k a(t, s) dt ds = 0.$$

This indicates that $\mathcal{H}_{(k)}(a)$ is also supported in $R = I \times J$. Hence

$$\int_{(YR)^c} |\mathcal{H}_{(k)}(a)(x, y)|^p dx dy = 0.$$

In Case 2, without loss of generality, we assume $I = [\alpha, \beta] \subset (0, \infty)$ and $J = [-r, r]$. We write

$$\begin{aligned} \mathcal{H}_{(k)}(a)(x, y) &= \frac{1}{x^{k+1}} \int_0^x t^k \int_1^\infty s^{-k-2} a(t, y/s) dt ds \\ &= \sum_{j=0}^\infty 2^{-j\sigma} \frac{1}{x^{k+1}} \int_0^x t^k \int_{2^j}^{2^{j+1}} 2^{j\sigma} s^{-k-2} a(t, y/s) dt ds \\ &= \sum_{j=0}^\infty 2^{-j\sigma} A_j(x, y), \end{aligned}$$

where

$$\sigma = k - 1/p + 1$$

is a positive number. The support of A_j is in

$$R_j = I \times [-2^{j+1}r, 2^{j+1}r].$$

To check the size condition of A_j , we write

$$A_j(x, y) = \int_1^\infty t^{-k-2} \int_{2^j}^{2^{j+1}} 2^{j\sigma} s^{-k-2} a(x/t, y/s) dt ds.$$

By the Minkowski integral inequality and a scaling argument, we have

$$\begin{aligned} \|A_j\|_{L^2} &\leq \int_1^\infty t^{-k-2} \int_{2^j}^{2^{j+1}} 2^{j\sigma} s^{-k-2} \|a(\cdot/t, \cdot/s)\|_{L^2} dt ds \\ &\leq \|a\|_{L^2} \int_{2^j}^{2^{j+1}} s^{-k-2+1/2+\sigma} ds \leq 2^{-kj+\sigma j-j/2} |R|^{1/2-1/p}. \end{aligned}$$

Thus indicates that,

$$\|A_j\|_{L^2} \leq |R_j|^{1/2-1/p}.$$

Also, it is easy to check

$$\int_{\mathbb{R}} x^\alpha A_j(x, y) dx = 0 \text{ for all } y \in \mathbb{R}$$

and

$$\int_{\mathbb{R}} A_j(x, y) y^\alpha dy = 0 \text{ for all } x \in \mathbb{R}$$

for all $\alpha = 0, 1, 2, \dots, s$. Thus, $A_j(x, y)$ is a rectangular $(p, 2, k)$ -atom with support in R_j .

For any $\gamma \geq 2$, we write

$$\mathcal{H}_{(k)}(a)(x, y) = \sum_{2^{j+1} > \gamma} 2^{-j\sigma} A_j(x, y) + \sum_{2^{j+1} \leq \gamma} 2^{-j\sigma} A_j(x, y).$$

Note

$$\text{supp}(A_j) \cap (\gamma R)^C = \emptyset$$

if $2^{j+1} \leq \gamma$. We have

$$\begin{aligned} &\int_{(\gamma R)^C} |\mathcal{H}_{(k)}(a)(x, y)|^p dx dy \\ &\leq \sum_{2^{j+1} > \gamma} 2^{-j\sigma p} \int_{(\gamma R)^C} |A_j(x, y)|^p dx dy \end{aligned}$$

$$\leq \sum_{2^{j+1} > \gamma} 2^{-j\sigma p} \leq \gamma^{-\sigma p}.$$

In Case 3, without loss of generality, we assume $I = [-\rho, \rho]$ and $J = [-r, r]$. We write

$$\begin{aligned} \mathcal{H}_{(k)}(a)(x, y) &= \int_1^\infty s^{-k-2} \int_1^\infty t^{-k-2} a(x/s, y/t) dt ds \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty 2^{-i\sigma} 2^{-j\sigma} \int_{2^i}^{2^{i+1}} 2^{j\sigma} s^{-k-2} \int_{2^j}^{2^{j+1}} 2^{i\sigma} t^{-k-2} a(x/s, y/t) ds dt \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty 2^{-i\sigma} 2^{-j\sigma} A_{i,j}(x, y). \end{aligned}$$

It is easy to check that the support of $A_{i,j}$ is in

$$R_{i,j} = [-2^{i+1}\rho, 2^{i+1}\rho] \times [-2^{j+1}r, 2^{j+1}r].$$

Again, by the Minkowski integral inequality and a scaling argument, we have

$$\begin{aligned} \|A_{i,j}\|_{L^2} &\leq \int_{2^i}^{2^{i+1}} s^{-k-2+\sigma} \int_{2^j}^{2^{j+1}} t^{-k-2+\sigma} \|a(\cdot/t, \cdot/s)\|_{L^2} dt ds \\ &\leq \|a\|_{L^2} \int_{2^i}^{2^{i+1}} s^{-k-2+1/2+\sigma} ds \int_{2^j}^{2^{j+1}} t^{-k-2+1/2+\sigma} dt \leq |R_{i,j}|^{1/2-1/p}. \end{aligned}$$

Thus, it is easy to check that each $A_{i,j}$ is a rectangular $(p, 2, k)$ -atom. We now write

$$\begin{aligned} \mathcal{H}_{(k)}(a)(x, y) &= \sum_{2^{i+1} > \gamma} \sum_{2^{j+1} \leq \gamma} 2^{-i\sigma} 2^{-j\sigma} A_{i,j}(x, y) + \sum_{2^{i+1} \leq \gamma} \sum_{2^{j+1} \leq \gamma} 2^{-i\sigma} 2^{-j\sigma} A_{i,j}(x, y) \\ &+ \sum_{2^{i+1} \leq \gamma} \sum_{2^{j+1} > \gamma} 2^{-i\sigma} 2^{-j\sigma} A_{i,j}(x, y) + \sum_{2^{i+1} > \gamma} \sum_{2^{j+1} > \gamma} 2^{-i\sigma} 2^{-j\sigma} A_{i,j}(x, y). \end{aligned}$$

Using the same argument as that in Case 2, we have

$$\int_{(\gamma R)^C} |\mathcal{H}_{(k)}(a)(x, y)|^p dx dy \leq \gamma^{-\sigma p}.$$

Thus, by Theorem A, we know that $\mathcal{H}_{(k)}$ is bounded from $H^p(\mathbb{R} \times \mathbb{R})$ to $L^p(\mathbb{R} \times \mathbb{R})$.

Finally, by the Hilbert transform characterization of H^p , we know that for $f \in L^2 \cap H^p$,

$$\|\mathcal{H}_{(k)}(f)\|_{H^p(\mathbb{R} \times \mathbb{R})} \simeq \|\mathfrak{R}_{1,1}\mathcal{H}_{(k)}(f)\|_{L^p(\mathbb{R} \times \mathbb{R})} + \sum_{m=0}^2 \|\mathfrak{R}_m\mathcal{H}_{(k)}(f)\|_{L^p(\mathbb{R} \times \mathbb{R})}.$$

Following the same argument as in [13], we have

$$\mathfrak{R}_{1,1}\mathcal{H}_{(k)}(f) = \mathcal{H}_{(k)}(\mathfrak{R}_{1,1}f)$$

and

$$\mathfrak{R}_m\mathcal{H}_{(k)}(f) = \mathcal{H}_{(k)}(\mathfrak{R}_m f), \quad m = 0, 1, 2$$

(one also easily sees the above commutating relations for Hausdorff operators and Hilbert transforms, since the Hilbert transforms commute with the dilation).

Thus, for $k > 1/p - 1$,

$$\begin{aligned} \|\mathcal{H}_{(k)}(f)\|_{H^p(\mathbb{R} \times \mathbb{R})} &\leq \|\mathfrak{R}_{1,1}(f)\|_{H^p(\mathbb{R} \times \mathbb{R})} + \sum_{m=0}^2 \|\mathfrak{R}_m(f)\|_{H^p(\mathbb{R} \times \mathbb{R})} \\ &\leq \|f\|_{H^p(\mathbb{R} \times \mathbb{R})}. \end{aligned}$$

The last inequality is because that the Hilbert transforms are bounded on $H^p(\mathbb{R} \times \mathbb{R})$.

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