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Remarks on a Subspace of Morrey Spaces

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Abstract. Let p, λ be real numbers such that $1 < p < \infty$, and $0 < \lambda < 1$. Also let $L^{p,\lambda}(\mathbf{T})$ be Morrey spaces on the unit circle $\mathbf{T}$, and $L_0^{p,\lambda}(\mathbf{T})$ the closure of $C(\mathbf{T})$ in $L^{p,\lambda}(\mathbf{T})$. Zorko [7] gave the predual $Z^{q,\lambda}(\mathbf{T})$ $(1/p + 1/q = 1)$ of $L^{p,\lambda}(\mathbf{T})$. In this article, we show a property of $L_0^{p,\lambda}(\mathbf{T})$ and prove in detail that $L_0^{p,\lambda}(\mathbf{T})$ is the predual of $Z^{q,\lambda}(\mathbf{T})$, whose fact is stated in Adams-Xiao [1].

1. Introduction and Main results

Let p be in $1 < p < \infty$, q the conjugate exponent of p, and $0 < \lambda < 1$. Also let $L^p(\mathbf{T})$ be the usual L^p-space on the unit circle $\mathbf{T}$ with respect to the normalized Haar measure. The Morrey spaces $L^{p,\lambda}(\mathbf{T})$ are defined by

$$L^{p,\lambda}(\mathbf{T}) = \left\{ f \;\middle|\; \|f\|_{p,\lambda} = \sup_{\substack{I \subset \mathbf{T} = [-\pi,\pi) \\ I \neq \emptyset : \text{interval}}} \left(\frac{1}{|I|^\lambda} \int_I |f|^p dx \right)^{1/p} < \infty \right\},$$

and $L_0^{p,\lambda}(\mathbf{T})$ the closure of $C(\mathbf{T})$ in $L^{p,\lambda}(\mathbf{T})$, where $C(\mathbf{T})$ is the set of all continuous functions on $\mathbf{T}$. Then it is easy to see that $L^{p,\lambda}(\mathbf{T})$ is a Banach space (cf. Kufner [3], Torchinsky [6; p. 215]). Also $Z^{q,\lambda}(\mathbf{T})$ $(1/p + 1/q = 1)$ are defined by $\{f \mid \|f\|_{Z^{q,\lambda}} < \infty\}$, where

$$\|f\|_{Z^{q,\lambda}} = \inf \left\{ \sum_{k=1}^{\infty} |c_k| \;\middle|\; f(x) = \sum_{k=1}^{\infty} c_k a_k(x),\ c_k \in \mathbf{C},\ a_k(x) : (q,\lambda)\text{-block} \right\},$$

where $a_k(x)$ is called (q,λ)-block, if

(1) supp $a_k \subset I$

(2) $\|a_k\|_q \leq \frac{1}{|I|^{\lambda/p}}$, where $1/p + 1/q = 1$,

for some interval I. In particular, $a_k(x)$ is called (q,λ)-atom, if a_k satisfies $\int_I a_k(x)dx = 0$, which is called cancellation property. $Z^{q,\lambda}(\mathbf{T})$ is a Banach space with the norm $\|\cdot\|_{Z^{q,\lambda}}$. Zorko

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[7] introduced the space $Z^{q,\lambda}(\mathbf{T})$, and proved that $Z^{q,\lambda}(\mathbf{T})$ is the predual of $L^{p,\lambda}(\mathbf{T})$. Also she [7] defined $L_0^{p,\lambda}(\mathbf{T})$, and remarked some properties. Adams-Xiao [1] pointed out that $L_0^{p,\lambda}(\mathbf{T})$ is the predual of $Z^{q,\lambda}(\mathbf{T})$, but they did not give the reason why they insisted that the proof is akin to that of H^1-VMO in Stein [5] (cf. [6]). Like Adams-Xiao [1], we think that $L^{p,\lambda}(\mathbf{T})$, $Z^{q,\lambda}(\mathbf{T})$, $L_0^{p,\lambda}(\mathbf{T})$ are similar to $BMO(\mathbf{T})$, $H^1(\mathbf{T})$, $VMO(\mathbf{T})$, respectively.

In this article, we show some properties of $L_0^{p,\lambda}(\mathbf{T})$, which is similar to that of $VMO(\mathbf{T})$. Next we give a detailed proof of the fact that $L_0^{p,\lambda}(\mathbf{T})$ is the predual of $Z^{q,\lambda}(\mathbf{T})$, by the method of Coifman-Weiss [2]. We expect that our proofs in the case of $\mathbf{T}$ may be available to Euclidean case $\mathbf{R}^n$.

Our results are as follows:

THEOREM 1.1. *Let* $1 \le p < \infty$, *and* $0 < \lambda < 1$. *Also let* ϕ *be an infinitely differentiable function such that* supp $\phi \subset [-1,1]$, $\frac{1}{2\pi}\int_{-\pi}^{\pi} \phi(x)dx = 1$ *and* $\phi \ge 0$, *and let* $\phi_j(x) = j\phi(jx)$ $(j = 1, 2, \dots)$. *Then, the following properties are equivalent*:

(1) $f \in L_0^{p,\lambda}(\mathbf{T})$
(2) $f \in L^{p,\lambda}(\mathbf{T})$ *and* $\|\tau_y f - f\|_{p,\lambda} \to 0$ $(y \to 0)$,
where $\tau_y f(x) = f(x-y)$
(3) $f \in L^{p,\lambda}(\mathbf{T})$ *and* $\|f - f * \phi_j\|_{p,\lambda} \to 0$ $(j \to \infty)$
(4) $\lim_{\delta\to 0} \sup_{|I|\le\delta, I\subset\mathbf{T}:\text{interval}} \frac{1}{|I|^\lambda}\int_I |f(x)|^p dx = 0$

THEOREM 1.2. *Let* $1 < p < \infty$, *and* $0 < \lambda < 1$. *Then* $L_0^{p,\lambda}(\mathbf{T})$ *is the predual of* $Z^{q,\lambda}(\mathbf{T})$, *where* $1/p + 1/q = 1$.

Throughout this paper, the dual space of a Banach space X is denoted by X^*. For an interval I, $|I|$ denotes the measure of I with respect to the normalized Haar measure of $\mathbf{T}$. Also the letter C stands for a constant not necessarily the same at each occurrence. $A \sim B$ stands for $C^{-1}A \le B \le CA$ for some $C > 0$.

2. Proofs of Main Theorems

2.1. Proof of Theorem 1.1

PROOF. According to Zorko [7], it is easy to prove that (1), (2) and (3) are equivalent. So, we omit their proofs. We show (4), when we assume (1). By the definition, for $f \in L_0^{p,\lambda}(\mathbf{T})$ and for any $\eta > 0$ there exists $g \in C(\mathbf{T})$ such that $\|f - g\|_{p,\lambda} < \eta$. Then for an interval $I \subset \mathbf{T}$ with $|I| \le \delta$, we have

$$\left(\frac{1}{|I|^\lambda}\int_I |f(x)|^p dx\right)^{1/p} \le \left(\frac{1}{|I|^\lambda}\int_I |f(x)-g(x)|^p dx\right)^{1/p} + \left(\frac{1}{|I|^\lambda}\int_I |g(x)|^p dx\right)^{1/p}$$

$$
\begin{aligned}
&\le \eta + \left(\frac{1}{|I|^\lambda} \int_I |g(x)|^p dx \right)^{1/p} \\
&\le \eta + |I|^{\frac{1-\lambda}{p}} \|g\|_{C(\mathbf{T})} \\
&\le \eta + \delta^{\frac{1-\lambda}{p}} \|g\|_{C(\mathbf{T})} ,
\end{aligned}
$$

and

$$
\lim_{\delta \to 0} \sup_{|I| \le \delta, I:\text{interval}} \frac{1}{|I|^\lambda} \int_I |f(x)|^p dx \le \eta^p .
$$

So we obtain (4). Next we show (3), when we assume (4). For any $\eta > 0$, there exists $\delta_0 > 0$ such that

$$
\sup_{|I| \le \delta_0, I:\text{interval}} \frac{1}{|I|^\lambda} \int_I |f(x)|^p dx < \eta^p .
$$

Then for $|I| \le \delta_0$, we have

$$
\begin{aligned}
\frac{1}{|I|^\lambda} \int_I |f * \phi_j(x)|^p dx &\le \frac{1}{|I|^\lambda} \int_I \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x-y)|^p \phi_j(y) dy \right) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi_j(y) \frac{1}{|I|^\lambda} \int_I |f(x-y)|^p dx dy \\
&\le \frac{1}{|I|^\lambda} \int_I |f(x)|^p dx \\
&< \eta^p
\end{aligned}
$$

by the Hölder inequality. Hence, for an interval $I \subset \mathbf{T}$ with $|I| \le \delta_0$, we have

$$
\begin{aligned}
&\left(\frac{1}{|I|^\lambda} \int_I |f(x) - f * \phi_j(x)|^p dx \right)^{1/p} \\
&\le \left(\frac{1}{|I|^\lambda} \int_I |f(x)|^p dx \right)^{1/p} + \left(\frac{1}{|I|^\lambda} \int_I |f * \phi_j(x)|^p dx \right)^{1/p} \\
&\le 2 \left(\sup_{|I| \le \delta_0, I:\text{interval}} \frac{1}{|I|^\lambda} \int_I |f(x)|^p dx \right)^{1/p} \\
&< 2\eta .
\end{aligned}
$$

On the other hand, for an interval $I \subset \mathbf{T}$ with $|I| > \delta_0$, we have

$$
\begin{aligned}
\frac{1}{|I|^\lambda} \int_I |f(x) - f * \phi_j(x)|^p dx &\le \frac{2\pi}{\delta_0^\lambda} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x) - f * \phi_j(x)|^p dx \\
&= \frac{2\pi}{\delta_0^\lambda} \|f - f * \phi_j\|_p^p .
\end{aligned}
$$

After all, we obtain

$$\sup_{I\subset\mathbf{T}:\text{interval}} \frac{1}{|I|^{\lambda}} \int_I |f(x) - f*\phi_j(x)|^p dx < (2\eta)^p + \frac{2\pi}{\delta_0^{\lambda}} \|f - f*\phi_j\|_p^p .$$

Therefore, we have

$$\lim_{j\to\infty} \|f - f*\phi_j\|_{p,\lambda} = 0 .$$

□

REMARK 2.1. Let f be in $Z^{q,\lambda}(\mathbf{T})$ such that $f = \sum_{k=1}^{\infty} c_k a_k$, where $\sum_k |c_k| < \infty$, a_k:(q,λ)-block. Then $f = \sum_k c_k a_k$ converges in $L^1(\mathbf{T})$ by the definition of $Z^{q,\lambda}(\mathbf{T})$ and Hölder's inequality.

2.2. Proof of Theorem 1.2. For the proof, we give some lemmas.

LEMMA 2.2 (Zorko [7]). *Let* $1 < p < \infty, 0 < \lambda < 1$ *and q the conjugate exponent of p. Then the dual space of* $Z^{q,\lambda}(\mathbf{T})$ *is* $L^{p,\lambda}(\mathbf{T})$.

LEMMA 2.3. *Let* $1 < p < \infty$ *and q be the conjugate exponent. Also let* $0 < \lambda < 1$. *Then every* $f \in Z^{q,\lambda}(\mathbf{T})$ *can be decomposed into a sum of block and atoms*:

$$f = c_0 a_0 + \sum_{k=1}^{\infty} c_k a_k ,$$

where $c_k \in \mathbf{C}$ *and* $|c_0| + \sum_{k=1}^{\infty} |c_k| \le C\|f\|_{Z^{q,\lambda}}$, a_0 *is a* (q,λ)*-block with* supp $a_0 \subset \mathbf{T}$, a_k'*s are* (q,λ)*-atoms such that* supp $a_k \subset I_k$ *satisfying* $|I_k| \le \frac{1}{4}$.

PROOF. Let $\mathbf{T} = [0, 2\pi)$, and $f \in Z^{q,\lambda}(\mathbf{T})$. Then, f is decomposed so that

$$f = \sum_{k=0}^{\infty} c_k' b_k ,$$

where $c_k' \in \mathbf{C}$, $\sum |c_k'| \le 2\|f\|_{Z^{q,\lambda}}$, and $\{b_k\}_{k=0}^{\infty}$ are (q,λ)-blocks. Let $b(x)$ be $b_k(x)$ for any $k \ge 0$, and A a set of functions defined by

$$A := \left\{ b_k \;\middle|\; \text{supp } b_k \subset I,\ \|b_k\|_q \le \frac{1}{|I|^{\lambda/p}},\ \text{and } |I| > \frac{1}{4} \right\}.$$

In the case of $|I| \le \frac{1}{4}$, we define b_1^1, b_2^1, I_1 by

$$b_1^1(x) = \frac{b(x) - b(x - |I|)}{2^{\frac{\lambda-1}{p}+1}} ,$$

$$b_2^1(x) = \frac{b(x) + b(x - |I|)}{2^{\frac{\lambda-1}{p}+1}} ,$$

$$I_1 = I \cup (I + |I|)\,.$$

Then, we have supp $b_j^1 \subset I_1$ $(j = 1, 2)$ and

$$\begin{aligned}
\left(\int_{I_1} |b_j^1(x)|^q dx\right)^{1/q} &= \left(2\int_I |b(x)|^q dx\right)^{1/q} 2^{-\frac{\lambda-1}{p}-1} \\
&\le 2^{\frac{1}{q}-\frac{\lambda-1}{p}-1}\frac{1}{|I|^{\lambda/p}} \\
&= 2^{-\lambda/p}\frac{1}{|I|^{\lambda/p}} = \frac{1}{|I_1|^{\lambda/p}} \quad (j = 1, 2)\,,
\end{aligned}$$

which shows that b_j^1 is a (q, λ)-block $(j = 1, 2)$. We also have

$$\int_0^{2\pi} b_1^1(x)\, dx = 0\,,$$

$$2^{\frac{\lambda-1}{p}} b_1^1(x) + 2^{\frac{\lambda-1}{p}} b_2^1(x) = \frac{b(x) - b(x - |I|)}{2} + \frac{b(x) + b(x - |I|)}{2} = b(x)\,.$$

So, b_1^1 is a (q, λ)-atom. When we set $\alpha = 2^{\frac{\lambda-1}{p}}$ and $a_k^1(x) = b_1^1(x)$, we have $b_k(x) = \alpha a_k^1(x) + \alpha b_2^1(x)$. Next, if we have $|I_1| \le \frac{1}{4}$, there exists a natural number $\ell \ge 3$ such that $\frac{1}{2^\ell} < |I_1| \le \frac{1}{2^{\ell-1}}$. So, we decompose $b_2^1(x)$ like $b(x)$ and define a_k^2, b_2^2, I_2 by

$$\begin{aligned}
a_k^2(x) &= \frac{b_2^1(x) - b_2^1(x - |I_1|)}{2^{\frac{\lambda-1}{p}+1}}\,, \\
b_2^2(x) &= \frac{b_2^1(x) + b_2^1(x - |I_1|)}{2^{\frac{\lambda-1}{p}+1}}\,, \\
I_2 &= I_1 \cup (I_1 + |I_1|)\,.
\end{aligned}$$

Then we have

$$\begin{aligned}
\int_0^{2\pi} a_k^2(x) dx &= 0\,, \\
b_2^1(x) &= \alpha a_k^2(x) + \alpha b_2^2(x)\,, \\
b_k(x) &= \alpha a_k^1(x) + \alpha b_2^1(x) \\
&= \alpha a_k^1(x) + \alpha^2 a_k^2(x) + \alpha^2 b_2^2(x)\,,
\end{aligned}$$

and hence, we see that a_k^1, a_k^2 are (q, λ)-atoms and b_2^2 is a (q, λ)-block. In fact,

$$\left(\int_{I_2} |b_2^2(x)|^q dx\right)^{1/q} \le 2^{-\lambda/p}|I_1|^{-\lambda/p} = |I_2|^{-\lambda/p}\,.$$

We repeat this process ℓ times until we have $|I_\ell| > \frac{1}{4}$. After all, we get

$$b_k(x) = \sum_{j=1}^{\ell} \alpha^j a_k^j(x) + \alpha^\ell b_2^\ell(x)\,,$$

where $\alpha = 2^{\frac{\lambda-1}{p}}$, a_k^j $(j = 1, \dots, \ell) : (q, \lambda)$-atoms with supp $a_k^j \subset I_j$, and $b_2^\ell : (q, \lambda)$-block with supp $b_k^\ell \subset I_\ell$. When we set $\ell_k = \ell$, we have

$$b_k(x) = \sum_{j=1}^{\ell_k} \alpha^j a_k^j(x) + \alpha^{\ell_k} b_2^{\ell_k}(x)\,.$$

After we repeat this process for b_k, we obtain

$$f(x) = \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} c_k' \alpha^\ell a_k^\ell(x) + \sum_{b_k \notin A} c_k' \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c_k' b_k(x)\,.$$

Noting $0 < \alpha < 1$, we have

$$\sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} |c_k'| \alpha^\ell + \sum_{b_k \notin A} |c_k'| \alpha^{\ell_k} + \sum_{b_k \in A} |c_k'| \le \left(\frac{1}{1-\alpha} + \alpha + 1 \right) \sum_{k=0}^{\infty} |c_k'|\,.$$

Also when we define

$$a_0(x) = \frac{\sum_{b_k \notin A} c_k' \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c_k' b_k(x)}{4^{\lambda/p} \left(\sum_{b_k \notin A} |c_k'| \alpha^{\ell_k} + \sum_{b_k \in A} |c_k'| \right)}\,,$$

we have that $\|a_0\|_q \le 1$, supp $a_0 \subset \mathbf{T} = [0, 2\pi)$ and $a_0 : (q, \lambda)$-block, since

$$\left(\frac{1}{2\pi} \int_0^{2\pi} \left| \sum_{b_k \notin A} c_k' \alpha^{\ell_k} b_2^{\ell_k}(x) + \sum_{b_k \in A} c_k' b_k(x) \right|^q dx \right)^{1/q} \le 4^{\lambda/p} \left(\sum_{b_k \notin A} |c_k'| \alpha^{\ell_k} + \sum_{b_k \in A} |c_k'| \right).$$

Moreover, we obtain

$$f(x) = 4^{\lambda/p} \left(\sum_{b_k \notin A} |c_k'| \alpha^{\ell_k} + \sum_{b_k \in A} |c_k'| \right) a_0(x) + \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} c_k' \alpha^\ell a_k^\ell(x)$$

and

$$4^{\lambda/p} \left(\sum_{b_k \notin A} |c_k'| \alpha^{\ell_k} + \sum_{b_k \in A} |c_k'| \right) + \sum_{b_k \notin A} \sum_{\ell=1}^{\ell_k} |c_k'| \alpha^\ell \le 2 \left(4^{\lambda/p} + \frac{1}{1-\alpha} \right) \|f\|_{Z^{q,\lambda}}\,.$$

$\square$

LEMMA 2.4. *Let n be any positive integer, $B_j^n = [\frac{j-1}{3^n}2\pi, \frac{j}{3^n}2\pi)$ $(j = 1, \dots, 3^n)$, and $\tilde{B}_j^n = 3B_j^n$, where the center of $\tilde{B}_j^n$ is the same as the center of B_j^n, and $|\tilde{B}_j^n| = 3|B_j^n|$. Also let $B^0 = B_1^0 = [0, 2\pi)$, and $\tilde{B}^0 = \tilde{B}_1^0 = [0, 2\pi)$. Then, $f \in Z^{q,\lambda}(\mathbf{T})$ has the representation*

$$f(x) = \lambda_0 a_0(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x)\,,$$

where a_0 : (q, λ)-block, a_j^n : (q, λ)-atoms, supp $a_0 \subset \mathbf{T}$, supp $a_j^n \subset \tilde{B}_j^n$, *and* $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq C\|f\|_{Z^{q,\lambda}}$.

PROOF. By Lemma 2.3, $f \in Z^{q,\lambda}(\mathbf{T})$ can be decomposed into a sum of block and atoms:

$$f = c_0 b_0 + \sum_{k=1}^{\infty} c_k b_k\,,$$

where $c_k \in \mathbf{C}$, $|c_0| + \sum_{k=1}^{\infty} |c_k| \leq C\|f\|_{Z^{q,\lambda}}$, and b_0 is a (q, λ)-block with supp $b_0 \subset \mathbf{T}$, and b_k's are (q, λ)-atoms such that supp $b_k \subset I_k$ satisfying $|I_k| \leq \frac{1}{4}$. For I_k with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$, there exists $j \in \{1, 2, 3\}$ such that $I_k \cap B_j^1 \neq \emptyset$. For B_1^1 we let Λ_1^1 be the index set $k \in \mathbf{N}$, determined by those b_k with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$ and $I_k \cap B_1^1 \neq \emptyset$. Then, we see that $I_k \subset \tilde{B}_1^1$ for $k \in \Lambda_1^1$ and

$$\Big\| \sum_{k\in\Lambda_1^1} c_k b_k \Big\|_q \leq \sum_{k\in\Lambda_1^1} |c_k|\ \|b_k\|_q \leq \sum_{k\in\Lambda_1^1} |c_k|\ |\tilde{B}_1^1|^{-\lambda/p} 3^{2\lambda/p}\,.$$

So, when we define

$$a_1^1 = \frac{\sum_{k\in\Lambda_1^1} c_k b_k}{3^{2\lambda/p} \sum_{k\in\Lambda_1^1} |c_k|} \text{ and } \lambda_1^1 = \sum_{k\in\Lambda_1^1} |c_k| 3^{2\lambda/p}\,,$$

we have supp $a_1^1 \subset \tilde{B}_1^1$, $\|a_1^1\|_q \leq \frac{1}{|\tilde{B}_1^1|^{\lambda/p}}$, and a_1^1 satisfies the cancellation property, that is, a_1^1 is a (q, λ)-atom supported by $\tilde{B}_1^1$, and

$$\lambda_1^1 a_1^1 = \sum_{k\in\Lambda_1^1} c_k b_k\,.$$

Next for B_2^1 we let Λ_2^1 be the index set determined by b_k in $\{b_j\}$ with $\frac{1}{3^2} < |I_k| \leq \frac{1}{3}$ and $I_k \cap B_2^1 \neq \emptyset$, excluding b_k which we have already chosen before. We construct (q, λ)-atom a_2^1 in the same way as for B_1^1. Similarly we construct (q, λ)-atom a_3^1 for B_3^1. We do this process for b_k with $\frac{1}{3^3} < |I_k| \leq \frac{1}{3^2}$, and obtain the index set Λ_j^2, (q,λ)-atoms a_j^2 with supp $a_j^2 \subset \tilde{B}_j^2$, and numbers λ_j^2 $(j = 1, \dots, 3^2)$, satisfying

$$\lambda_j^2 a_j^2 = \sum_{k \in \Lambda_j^2} c_k b_k \,.$$

After that, we repeat this process. In the n-th step, for b_k with $\frac{1}{3^{n+1}} < |I_k| \leq \frac{1}{3^n}$ we obtain the index set Λ_j^n, (q,λ)-atoms a_j^n with supp $a_j^n \subset \tilde{B}_j^n$, and numbers λ_j^n $(j = 1, \dots, 3^n)$, satisfying

$$\lambda_j^n a_j^n = \sum_{k \in \Lambda_j^n} c_k b_k \,.$$

By the construction of a_j^n and λ_j^n, we have

$$f(x) = \lambda_0 a_0(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x),$$

where $a_0 = b_0 : (q,\lambda)$-block, $\lambda_0 = c_0$, $a_j^n : (q,\lambda)$-atoms, supp $a_0 \subset \mathbf{T}$, supp $a_j^n \subset \tilde{B}_j^n$, and $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq 2 \cdot 3^{2\lambda/p} \|f\|_{Z^{q,\lambda}}$. □

LEMMA 2.5. *Suppose* $\|f_k\|_{Z^{q,\lambda}} \leq 1, k = 1, 2, \dots$. *Then there exist* $f \in Z^{q,\lambda}(\mathbf{T})$ *and a subsequence* $\{f_{k_j}\}$ *such that*

$$\lim_{j\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f_{k_j}(x) v(x) dx = \frac{1}{2\pi} \int_0^{2\pi} f(x) v(x) dx$$

for all $v \in C(\mathbf{T})$.

PROOF. By Lemma 2.4, we may assume that $f_k \in Z^{q,\lambda}(\mathbf{T})$ has the representation

$$f_k(x) = \lambda_0(k) a_0(k)(x) + \sum_{n=1}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n(k) a_j^n(k)(x) \,,$$

where $a_0(k) : (q,\lambda)$-block, $a_j^n(k) : (q,\lambda)$-atoms, supp $a_0(k) \subset \mathbf{T}$, supp $a_j^n(k) \subset \tilde{B}_j^n$, and $|\lambda_0(k)| + \sum_{j,n} |\lambda_j^n(k)| \leq C$. Also we may assume that $\lambda_0(k)$, $\lambda_j^n(k) \geq 0$, $\|a_j^n(k)\|_q \leq |\tilde{B}_j^n|^{-\lambda/p}$, and that there exist λ_0, λ_j^n such that $\lim_{k\to\infty} \lambda_0(k) = \lambda_0$, $\lim_{k\to\infty} \lambda_j^n(k) = \lambda_j^n$ $(j, n \geq 1)$, and $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq C$. Let $L^q(\tilde{B}_j^n) = (L^p(\tilde{B}_j^n))^*$ be the dual space of $L^p(\tilde{B}_j^n)$ (L^p-space on $\tilde{B}_j^n$). By $a_j^n(k) \in L^q(\tilde{B}_j^n)$ and the diagonal argument, there exists an increasing sequence of natural numbers, $k_1 < k_2 < \cdots < k_n < \cdots$ and $a_0 \in L^q(\tilde{B}^0)$,

$a_j^n \in L^q(\tilde{B}_j^n)$ such that for $\phi \in L^p(\mathbf{T})$

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)\phi(x)dx = \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)\phi(x)dx$$

and

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_0^{2\pi} a_0(k_\ell)(x)\phi(x)dx = \frac{1}{2\pi} \int_0^{2\pi} a_0(x)\phi(x)dx \,,$$

that is, $a_j^n(k_\ell) \to a_j^n$ $(\ell \to \infty)$ in the weak*-topology of $\sigma(L^q(\tilde{B}_j^n), L^p(\tilde{B}_j^n))$ $(j, n \geq 1)$ and $a_0(k_\ell) \to a_0$ $(\ell \to \infty)$ in the weak*-topology of $\sigma(L^q(\tilde{B}^0), L^p(\tilde{B}^0))$. Here, we define f by

$$f(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{3^n} \lambda_j^n a_j^n(x) \,,$$

where $a_1^0 = a_0$ and $\lambda_1^0 = \lambda_0$. Then f is in $Z^{q,\lambda}(\mathbf{T})$ and a_j^n are (q, λ)-atoms, since supp $a_j^n \subset \tilde{B}_j^n$, $\|a_j^n\|_q \leq |\tilde{B}_j^n|^{-\lambda/p}$, $|\lambda_0| + \sum_{j,n} |\lambda_j^n| \leq C$, and $\int_{\tilde{B}_j^n} a_j^n(x)dx = 0$. Let $v \in C(\mathbf{T})$, and $a_1^0(k_\ell) = a_0(k_\ell)$, $\lambda_1^0(k_\ell) = \lambda_0(k_\ell)$. We define

$$J_{k_\ell} = \frac{1}{2\pi} \int_0^{2\pi} f_{k_\ell}(x)v(x)dx = \sum_{n=0}^{\infty} \sum_j \lambda_j^n(k_\ell) \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx \,,$$

and

$$J = \frac{1}{2\pi} \int_0^{2\pi} f(x)v(x)dx = \sum_{n=0}^{\infty} \sum_j \lambda_j^n \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)v(x)dx \,.$$

Also, for any integer N we define

$$J_{k_\ell}^N = \sum_{n=0}^{N} \sum_j \lambda_j^n(k_\ell) \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx \,,$$

$$J_{k_\ell}^{N,\infty} = \sum_{n=N+1}^{\infty} \sum_j \lambda_j^n(k_\ell) \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx \,,$$

$$J^N = \sum_{n=0}^{N} \sum_j \lambda_j^n \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)v(x)dx \,,$$

and

$$J^{N,\infty} = \sum_{n=N+1}^{\infty} \sum_j \lambda_j^n \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)v(x)dx \,.$$

Moreover, when the center of $\tilde{B}_j^n$ $(j, n \geq 1)$ is denoted by x_j^n, we have

$$J_{k_\ell}^{N,\infty} = \sum_{n=N+1}^{\infty} \sum_j \lambda_j^n(k_\ell) \frac{1}{2\pi} \int_{\tilde{B}_j^n} a_j^n(k_\ell)(x)(v(x) - v(x_j^n))dx ,$$

since $a_j^n(k)$ $(j, n \geq 1)$ are (q, λ)-atoms. Here, we remark that v is uniformly continuous on $\mathbf{T}$. Hence, for any $\varepsilon > 0$ there exists N_0 such that

$$|J_{k_\ell}^{N_0,\infty}| \leq \varepsilon \sum_{n=N_0+1}^{\infty} \sum_j \lambda_j^n(k_\ell) |\tilde{B}_j^n|^{\frac{1-\lambda}{p}} \leq C\varepsilon .$$

The same conclusion can be drawn for $J^{N_0,\infty}$, since a_j^n are (q, λ)-atoms. Also we have

$$\begin{aligned}
&\left|\sum_{n=0}^{N_0} \sum_{j=1}^{3^n} \left(\lambda_j^n(k_\ell) \frac{1}{2\pi} \int_0^{2\pi} a_j^n(k_\ell)(x)v(x)dx - \lambda_j^n \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)v(x)dx \right)\right| \\
&\leq \sum_{n=0}^{N_0} \sum_{j=1}^{3^n} \left\{ \lambda_j^n(k_\ell) \left| \frac{1}{2\pi} \int_0^{2\pi} (a_j^n(k_\ell)(x) - a_j^n(x))v(x)dx \right| \right. \\
&\quad \left. + |\lambda_j^n(k_\ell) - \lambda_j^n| \left| \frac{1}{2\pi} \int_0^{2\pi} a_j^n(x)v(x)dx \right| \right\} \\
&\quad \to 0 ,
\end{aligned}$$

as $\ell \to \infty$. Moreover, we obtain

$$\begin{aligned}
J_{k_\ell} - J &= (J_{k_\ell}^{N_0} - J^{N_0}) + (J_{k_\ell}^{N_0,\infty} - J^{N_0,\infty}) , \\
|J_{k_\ell}^{N_0,\infty} - J^{N_0,\infty}| &\leq |J_{k_\ell}^{N_0,\infty}| + |J^{N_0,\infty}| \\
&\leq 2C\varepsilon .
\end{aligned}$$

Hence, we have $\limsup_{\ell\to\infty} |J_{k_\ell} - J| \leq 2C\varepsilon$, and $\lim_{\ell\to\infty} J_{k_\ell} = J$. Therefore, we get the result:

$$\lim_{\ell\to\infty} \frac{1}{2\pi} \int_0^{2\pi} f_{k_\ell}(x)v(x)dx = \frac{1}{2\pi} \int_0^{2\pi} f(x)v(x)dx \ (v \in C(\mathbf{T})) .$$

□

LEMMA 2.6. *Let f be in $Z^{q,\lambda}(\mathbf{T})$. Then we have*

$$\|f\|_{Z^{q,\lambda}} \sim \|f\|_{(L_0^{p,\lambda})^*} .$$

PROOF. Let $A = \|f\|_{Z^{q,\lambda}} > 0$. Then there exists $g \in L^{p,\lambda}(\mathbf{T})$ such that

$$\left| \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx \right| \geq \frac{A}{2}, \ \|g\|_{p,\lambda} \leq 1 .$$

By $f \in Z^{q,\lambda}(\mathbf{T})$, we may assume that

$$f(x) = \sum_{k=0}^{\infty} c_k a_k(x)\,,$$

where a_k : (q, λ)-block, supp $a_k \subset B_k$ for some interval B_k, and $\sum_{k=0}^{\infty} |c_k| \le 2\|f\|_{Z^{q,\lambda}}$. Also for any $\varepsilon > 0$ let $\phi_\varepsilon(x) = \frac{1}{|I_\varepsilon|}\chi_{I_\varepsilon}(x)$, where $I_\varepsilon = [-\varepsilon, \varepsilon]$ and χ_E denotes the characteristic function of E. When we define $g_\varepsilon(x) = g * \phi_\varepsilon(x)$ for $g \in L^{p,\lambda}(\mathbf{T})$, it is easy to see $g_\varepsilon \in C(\mathbf{T})$ and $\|g_\varepsilon\|_{p,\lambda} \le \|g\|_{p,\lambda}$. Now for any integer $N \ge 1$ and $g \in L^{p,\lambda}(\mathbf{T})$, we define

$$I_\varepsilon^N = \sum_{k=0}^{N} c_k \frac{1}{2\pi}\int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx\,,$$

and

$$II_\varepsilon^N = \sum_{k=N+1}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx\,.$$

Then, we have

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(x)(g(x) - g_\varepsilon(x))dx &= \sum_{k=0}^{\infty} c_k \frac{1}{2\pi}\int_0^{2\pi} a_k(x)(g(x) - g_\varepsilon(x))dx \\ &= I_\varepsilon^N + II_\varepsilon^N\,.\end{aligned}$$

By $\|g_\varepsilon\|_{p,\lambda} \le \|g\|_{p,\lambda}$, we obtain

$$\begin{aligned}|II_\varepsilon^N| &\le \sum_{k=N+1}^{\infty} |c_k|\ \|a_k\|_{Z^{q,\lambda}}\|g - g_\varepsilon\|_{p,\lambda} \\ &\le 2\sum_{k=N+1}^{\infty} |c_k|\,.\end{aligned}$$

Also for any $\eta > 0$, there exists N_0 a positive integer such that $\sum_{k=N_0+1}^{\infty} |c_k| < \frac{\eta}{2}$. Hence, we have $|II_\varepsilon^{N_0}| < \eta$ for all $\varepsilon > 0$. Moreover, we have

$$\begin{aligned}|I_\varepsilon^{N_0}| &\le \sum_{k=0}^{N_0} |c_k|\ \|a_k\|_q\|g - g_\varepsilon\|_p \\ &= \sum_{k=0}^{N_0} |c_k|\ \|a_k\|_q\|g - g * \phi_\varepsilon\|_p \\ &\to 0\,,\end{aligned}$$

as $\varepsilon \to 0$. Therefore, we get

$$\limsup_{\varepsilon\to 0}\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)g_\varepsilon(x)dx - \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx\right| \le \eta\,,$$

and

$$\lim_{\varepsilon\to 0}\frac{1}{2\pi}\int_0^{2\pi} f(x)g_\varepsilon(x)dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx\,.$$

Hence, there exists $\varepsilon_0 > 0$ such that $|\frac{1}{2\pi}\int_0^{2\pi} f(x)g_{\varepsilon_0}(x)dx| \ge \frac{A}{3}$. So we obtain

$$\sup_{\|g\|_{p,\lambda}\le 1,\, g\in L_0^{p,\lambda}}\left|\frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx\right| \ge \frac{A}{3}\,.$$

Therefore, we have $\|f\|_{Z^{q,\lambda}} \le 3\|f\|_{(L_0^{p,\lambda})^*}$. Since the converse is trivial, we get the desired result. □

Now we are ready to prove Theorem 1.2.

PROOF OF THEOREM 1.2. First we have $Z^{q,\lambda}(\mathbf{T}) \subset (L_0^{p,\lambda}(\mathbf{T}))^*$ by Lemma 2.2. Since $\big(Z^{q,\lambda}(\mathbf{T})\big)^* = L^{p,\lambda}(\mathbf{T}) \supset L_0^{p,\lambda}(\mathbf{T})$, we see that the annihilator of $Z^{q,\lambda}(\mathbf{T})$ is $\{0\}$, and hence $Z^{q,\lambda}(\mathbf{T})$ is weak*-dense in $(L_0^{p,\lambda}(\mathbf{T}))^*$ (see Theorem 4.7 (b) in Rudin [4]). By the Banach-Alaoglu theorem and the separability of $L_0^{p,\lambda}(\mathbf{T})$ we see that the unit ball of $(L_0^{p,\lambda}(\mathbf{T}))^*$ is weak*-compact and metrizable (see Theorem 3.16 in Rudin [4]). Thus, if T is in $(L_0^{p,\lambda}(\mathbf{T}))^*$ with $\|T\|_{(L_0^{p,\lambda}(\mathbf{T}))^*} \le 1$, then there exists a sequence $\{f_k\} \subset Z^{q,\lambda}(\mathbf{T})$ with $\|f_k\|_{(L_0^{p,\lambda}(\mathbf{T}))^*} \le 1$ such that $f_k \to T$ in the weak*-topology of $(L_0^{p,\lambda}(\mathbf{T}))^*$. Here, we may assume $\|f_k\|_{Z^{q,\lambda}(\mathbf{T})} \le 1$ by Lemma 2.6. Hence, by Lemma 2.5, there exist $f \in Z^{q,\lambda}(\mathbf{T})$ and a subsequence $\{f_{k_j}\}$ $(k_1 < k_2 < \dots)$ such that $\|f_{k_j}\|_{Z^{q,\lambda}} \le 1$ and

$$\lim_{j\to\infty}\frac{1}{2\pi}\int_0^{2\pi} f_{k_j}(x)g(x)dx = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx$$

for all $g \in C(\mathbf{T})$. Hence, we have

$$\langle T, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx$$

for all $g \in C(\mathbf{T})$. Therefore we get the desired result. □

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