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Vertex Unfoldings of Tight Polyhedra

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Abstract. The unfolding of a polyhedron along its edges is known as a vertex unfolding if adjacent faces are allowed to be connected not only at an edge but also at a vertex. Demaine et al. [1] showed that every triangulated polyhedron has a vertex unfolding. We extend this result to a tight polyhedron, where a polyhedron is tight if its non-triangular faces are mutually non-incident.

1. Introduction

We investigate a procedure to cut open a polyhedron homeomorphic to the 2-sphere along its edges and unfold it to a connected flat piece without overlap. The unfolding needs to consist of the faces of the polyhedron joined along the edges. This type of unfolding has been referred to as an *edge unfolding* or simply an *unfolding*. It is known that some non-convex polyhedra have no edge unfoldings. However, no example of a convex polyhedron that has no edge unfolding is known. The determination of whether every convex polyhedron has an edge unfolding is a long-standing open problem. The difficulty of this question led to the exploration of other unfoldings that have a broader definition of edge unfolding. We pay attention to a *vertex unfolding* that permits two faces joined not only at an edge but also at a vertex, that is, the resulting piece may have a disconnected interior. See [2, §22] for details of edge unfolding and vertex unfolding.

In [1], Demaine et al. showed the following, where they proved conclusively that $\mathcal{P}$ does not need to be a spherical polyhedron, but may be a connected triangulated 2-manifold, possibly with boundaries.

THEOREM 1 (Demaine et al [1]). *Let $\mathcal{P}$ be a polyhedron. If $\mathcal{P}$ is triangulated, then it has a vertex unfolding.*

We broadly describe the proof of Theorem 1 here and describe it in detail in Sections 2-4. Their algorithm [1] first finds a spanning face path from triangle to triangle on the surface of the polyhedron, connecting through common vertices. Although the face path might "cross"

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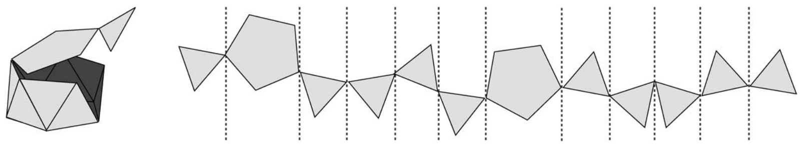

FIGURE 1. Vertex unfolding of the pentagonal antiprism

at some vertices, the algorithm converts it into a non-crossing one (see Section 3 for the definition of "cross"), and further, lays out the triangles along a line without overlap.

Their method is based on the condition that all faces are triangular, and the existences of the face path and the line-layout of it might actually fail for a polyhedron with non-triangular faces. For example, the truncated cube has no face path since its six octagons are inadequate to lay out eight triangles along a line, and if a face path consists of isosceles trapezoids, a local overlap might occur in a long strip.

In this paper, we fix these problems and make progress on Theorem 1 for a polyhedron with non-triangular faces. A (possibly non-convex) polyhedron $\mathcal{P}$ is *tight* if no two non-triangular faces share a vertex. Here, a non-triangular face needs not necessarily be a convex polygon. Examples of tight polyhedra are the snub cube, snub dodecahedron, pyramids, and antiprisms. The main theorem in this paper is as follows.

THEOREM 2. *Let $\mathcal{P}$ be a polyhedron. If $\mathcal{P}$ is tight, then it has a vertex unfolding.*

Figure 1 shows a vertex unfolding of the pentagonal antiprism. Our proof basically depend on the method in [1]. Our new result is a graph theoretical part of it, which is contained in Section 2.

2. Hamiltonian vertex-face tour

In this section, we observe tight polyhedra from a graph theoretical standpoint. We use standard terminology and notations of graph theory, for example, see [3]. By Steinitz's theorem, a surface of a polyhedron corresponds to a 3-connected plane graph. Thus, we also call a 3-connected plane graph *tight* if its non-triangular faces are mutually non-incident. We prepare some more definitions.

Let G be a tight graph. A disjoint union T of closed alternating sequences of vertices v_i and faces f_i of G is called a *spanning vertex-face tour* if each face of G appears exactly once in T and each closed component $(v_1, f_1, v_2, f_2, \dots, v_k, f_k, v_1)$ satisfies that v_i and v_{i+1} are distinct and both are incident to the face f_i for $i = 1, 2, \dots, k$ (indices are taken modulo k). Some vertex of G may be repeated in T; conversely, some vertex may not appear in T. If a spanning vertex-face tour T is connected, then T is called a *Hamiltonian vertex-face tour*.

Next, we define two operations on a spanning vertex-face tour T. Let $f = uvx$ and $f' = uvy$ be two adjacent triangular faces of G. We refer to the operation of replacing (u, f, x)

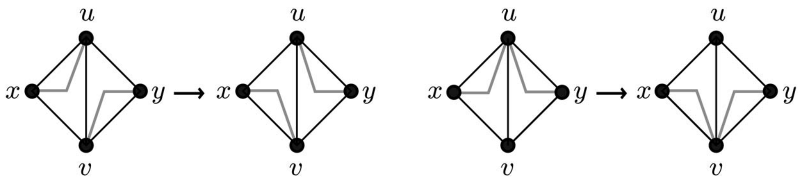

FIGURE 2. Switching operation (left) and reflecting operation (right)

and (v, f', y) with (v, f, x) and (u, f', y), respectively, as the *switching* operation, and the operation of replacing (u, f, x) and (u, f', y) with (v, f, x) and (v, f', y), respectively, as the *reflecting* operation (Figure 2). Note that simultaneous changing of combinations between vertices and faces at several triangular faces of T, such as in a switching operation or reflecting operation, may produce another spanning vertex-face tour T'. In general, we refer to such operations as *triangular recombinations*.

First, we prove the following lemma.

LEMMA 3. *Let G be a tight graph. Let $F^* = \{f_1, f_2, \ldots, f_m\}$ be the set of all non-triangular faces of G, and let v_i and v'_i be distinct vertices of f_i for $i = 1, 2, \ldots, m$. If G has a spanning vertex-face tour T, then G has a spanning vertex-face tour T' containing each (v_i, f_i, v'_i) for $i = 1, 2, \ldots, m$.*

PROOF. For simplicity, let $f = v_1 v_2 \cdots v_n$ denotes any non-triangular face of G. Let g_i be the triangular face adjacent to f by sharing $v_i v_{i+1}$ for $i = 1, 2, \ldots, n$ (indices are taken modulo n), and let u_i be the remaining vertex of g_i for $i = 1, 2, \ldots, n$.

We only have to show that if T contains (v_1, f, v_k) for some $2 \le k \le n-1$, then T can be converted to a spanning vertex-face tour T' containing (v_1, f, v_{k+1}) instead of (v_1, f, v_k) by performing only triangular recombinations.

Case 1: T contains (u_k, g_k, v_k) or (u_k, g_k, v_{k+1}).

In this case, we can obtain T' from T by replacing (v_1, f, v_k) with (v_1, f, v_{k+1}), and simultaneously by replacing (u_k, g_k, v_k) with (u_k, g_k, v_{k+1}) in the former case and (u_k, g_k, v_{k+1}) with (u_k, g_k, v_k) in the latter case.

Case 2: T contains (v_k, g_k, v_{k+1}).

In this case, we check the triangular faces incident to v_i from $i = k+1, k+2, \ldots, n-1, n, 1, 2, \ldots, k$ in turn. For $i = k+1, k+2, \ldots, n-1, n, 1, 2, \ldots, k$, if they exist, let $h_i^1, \ldots, h_i^{p_i-3}$ be the triangular faces incident to v_i between g_{i-1} and g_i and opposite to f in cyclic order, where $p_i = \deg v_i$, and let $w_i^1, w_i^2, \ldots, w_i^{p_i-4}$ be the vertices incident to v_i from u_{i-1} to u_i.

First, we examine the triangular faces incident to v_{k+1}. If T contains $(u_k, h_{k+1}^1, w_{k+1}^1)$, then the switching operation at g_k and h_{k+1}^1 leads this case to Case 1. If T contains $(v_{k+1}, h_{k+1}^1, w_{k+1}^1)$, then the reflecting operation at g_k and h_{k+1}^1 again leads this case to

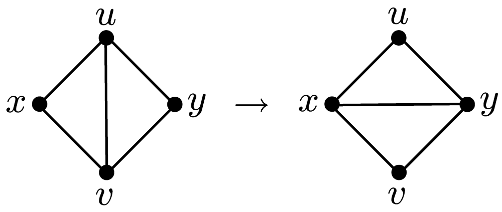

FIGURE 3. Diagonal flip

Case 1. Thus, T must contain $(v_{k+1}, h^1_{k+1}, u_k)$. By repeating this argument, we can say that T contains $(v_{k+1}, g_{k+1}, u_{k+1})$.

Second, we check the triangular faces incident to v_{k+2}, and we can say that T contains $(v_{k+2}, g_{k+2}, u_{k+2})$. Repeating this argument, we finally deduce that T contains $(v_k, h_k^{p_k-3}, w_k^{p_k-4})$. Thus, we can apply the reflecting operation at $h_k^{p_k-3}$ and g_k, which leads this case to Case 1. □

REMARK 4. In the proof of Lemma 3, we can choose a triangular face f as a member of F^* if the faces incident to f are all triangular faces.

Next, we prove the following lemma.

LEMMA 5. *Let G be a plane triangulation. Then G has a spanning vertex-face tour T.*

In order to prove Lemma 5, we use Wagner's theorem [4], which states that every triangulation can be transformed into the standard triangulation by a finite sequence of diagonal flips. Here, the operation *diagonal flip* is defined as follows. Let uv be an edge of a triangulation G. Let uvx and uvy be the faces incident to uv. Then x and y are distinct vertices unless $G = K_3$. If x and y are not adjacent, then a diagonal flip is performed to obtain a new triangulation G' from G by deleting uv and adding the edge xy (Figure 3). The standard triangulation is defined as illustrated in Figure 4. Note that the standard triangulation has a Hamiltonian vertex-face tour.

PROOF. Let $f_1 = v_1v_2v_3$ and $f_2 = v_3v_4v_1$ be two adjacent faces of G, G' be the triangulation obtained from G by performing the diagonal flip at v_1v_3, and $f_1' = v_2v_3v_4$ and $f_2' = v_4v_1v_2$ be the new faces of G'. >From Wagner's theorem and the fact in Figure 4, we only have to show that if G has a spanning vertex-face tour T then G' has a spanning vertex-face tour T'.

Let g_1 and g_2 be triangular faces of G that are adjacent to f_1 by sharing v_1v_2 and v_2v_3, respectively, let g_3 and g_4 be triangular faces of G that are adjacent to f_2 by sharing v_3v_4 and v_4v_1, respectively, and let u_i be the remaining vertex of g_i for $i = 1, 2, 3, 4$.

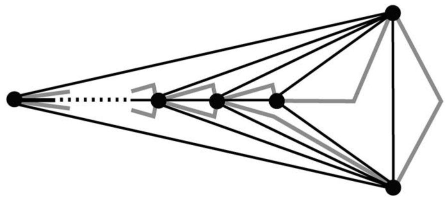

FIGURE 4. Standard triangulation and Hamiltonian vertex-face tour (gray line)

If they exist, let $h_i^1, h_i^2, \dots, h_i^{p_i-3}$ for $i = 2, 4$ and $h_i^1, h_i^2, \dots, h_i^{p_i-4}$ for $i = 1, 3$ be the triangular faces that are incident to v_i and between g_{i-1} and g_i but opposite to f_1 and f_2 in cyclic order, where $p_i = \deg_G v_i$. Let $w_i^1, w_i^2, \dots, w_i^{p_i-4}$ for $i = 2, 4$ and $w_i^1, w_i^2, \dots, w_i^{p_i-5}$ for $i = 1, 3$ be the vertices incident to v_i from u_{i-1} to u_i. We divide the proof into four cases.

Case 1: T contains (v_2, f_1, v_3) and (v_4, f_2, v_1).

In this case, we can obtain T' from T by replacing (v_2, f_1, v_3) and (v_4, f_2, v_1) with (v_2, f_1', v_3) and (v_4, f_2', v_1), respectively.

Case 2: T contains (v_1, f_1, v_3) and (v_4, f_2, v_1).

In this case, we can obtain T' from T by replacing (v_1, f_1, v_3) and (v_4, f_2, v_1) with (v_3, f_1', v_2) and (v_2, f_2', v_4), respectively.

Case 3: T contains (v_1, f_1, v_2) and (v_4, f_2, v_1).

In this case, we consider the triangular faces incident to v_3 from g_2 to g_3 in turn. If T contains (u_2, g_2, v_3), then the switching operation at f_1 and g_2 leads the case to Case 2. If T contains (u_2, g_2, v_2), then the reflecting operation at f_1 and g_2 again leads the case to Case 2. Thus, T must contain (v_3, g_2, v_2). Similarly, we can say that T contains $(v_3, h_3^1, u_2), (v_3, h_3^2, w_3^1), \dots, (v_3, g_3, u_3)$. Thus, if we perform the switching operation at g_3 and f_2, the situation becomes a symmetric version of Case 2.

Case 4: T contains (v_1, f_1, v_3) and (v_3, f_2, v_1).

In this case, we can obtain T' from T by replacing (v_1, f_1, v_3) and (v_3, f_2, v_1) with (v_2, f_1', v_4) and (v_4, f_2', v_2), respectively. □

LEMMA 6. *Let G be a tight graph. Then G has a spanning vertex-face tour T.*

PROOF. We prove this by applying a double-induction on the size and the number of the maximum face of G.

Case 1: G has no non-triangular faces.

This case follows from Lemma 5.

Case 2: The maximum face size of G is at least four.

Let $f = v_1 v_2 \cdots v_n$ be a face with the maximum size ($n \geq 4$). From the planarity of G, we may assume that $G' = G + v_1 v_3$ is tight. Let $f' = v_1 v_2 v_3$ and $f'' = v_3 v_4 \cdots v_n v_1$ be the new faces of G'. From the inductive hypothesis, G' has a spanning vertex-face tour T'. We show that G has a spanning vertex-face tour T.

From Lemma 3 and Remark 4, we may assume that T' contains (v_1, f'', v_3). Therefore, if T' contains (v_2, f', v_1), then we can obtain T from T' by replacing (v_2, f', v_1) and (v_1, f'', v_3) with (v_2, f, v_3); the case where T' contains (v_2, f', v_3) is similar. Thus, we may assume that T' contains (v_1, f', v_3).

We consider the triangular faces incident to v_i from $i = 3, 4, \ldots, n$ in turn. Let g_2 be the triangular face adjacent to f' sharing v_2v_3, and for $i = 3, 4, \ldots, n$, let g_i be the triangular face adjacent to f'' sharing v_iv_{i+1} (indices are taken modulo n). Further, let u_i be the remaining vertex of g_i for $i = 2, 3, \ldots, n$.

First, we examine the triangular faces incident to v_3 from g_2 to g_3. If they exist, let $h_3^1, h_3^2, \ldots, h_3^{p_3-4}$ be the triangular faces between g_2 and g_3 in cyclic order, where $p_3 = \deg_{G'} v_3$, and let $h_3^1 = v_3u_2w_3^1$, $h_3^j = v_3w_3^{j-1}w_3^j$ for $j = 2, 3, \ldots, p-6$, and $h_3^{p-4} = v_3w_3^{p-5}u_3$. If T' contains (v_2, g_2, u_2), then the switching operation at f' and g_2 yields a new spanning vertex-face tour containing (v_1, f', v_2), and if T' contains (v_3, g_2, u_2), then the reflecting operation at f' and g_2 yields a new vertex-face tour containing (v_1, f', v_2), and in both cases, we can obtain T by replacing (v_1, f', v_2) and (v_1, f'', v_3) with (v_2, f, v_3). Thus, T' must contain (v_3, g_2, v_2). By repeating this argument from h_3^1 to g_3, we can say that T' contains (v_3, g_3, u_3). Thus, we can obtain a new vertex-face tour by replacing (v_1, f'', v_3) and (v_3, g_3, u_3) with (v_1, f'', v_4) and (v_4, g_3, u_3), respectively. In this case, we can obtain T by replacing (v_1, f', v_3), (v_1, f'', v_4) with (v_3, f, v_4). $\square$

LEMMA 7. *Let G be a tight graph. Let $F^* = \{f_1, f_2, \ldots, f_m\}$ be the set of all non-triangular faces of G, and let v_i and v_i' be distinct vertices of f_i for $i = 1, 2, \ldots, m$. If G has a spanning vertex-face tour T' containing each (v_i, f_i, v_i') for $i = 1, 2, \ldots, m$, then G has a Hamiltonian vertex-face tour T'' containing each (v_i, f_i, v_i') for $i = 1, 2, \ldots, m$.*

PROOF. We show that T' can be converted to be a connected spanning vertex-face tour by performing only a series of triangular recombinations. Suppose that T' is disconnected at two adjacent faces g and f. We may assume that $g = v_1v_2u_1$ is a triangular face. Let $f = v_1v_2 \cdots v_n$. We divide the proof into two cases.

Case 1: $n = 3$.

In this case, we may assume that two components of T' containing (v_1, g, u_1) and (v_2, f, v_3) are disconnected. Thus, we can make the two components connected by performing the switching operation at g and f.

Case 2: $n \geq 4$.

Suppose that T' contains (v_{k_1}, f, v_{k_2}) for some k_1 and k_2. Then T' must contain $(v_{k_1}, h_{k_1}^l, w_l)$ for some triangular face $h_{k_1}^l$ incident to v_{k_1}, and for some vertex w_l of $h_{k_1}^l$. Now, $h_{k_1}^l$ is connected to g by a path of triangular faces, and it holds from Case 1 that they can become connected by triangular recombinations. $\square$

Our goal is the following.

THEOREM 8. *Let G be a tight graph. Let $F^* = \{f_1, f_2, \ldots, f_m\}$ be the set of all non-triangular faces of G, and let v_i and v_i' be distinct vertices of f_i for $i = 1, 2, \ldots, m$. Then G has a Hamiltonian vertex-face tour containing each (v_i, f_i, v_i') for $i = 1, 2, \ldots, m$.*

PROOF. Let G be a tight graph. >From Lemma 6, G has a spanning vertex-face tour T. Then, from Lemma 3, G has a spanning vertex-face tour T' containing each (v_i, f_i, v_i'). Thus, from Lemma 7, G has a Hamiltonian vertex-face tour T'' containing each (v_i, f_i, v_i'). □

3. Non-crossing Hamiltonian face path

For a polyhedron $\mathcal{P}$ and its graph G, a Hamiltonian vertex-face tour of G guarantees an existence of a path of the faces of $\mathcal{P}$. We call it a *Hamiltonian face path* of $\mathcal{P}$. However, the path might cross itself in the sense that it contains the pattern $(\ldots, f_1, v, f_3, \ldots, f_2, v, f_4, \ldots)$ with the faces f_1, f_2, f_3, f_4 incident to a vertex v appearing in cyclic order. This make it physically impossible for the faces of an unfolding to be a single piece. Hence, we need to detect a non-crossing path. A face path of $\mathcal{P}$ (likewise, a vertex-face tour of G) is *non-crossing* if it has no patterns as that described above.

LEMMA 9. *In Theorem* 8, *any Hamiltonian vertex-face tour of G can be converted to a non-crossing one.*

PROOF. This is contained in [1]. The key point of the proof is as follows. Suppose that a Hamiltonian vertex-face tour T crosses at a vertex v. Let $f_1, f_2, \ldots$ be the faces passing through v in T in cyclic order. We remove the face path of $(\ldots, f_1, v, f_2, \ldots)$ from T as depicted in Figure 5. If the resulting tour is disconnected, then we remove the face path of $(\ldots, f_2, v, f_3, \ldots)$ instead of the above from T such that the resulting tour is connected. By repeating this operation at every vertex of G, we obtain a non-crossing Hamiltonian vertex-face tour. □

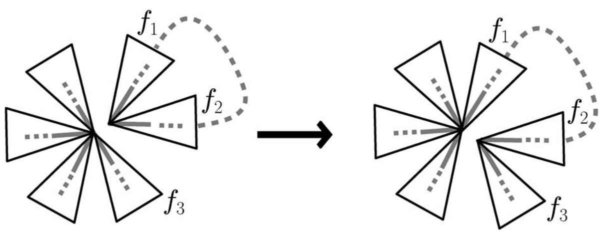

FIGURE 5. Converting a Hamiltonian vertex-face tour into a non-crossing one

4. Layout of a face path

In this section, we exhibit the procedure to lay out the faces of a tight polyhedron $\mathcal{P}$ to form a vertex unfolding. First, we show the following.

LEMMA 10. *Let $\mathcal{F}$ be a (possibly non-convex) polygon with four sides or more. Then, there are two vertices u, v of $\mathcal{F}$ and an arrangement of $\mathcal{F}$ in a vertical interval of the plane with u and v on its left and right boundaries, respectively.*

PROOF. We only have to choose u and v such that the length of segment $\overline{uv}$ is longest among all diagonals and edges of $\mathcal{F}$. □

PROOF OF THEOREM 2. Let $\mathcal{P}$ be a tight polyhedron. Consider the graph G of $\mathcal{P}$. From Lemma 9, G has a non-crossing Hamiltonian vertex-face tour T. Let $\mathcal{T}$ be the corresponding face path of $\mathcal{P}$. We may assume from Theorem 8 that $\mathcal{T}$ uses the vertices of Lemma 10 in each non-triangular face.

Now, we can arrange the faces as follows; this is a consequence of Lemma 22.6.2 in textbook [2]. Suppose inductively that $\mathcal{P}$ has been laid out along a line up to face f_{i-1} with all faces left of vertex v_i, which is the rightmost vertex of f_{i-1}. Let (v_i, f_i, v_{i+1}) be the next face in $\mathcal{T}$. If f_i is a triangular face, rotate f_i around v_i such that f_i lies horizontally between or at the same horizontal coordinate as v_i and v_{i+1}. If f_i is a non-triangular face, we can use Lemma 10. Repeating this process along $\mathcal{T}$ produces a non-overlapping layout of the faces of $\mathcal{P}$. Thus, $\mathcal{P}$ has a vertex unfolding. □

References

[1] E. D. DEMAINE, D. EPPSTEIN, G. W. HART and J. O'ROURKE, Vertex-unfoldings of simplicial manifolds, in Discrete Geometry (ed. Andras Bezdek), Marcel Dekker, New York, 2003, 215–228.

[2] E. D. DEMAINE and J. O'ROURKE, Geometric Folding Algorithms: Linkages, Origami, Polyhedra, Cambridge University Press, Cambridge (2007).

[3] R. DIESTEL, *Graph Theory* (3e), Graduate Texts in Mathematics No. 173, Springer (2005).

[4] K. WAGNER, Bemerkung zum Vierfarbenproblem, Jber. Deutsch. Math.-Verein. **46** (1936), 26–32.

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