TOKYO J. MATH.
VOL. 37, NO. 1, 2014

Rank Two Prolongations of Second-order PDE and Geometric Singular Solutions

Takahiro NODA and Kazuhiro SHIBUYA

Nagoya University and Hiroshima University

(Communicated by K. Yoshitomi)

Abstract. We study geometric structures of rank two prolongations of implicit second-order partial differential equations (PDEs) for two independent and one dependent variables and characterize the type of these PDEs by the topology of fibers of the rank two prolongations. Moreover, by using properties of these prolongations, we give explicit expressions of geometric singular solutions of second-order PDEs from the point of view of contact geometry of second order.

1. Introduction

Let us start by recalling the geometric construction of the 2-jet bundle for two independent and one dependent variables, following [15], [16] and [18].

First, let M be a manifold of dimension 3. We consider the space of 2-dimensional contact elements to M, i.e., the Grassmann bundle $J(M,2)$ over M consisting of 2-dimensional subspaces of tangent spaces to M, namely, $J(M,2)$ is defined by

$$J(M,2) = \bigcup_{x\in M} J_x\,, \quad J_x = \mathrm{Gr}(T_x(M),2)\,,$$

where $\mathrm{Gr}(T_x(M),2)$ denotes the Grassmann manifold of 2-dimensional subspaces in $T_x(M)$. Let $\pi : J(M,2) \to M$ be the bundle projection. The canonical system C on $J(M,2)$ is, by definition, the differential system of codimension 1 on $J(M,2)$ defined by

$$C(u) = \pi_*^{-1}(u) = \{v \in T_u(J(M,2)) \mid \pi_*(v) \in u\} \subset T_u(J(M,2)) \xrightarrow{\pi_*} T_x(M)\,,$$

where $\pi(u) = x$ for $u \in J(M,2)$. The differential system $(J(M,2),C)$ is the (geometric) 1-jet space, also called contact manifold of dimension 5. In general, by a differential system (R,D), we mean a distribution D on a manifold R, that is, D is a subbundle of the tangent bundle TR of R.

Received December 11, 2012; revised July 8, 2013
Mathematics Subject Classification: 58A15, 58A17
Key words and phrases: Implicit second order PDEs, Differential systems, Rank two prolongations, Geometric singular solutions

Next, we should start from a contact manifold (J, C) of dimension 5, which is locally a space of 1-jet for two independent and one dependent variables. Then we can construct the geometric second-order jet space $(L(J), E)$ as follows: We consider the Lagrange-Grassmann bundle $L(J)$ over J consisting of all 2-dimensional integral elements of (J, C), namely,

$$L(J) = \bigcup_{u \in J} L_u \subset J(J, 2)\,,$$

where L_u is the Grassmann manifold of all Lagrangian (or Legendrian) subspaces of the symplectic vector space $(C(u), d\varpi)$ for any $u \in J$. Here ϖ is a local contact form on J. Namely, $v \in J(J, 2)$ is an integral element if and only if $v \subset C(u)$ and $d\varpi|_v = 0$, where $u = \pi(v)$. Then the canonical system E on $L(J)$ is defined by

$$E(v) = \pi_*^{-1}(v) \subset T_v(L(J)) \xrightarrow{\pi_*} T_u(J)\,,$$

where $\pi(v) = u$ for $v \in L(J)$ and $\pi : L(J) \to J$ is the projection. The geometric jet space of second order $(L(J), E)$ is locally a space of 2-jets for two independent and one dependent variables $(J^2(\mathbf{R}^2, \mathbf{R}), C^2)$. Here, the 2-jet space $(J^2(\mathbf{R}^2, \mathbf{R}), C^2)$ is defined as follows:

$$J^2(\mathbf{R}^2, \mathbf{R}) := \left\{(x, y, z, p, q, r, s, t)\right\} \tag{1}$$

and $C^2 := \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by the following 1-forms:

$$\varpi_0 := dz - pdx - qdy\,, \quad \varpi_1 := dp - rdx - sdy\,, \quad \varpi_2 := dq - sdx - tdy\,.$$

In this paper, we identify $(L(J), E)$ with $(J^2(\mathbf{R}^2, \mathbf{R}), C^2)$ since we only consider the local geometry of jet spaces.

Now we consider single PDEs $F(x, y, z, p, q, r, s, t) = 0$, where F is a smooth function on $J^2(\mathbf{R}^2, \mathbf{R})$. We set $R = \{F = 0\} \subset J^2(\mathbf{R}^2, \mathbf{R})$, and restrict the canonical differential system C^2 to R. We denote it by $D(:= C^2|_R)$. We consider a PDE $R = \{F = 0\}$ with the condition $(F_r, F_s, F_t) \neq (0, 0, 0)$ which we call the *regularity condition*. Thus, R is a smooth hypersurface, and also the restriction $\pi^2_1|_R : R \to J^1(\mathbf{R}^2, \mathbf{R})$ of the natural projection $\pi^2_1 : J^2(\mathbf{R}^2, \mathbf{R}) \to J^1(\mathbf{R}^2, \mathbf{R})$ is a submersion. Due to the regularity condition, restricted 1-forms $\varpi_i|_R$ $(i = 0, 1, 2)$ on R are linearly independent. Therefore, we have the induced differential system $D = \left\{\varpi_0|_R = \varpi_1|_R = \varpi_2|_R = 0\right\}$ on R. Then, D is a vector bundle of rank 4 on R. For brevity, we denote each restricted generator 1-form $\varpi_i|_R$ of D by ϖ_i in the following. For such an equation $F = 0$, we consider the discriminant $\Delta := F_r F_t - {F_s}^2/4$.

DEFINITION 1. Let $R = \{F = 0\}$ be a single second-order regular PDE. For the discriminant Δ of F, a point $w \in R$ is said to be *hyperbolic* or *elliptic* if $\Delta(w) < 0$ or $\Delta(w) > 0$, respectively. Moreover, a point $w \in R$ is said to be *parabolic* if $(F_r(w), F_s(w), F_t(w)) \neq (0, 0, 0)$ and $\Delta(w) = 0$.

For second-order regular PDEs, we are interested in geometric singular solutions. Here, the notion of geometric solutions including singular solutions is defined as follows (see [8]).

DEFINITION 2. Let (R, D) be a second-order regular PDE. For a 2-dimensional integral manifold S of R, if the restriction $\pi^2_1|_R : R \to J^1$ of the natural projection $\pi^2_1 : J^2 \to J^1$ is an immersion on an open dense subset in S, then we call S a *geometric solution* of (R, D). If all points of a geometric solution S are immersive points, then we call S a *regular solution*. On the other hand, a geometric solution S have a singular point, then we call S a *singular solution*.

From the definition, images $\pi^2_1(S)$ of geometric solutions S by the projection π^2_1 are Legendrian in $J^1(\mathbf{R}^2, \mathbf{R})$, i.e., $\varpi_0|_{\pi^2_1(S)} = d\varpi_0|_{\pi^2_1(S)} = 0$. We will investigate the method of the construction of these singular solutions. For this purpose, we define the notion of rank n prolongations of differential systems, in general, as follows:

DEFINITION 3. Let (R, D) be a differential system given by $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$. An n-dimensional *integral element* of D at $x \in R$ is an n-dimensional subspace v of $T_x R$ such that $\varpi_i|_v = d\varpi_i|_v = 0$ $(i = 1, \ldots, s)$. Namely, n-dimensional integral elements are candidates for the tangent spaces at x to n-dimensional integral manifolds of D. It follows that the *rank n prolongation of* (R, D) is defined by

$$\Sigma(R) := \bigcup_{x \in R} \Sigma_x \,, \tag{2}$$

where $\Sigma_x = \{v \subset T_x R \mid v \text{ is an } n\text{-dimensional integral element of } (R, D)\}$. We define the canonical system $\hat{D}$ on $\Sigma(R)$ by

$$\hat{D}(u) : = p_*^{-1}(u) = \{v \in T_u(\Sigma(R)) \mid p_*(v) \in u\} \,,$$

where $u \in \Sigma(R)$ is a smooth point of $\Sigma(R)$ and $p : \Sigma(R) \to R$ is the projection.

This space $\Sigma(R)$ is a subset of the Grassmann bundle over R

$$J(D, n) := \bigcup_{x \in R} J_x \tag{3}$$

where $J_x := \{v \subset T_x R \mid v \text{ is an } n\text{-dimensional subspace of } D(x)\}$. In general, the rank n prolongations $\Sigma(R)$ have singular points, that is, $\Sigma(R)$ is not a smooth manifold. This kind of prolongation is very useful to study geometric structures of equations (R, D) or their solutions. In this paper, we only consider in the case of $n = 2$.

Let us now proceed to the description of the various sections and explain the main results in the present paper. In section 2, we investigate the fiber topology of rank 2 prolongations $(\Sigma(R), \hat{D})$ of differential systems (R, D) induced by hyperbolic, parabolic and elliptic equations. One of the main results of this paper is that the type of equations defined by local structure is characterized by the topology of fibers of the prolongation $p : \Sigma(R) \to R$. Namely, we obtain that the topology of fibers of the prolongations $p : \Sigma(R) \to R$ of differential systems (R, D) associated with hyperbolic, parabolic or elliptic equations is torus, pinched torus or sphere, respectively (Corollary 1). In section 3, we study structures of the canonical systems $\hat{D}$ on the rank 2 prolongations $\Sigma(R)$ for hyperbolic, parabolic and elliptic equations (R, D) as differential systems. More precisely, obtained results in this section clarify the structure of nilpotent graded Lie algebras (symbol algebras) of the canonical systems on the rank 2 prolongations for hyperbolic, parabolic and elliptic equations. Here, the symbol algebra is a fundamental invariant of differential systems under contact transformations (see section 3.2). In section 4, we research an approach to construct geometric singular solutions of hyperbolic, parabolic, elliptic equations defined by Definition 2. Especially, we give the explicit integral representation of these singular solutions of model equations for each class of single equations. In section 5, we introduce hyperbolic, parabolic and elliptic rank 4 distributions which are generalizations of hyperbolic, parabolic and elliptic PDEs and prove the topology of fibers of the prolongation of these rank 4 distributions is torus, pinched torus or sphere, respectively (Proposition 9). This result is a generalization of a part of Theorem 18 in [3], [4], [5]. We also prove that the procedure of prolongations of these distributions preserves their types, namely, the rank 2 prolongation of hyperbolic, parabolic or elliptic rank 4 distributions is also a rank 4 distribution of the type of hyperbolic, parabolic or elliptic, respectively (Theorem 4). It follows that, by successive prolongations of these rank 4 distributions, we can define the notion of k-th rank 2 prolongations as a generalization of k-th rank 1 prolongations introduced previously in [7] or [11] (these are called "Monster Goursat manifolds" in [7]).

2. Rank 2 prolongations of regular PDEs

In this section, we show that the types of equations are characterized by the topology of fibers of the rank 2 prolongations of equations. For this purpose, we provide the rank 2 prolongations of hyperbolic, parabolic and elliptic PDEs by using inhomogeneous Grassmann coordinates.

2.1. Rank 2 prolongations of hyperbolic equations. Let (R, D) be a locally hyperbolic equation. Then, there exists a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{11}, \pi_{22}\}$ around $x \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod} \;\; \varpi_0 \,, \\ d\varpi_1 &\equiv \omega_1 \wedge \pi_{11} \quad \text{mod} \;\; \varpi_0, \varpi_1, \varpi_2 \,, \\ d\varpi_2 &\equiv \omega_2 \wedge \pi_{22} \quad \text{mod} \;\; \varpi_0, \varpi_1, \varpi_2 \,. \end{aligned} \tag{4}$$

In terms of this structure equation, we construct the rank 2 prolongation of (R, D) by taking integral elements.

THEOREM 1. *Let (R, D) be a locally hyperbolic equation. Then, the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D, 2)$, and it is a $T^2 = S^1 \times S^1$-bundle over R.*

PROOF. First, we show that $\Sigma(R)$ is a submanifold of $J(D, 2)$. Let $\pi : J(D, 2) \to R$

be the projection and U an open set in R. Then $\pi^{-1}(U)$ is covered by 6 open sets in $J(D,2)$:

$$\pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{11}} \cup U_{\omega_1\pi_{22}} \cup U_{\omega_2\pi_{11}} \cup U_{\omega_2\pi_{22}} \cup U_{\pi_{11}\pi_{22}}\,, \tag{5}$$

where

$$\begin{aligned} U_{\omega_1\omega_2} &:= \left\{v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \omega_2|_v \neq 0\right\},\\ U_{\omega_1\pi_{11}} &:= \left\{v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \pi_{11}|_v \neq 0\right\},\\ U_{\omega_1\pi_{22}} &:= \left\{v \in \pi^{-1}(U) \mid \omega_1|_v \wedge \pi_{22}|_v \neq 0\right\},\\ U_{\omega_2\pi_{11}} &:= \left\{v \in \pi^{-1}(U) \mid \omega_2|_v \wedge \pi_{11}|_v \neq 0\right\},\\ U_{\omega_2\pi_{22}} &:= \left\{v \in \pi^{-1}(U) \mid \omega_2|_v \wedge \pi_{22}|_v \neq 0\right\},\\ U_{\pi_{11}\pi_{22}} &:= \left\{v \in \pi^{-1}(U) \mid \pi_{11}|_v \wedge \pi_{22}|_v \neq 0\right\}. \end{aligned}$$

In the following, we explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, \ldots, U_{\pi_{11}\pi_{22}}$.

(I) On $U_{\omega_1\omega_2}$:

For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, $p(w) = v$. Hence, by restricting π_{11}, π_{22} to w, we can introduce the inhomogeneous coordinate p^1_{ij} of fibers of $J(D,2)$ around w with $\pi_{11}|_w = p^1_{11}(w)\omega_1|_w + p^1_{12}(w)\omega_2|_w$, $\pi_{22}|_w = p^1_{21}(w)\omega_1|_w + p^1_{22}(w)\omega_2|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge (p^1_{11}(w)\omega_1|_w + p^1_{12}(w)\omega_2|_w) \equiv p^1_{12}(w)\omega_1|_w \wedge \omega_2|_w\,,\\ d\varpi_2|_w &\equiv \omega_2|_w \wedge (p^1_{21}(w)\omega_1|_w + p^1_{22}(w)\omega_2|_w) \equiv -p^1_{21}(w)\omega_1|_w \wedge \omega_2|_w\,. \end{aligned}$$

Hence, we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D,2)$, where $f_1 = p^1_{12}$, $f_2 = p^1_{21}$, that is, $\left\{f_1 = f_2 = 0\right\} \subset U_{\omega_1\omega_2}$. Then df_1, df_2 are independent on $\left\{f_1 = f_2 = 0\right\}$.

(II) On $U_{\omega_1\pi_{11}}$:

For $w \in U_{\omega_1\pi_{11}}$, by restricting ω_2, π_{22} to w, we introduce the inhomogeneous coordinate p^2_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^2_{11}(w)\omega_1|_w + p^2_{12}(w)\pi_{11}|_w$, $\pi_{22}|_w = p^2_{21}(w)\omega_1|_w + p^2_{22}(w)\pi_{11}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w \not\equiv 0$. Thus, there does not exist integral element, that is, $U_{\omega_1\pi_{11}} \cap p^{-1}(U) = \emptyset$.

(III) On $U_{\omega_1\pi_{22}}$:

For $w \in U_{\omega_1\pi_{22}}$, by restricting ω_2, π_{11} to w, we introduce the inhomogeneous coordinate p^3_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^3_{11}(w)\omega_1|_w + p^3_{12}(w)\pi_{22}|_w$, $\pi_{11}|_w = p^3_{21}(w)\omega_1|_w + p^3_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge \pi_{11}|_w \equiv p^3_{22}(w)\omega_1|_w \wedge \pi_{22}|_w\,,\\ d\varpi_2|_w &\equiv \omega_2|_w \wedge \pi_{22}|_w \equiv p^3_{11}(w)\omega_1|_w \wedge \pi_{22}|_w\,. \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are independent in the same as (I).

(IV) On $U_{\omega_2\pi_{11}}$:

For $w \in U_{\omega_2\pi_{11}}$, by restricting ω_1, π_{22} to w, we introduce the inhomogeneous coordinate p^4_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^4_{11}(w)\omega_2|_w + p^4_{12}(w)\pi_{11}|_w$, $\pi_{22}|_w = p^4_{21}(w)\omega_2|_w + p^4_{22}(w)\pi_{11}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w \equiv p^4_{11}(w)\omega_2|_w \wedge \pi_{11}|_w\,,$$

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi_{22}|_w \equiv p^4_{22}(w)\omega_2|_w \wedge \pi_{11}|_w\,.$$

Then the defining functions of $\Sigma(R)$ are independent in the same way as in (I).

(V) On $U_{\omega_2\pi_{22}}$:

For $w \in U_{\omega_2\pi_{22}}$, by restricting ω_1, π_{11} to w, we introduce the inhomogeneous coordinate p^5_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^5_{11}(w)\omega_2|_w + p^5_{12}(w)\pi_{22}|_w$, $\quad \pi_{11}|_w = p^5_{21}(w)\omega_2|_w + p^5_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_2|_w \equiv \omega_2|_w \wedge \pi_{22}|_w \not\equiv 0$. Thus, there does not exist integral element, that is, $U_{\omega_2\pi_{22}} \cap p^{-1}(U) = \emptyset$.

(VI) On $U_{\pi_{11}\pi_{22}}$:

For $w \in U_{\pi_{11}\pi_{22}}$, by restricting ω_1, ω_2 to w, we introduce the inhomogeneous coordinate p^6_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^6_{11}(w)\pi_{11}|_w + p^6_{12}(w)\pi_{22}|_w$, $\omega_2|_w = p^6_{21}(w)\pi_{11}|_w + p^6_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w \equiv p^6_{12}(w)\pi_{22}|_w \wedge \pi_{11}|_w,$$

$$d\varpi_2|_w \equiv \omega_2|_w \wedge \pi_{22}|_w \equiv p^6_{21}(w)\pi_{11}|_w \wedge \pi_{22}|_w\,.$$

Then the defining functions of $\Sigma(R)$ are independent in the same way as in (I).

Under these discussions, the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D,2)$.

Next, we show that the topology of fibers of $\Sigma(R)$ is torus. In the above discussion, we have $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}} \cup P_{\omega_2\pi_{11}} \cup P_{\pi_{11}\pi_{22}}$, where $P_{\omega_1\omega_2} := p^{-1}(U) \cap U_{\omega_1\omega_2}$, $P_{\omega_1\pi_{22}} := p^{-1}(U) \cap U_{\omega_1\pi_{22}}$, $P_{\omega_2\pi_{11}} := p^{-1}(U) \cap U_{\omega_2\pi_{11}}$, and $P_{\pi_{11}\pi_{22}} := p^{-1}(U) \cap U_{\pi_{11}\pi_{22}}$. From Definition 3, we have the canonical system $\hat{D}$ on each open set. To prove our assertion, we investigate the gluing of $(\Sigma(R), \hat{D})$. For instance, we construct the transition functions on $U_{\omega_1\omega_2} \cap U_{\omega_1\pi_{22}}$ in the following. On $U_{\omega_1\omega_2}$, the canonical system $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\pi_{11}} = \varpi_{\pi_{22}} = 0\}$ is given by $\varpi_{\pi_{11}} := \pi_{11} - p^1_{11}\omega_1$, $\quad \varpi_{\pi_{22}} := \pi_{22} - p^1_{22}\omega_2$. On the other hand, the canonical system $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = \varpi_{\pi_{11}} = 0\}$ on $U_{\omega_1\pi_{22}}$ is given by $\varpi_{\omega_2} := \omega_2 - p^3_{12}\pi_{22}$, $\quad \varpi_{\pi_{11}} := \pi_{11} - p^3_{21}\omega_1$. Then, the transition functions $\phi : U_{\omega_1\omega_2} \cap U_{\omega_1\pi_{22}} \to U_{\omega_1\omega_2} \cap U_{\omega_1\pi_{22}}$ is given by

$$\phi(v, p^1_{11}, p^1_{22}) = \left(v, p^3_{12} := \frac{1}{p^1_{22}}, p^3_{21} := p^1_{11}\right) \text{ for } p^1_{22} \neq 0\,,$$

where v is a local coordinate on R. We also have similar transition functions for the other intersection open sets $U_{\omega_1\omega_2} \cap U_{\omega_2\pi_{11}}$, $U_{\omega_1\pi_{22}} \cap U_{\pi_{11}\pi_{22}}$, $U_{\omega_2\pi_{11}} \cap U_{\pi_{11}\pi_{22}}$. Consequently, the topological structure of fibers is $T^2 = S^1 \times S^1$. □

REMARK 1. In fact, this result (i.e. $\Sigma(R)$ is a torus bundle) is known by Bryant, Griffiths and Hsu in [2]. They obtained this result for the hyperbolic exterior differential system which is a generalization of distributions corresponding to hyperbolic equations (see Remark 3). However, we will also consider parabolic and elliptic cases and our method is distinct one. We will use the structure of this covering in $\Sigma(R)$ when we will study singular solutions (see, section 5). Thus, we need to prove in the above way.

2.2. Rank 2 prolongations of parabolic equations. Let (R, D) be a locally parabolic equation. Then, there exists a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{12}, \pi_{22}\}$ around $x \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds:

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \mathrm{mod} \;\; \varpi_0\,, \\ d\varpi_1 &\equiv \qquad\qquad\;\; \omega_2 \wedge \pi_{12} \quad \mathrm{mod} \;\; \varpi_0, \varpi_1, \varpi_2\,, \\ d\varpi_2 &\equiv \omega_1 \wedge \pi_{12} + \omega_2 \wedge \pi_{22} \quad \mathrm{mod} \;\; \varpi_0, \varpi_1, \varpi_2\,. \end{aligned} \tag{6}$$

From this structure equation, we clarify the rank 2 prolongation $\Sigma(R)$.

LEMMA 1. *Let (R, D) be a locally parabolic equation. Then, the rank 2 prolongation $\Sigma(R)$ has singular points, that is, $\Sigma(R)$ is not a smooth manifold.*

PROOF. Let U be an open set in R, and $\pi : J(D, 2) \to R$ be the projection. Then $\pi^{-1}(U)$ is covered by 6 open sets in $J(D, 2)$:

$$\pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{12}} \cup U_{\omega_1\pi_{22}} \cup U_{\omega_2\pi_{12}} \cup U_{\omega_2\pi_{22}} \cup U_{\pi_{12}\pi_{22}}\,, \tag{7}$$

where each open set is given in the same way as the hyperbolic case (5). Now we explicitly describe the defining equation of $\Sigma(R)$ on each open set.

(I) On $U_{\omega_1\omega_2}$:

For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, $p(w) = v$. Hence, by restricting π_{12}, π_{22} to w, we can introduce the inhomogeneous coordinate p^1_{ij} of fibers of $J(D, 2)$ around w with $\pi_{12}|_w = p^1_{11}(w)\omega_1|_w + p^1_{12}(w)\omega_2|_w$, $\pi_{22}|_w = p^1_{21}(w)\omega_1|_w + p^1_{22}(w)\omega_2|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_2|_w \wedge \pi_{12}|_w \equiv p^1_{11}(w)\omega_2|_w \wedge \omega_1|_w\,, \\ d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w + \omega_2|_w \wedge \pi_{22}|_w \equiv (p^1_{12}(w) - p^1_{21}(w))\omega_1|_w \wedge \omega_2|_w\,. \end{aligned}$$

Hence we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D, 2)$, where $f_1 = p^1_{11}$, $f_2 = p^1_{12} - p^1_{21}$, that is, $\{f_1 = f_2 = 0\} \subset U_{\omega_1\omega_2}$. Then df_1, df_2 are independent on $\{f_1 = f_2 = 0\}$.

(II) On $U_{\omega_1\pi_{12}}$:

For $w \in U_{\omega_1\pi_{12}}$, by restricting ω_2, π_{22} to w, we we introduce the inhomogeneous coordinate p^2_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^2_{11}(w)\omega_1|_w + p^2_{12}(w)\pi_{12}|_w$, $\pi_{22}|_w = p^2_{21}(w)\omega_1|_w + p^2_{22}(w)\pi_{12}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_2|_w \wedge \pi_{12}|_w \equiv p^2_{11}(w)\omega_1|_w \wedge \pi_{12}|_w\,,$$

$$\begin{aligned} d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w + \omega_2|_w \wedge \pi_{22}|_w \\ &\equiv (1 + p^2_{11}(w)p^2_{22}(w) - p^2_{12}(w)p^2_{21}(w))\omega_1|_w \wedge \pi_{12}|_w\,. \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are independent.

(III) On $U_{\omega_1\pi_{22}}$:

For $w \in U_{\omega_1\pi_{22}}$, by restricting ω_2, π_{12} to w, we introduce the inhomogeneous coordinate p^3_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^3_{11}(w)\omega_1|_w + p^3_{12}(w)\pi_{22}|_w$, $\pi_{12}|_w = p^3_{21}(w)\omega_1|_w + p^3_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_2|_w \wedge \pi_{12}|_w \equiv (p^3_{11}(w)p^3_{22}(w) - p^3_{12}(w)p^3_{21}(w))\omega_1|_w \wedge \pi_{22}|_w\,,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi_{12}|_w + \omega_2|_w \wedge \pi_{22}|_w \equiv (p^3_{11}(w) + p^3_{22}(w))\omega_1|_w \wedge \pi_{22}|_w\,.$$

Therefore, we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(R)$ in $U_{\omega_1\pi_{22}}$ of $J(D,2)$, where $f_1 = p^3_{11}p^3_{22} - p^3_{12}p^3_{21}$, $f_2 = p^3_{11} + p^3_{22}$, that is, $\{f_1 = f_2 = 0\} \subset U_{\omega_1\pi_{22}}$. Then, df_1, df_2 are linearly dependent on $S := \left\{p^3_{11} = p^3_{22}, p^3_{12} = p^3_{21} = 0\right\}$. Hence, $S \cap \Sigma(R) = \{p^3_{11} = p^3_{22} = p^3_{12} = p^3_{21} = 0\}$ which is a point on each fiber is a singular subset in $\Sigma(R)$.

(IV) On $U_{\omega_2\pi_{12}}$:

For $w \in U_{\omega_2\pi_{12}}$, by restricting ω_1, π_{22} to w, we introduce the inhomogeneous coordinate p^4_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^4_{11}(w)\omega_2|_w + p^4_{12}(w)\pi_{12}|_w$, $\pi_{22}|_w = p^4_{21}(w)\omega_2|_w + p^4_{22}(w)\pi_{12}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$. However, we have $d\varpi_1|_w \equiv \omega_2|_w \wedge \pi_{12}|_w \not\equiv 0$. Hence, there does not exist integral element, that is, $U_{\omega_2\pi_{12}} \cap p^{-1}(U) = \emptyset$.

(V) On $U_{\omega_2\pi_{22}}$:

For $w \in U_{\omega_2\pi_{22}}$, by restricting ω_1, π_{12} to w, we can introduce the inhomogeneous coordinate p^5_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^5_{11}(w)\omega_2|_w + p^5_{12}(w)\pi_{22}|_w$, $\pi_{12}|_w = p^5_{21}(w)\omega_2|_w + p^5_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_2|_w \wedge \pi_{12}|_w \equiv p^5_{22}(w)\omega_2|_w \wedge \pi_{22}|_w\,,$$

$$\begin{aligned} d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w + \omega_2|_w \wedge \pi_{22}|_w \\ &\equiv (1 + p^5_{11}(w)p^5_{22}(w) - p^5_{12}(w)p^5_{21}(w))\omega_2|_w \wedge \pi_{22}|_w\,. \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are independent in the same as (I).

(VI) On $U_{\pi_{12}\pi_{22}}$:

For $w \in U_{\pi_{12}\pi_{22}}$, by restricting ω_1, ω_2 to w, we introduce the inhomogeneous coordinate p^6_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^6_{11}(w)\pi_{12}|_w + p^6_{12}(w)\pi_{22}|_w,\ \omega_2|_w = p^6_{21}(w)\pi_{12}|_w + p^6_{22}(w)\pi_{22}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_2|_w \wedge \pi_{12}|_w \equiv p^6_{22}(w)\pi_{22}|_w \wedge \pi_{12}|_w\,,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi_{12}|_w + \omega_2|_w \wedge \pi_{22}|_w \equiv (p^6_{21}(w) - p^6_{12}(w))\pi_{12}|_w \wedge \pi_{22}|_w\,.$$

Then the defining functions of $\Sigma(R)$ are also independent.

Summarizing these discussions, the rank 2 prolongations $\Sigma(R)$ for locally parabolic equations R has singular points, that is, these are not smooth. □

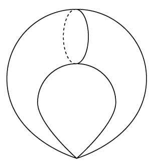
We set $P_{\omega_1\omega_2} := p^{-1}(U) \cap U_{\omega_1\omega_2},\ P_{\omega_1\pi_{12}} := p^{-1}(U) \cap U_{\omega_1\pi_{12}},\ P_{\omega_1\pi_{22}} := p^{-1}(U) \cap U_{\omega_1\pi_{22}},\ P_{\omega_2\pi_{22}} := p^{-1}(U) \cap U_{\omega_2\pi_{22}}$, and $P_{\pi_{12}\pi_{22}} := p^{-1}(U) \cap U_{\pi_{12}\pi_{22}}$.

LEMMA 2. *We have* $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}} \cup P_{\pi_{12}\pi_{22}}$.

PROOF. From the discussion of the proof of the previous lemma, we have $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{12}} \cup P_{\omega_1\pi_{22}} \cup P_{\omega_2\pi_{22}} \cup P_{\pi_{12}\pi_{22}}$. Hence, it is sufficient to prove $P_{\omega_1\pi_{12}},\ P_{\omega_2\pi_{22}} \subset P_{\omega_1\omega_2}$. For the open set $P_{\omega_1\pi_{12}}$, we prove this property. Let w be any point in $P_{\omega_1\pi_{12}} \subset p^{-1}(U)$. Here, if $w \notin P_{\omega_1\omega_2}$, then $\omega_1|_w \wedge \omega_2|_w = 0$. Hence, by $\omega_1|_w \wedge \omega_2|_w = p^2_{12}(w)\omega_1|_w \wedge \pi_{12}|_w$, we have the condition $p^2_{12}(w) = 0$. However, w is an integral element, and we have $p^2_{12}(w) \neq 0$. Thus, we have $P_{\omega_1\pi_{12}} \subset P_{\omega_1\omega_2}$. For the open set $P_{\omega_2\pi_{22}}$, we also obtain the statement from the same argument. □

THEOREM 2. *Let* (R, D) *be a locally parabolic equation. Then, the rank* 2 *prolongation* $\Sigma(R)$ *has singular points, and it has the structure of pinched torus fibration.*

PROOF. By the above lemma, note that the fiber $p^{-1}(w)$ at $w \in R$ decompose to the disjoint union $p^{-1}(w) = \mathbf{R}^2 \cup \mathbf{R} \cup \{\text{a point}\}$ as a set. Moreover, by gluing on $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}} \cup P_{\pi_{12}\pi_{22}}$ in the proof of Lemmas 1 and 2, we obtain the statement. □



2.3. Rank 2 prolongations of elliptic equations. Let (R, D) be a locally elliptic equation. Then, there exists a local coframe $\{\varpi_0, \varpi_1, \varpi_2, \omega_1, \omega_2, \pi_{11}, \pi_{12}\}$ around $x \in R$ such that $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ and the following structure equation holds:

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod } \varpi_0\,,$$

$$d\varpi_1 \equiv \omega_1 \wedge \pi_{11} + \omega_2 \wedge \pi_{12} \quad \text{mod } \varpi_0, \varpi_1, \varpi_2\,, \tag{8}$$

$$d\varpi_2 \equiv \omega_1 \wedge \pi_{12} - \omega_2 \wedge \pi_{11} \mod \varpi_0, \varpi_1, \varpi_2 .$$

From this structure equation, we investigate the rank 2 prolongation $\Sigma(R)$. Let U be an open set in R, and $\pi : J(D,2) \to R$ the projection. Then $\pi^{-1}(U)$ is covered by 6 open sets in $J(D,2)$:

$$\pi^{-1}(U) = U_{\omega_1\omega_2} \cup U_{\omega_1\pi_{11}} \cup U_{\omega_1\pi_{12}} \cup U_{\omega_2\pi_{11}} \cup U_{\omega_2\pi_{12}} \cup U_{\pi_{11}\pi_{12}} , \tag{9}$$

where each open set is also given in the same way as hyperbolic case (5). Now we explicitly describe the defining equation of $\Sigma(R)$ in terms of the inhomogeneous Grassmann coordinate of fibers in $U_{\omega_1\omega_2}, \dots, U_{\pi_{11}\pi_{12}}$.

(I) On $U_{\omega_1\omega_2}$:

For $w \in U_{\omega_1\omega_2}$, w is a 2-dimensional subspace of $D(v)$, $p(w) = v$. Hence, by restricting π_{11}, π_{12} to w, we introduce the inhomogeneous coordinate p^1_{ij} of fibers of $J(D,2)$ around w with $\pi_{11}|_w = p^1_{11}(w)\omega_1|_w + p^1_{12}(w)\omega_2|_w,\ \pi_{12}|_w = p^1_{21}(w)\omega_1|_w + p^1_{22}(w)\omega_2|_w$. Moreover w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \equiv (p^1_{12}(w) - p^1_{21}(w))\omega_1|_w \wedge \omega_2|_w ,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w \equiv (p^1_{11}(w) + p^1_{22}(w))\omega_1|_w \wedge \omega_2|_w .$$

Hence we obtain the defining equations $f_1 = f_2 = 0$ of $\Sigma(R)$ in $U_{\omega_1\omega_2}$ of $J(D,2)$, where $f_1 = p^1_{12} - p^1_{21}$, $f_2 = p^1_{11} + p^1_{22}$, that is, $\{f_1 = f_2 = 0\} \subset U_{\omega_1\omega_2}$. Then df_1, df_2 are independent on $\{f_1 = f_2 = 0\}$.

(II) On $U_{\omega_1\pi_{11}}$:

For $w \in U_{\omega_1\pi_{11}}$, by restricting ω_2, π_{12} to w, we introduce the inhomogeneous coordinate p^2_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^2_{11}(w)\omega_1|_w + p^2_{12}(w)\pi_{11}|_w,\ \pi_{12}|_w = p^2_{21}(w)\omega_1|_w + p^2_{22}(w)\pi_{11}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \\ &\equiv (1 + p^2_{11}(w)p^2_{22}(w) - p^2_{12}(w)p^2_{21}(w))\omega_1|_w \wedge \pi_{11}|_w , \\ d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w \\ &\equiv (-p^2_{11}(w) + p^2_{22}(w))\omega_1|_w \wedge \pi_{11}|_w . \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are independent in the same as (I).

(III) On $U_{\omega_1\pi_{12}}$:

For $w \in U_{\omega_1\pi_{12}}$, by restricting ω_2, π_{11} to w, we introduce the inhomogeneous coordinate p^3_{ij} of fibers of $J(D,2)$ around w with $\omega_2|_w = p^3_{11}(w)\omega_1|_w + p^3_{12}(w)\pi_{12}|_w,\ \pi_{11}|_w = p^3_{21}(w)\omega_1|_w + p^3_{22}(w)\pi_{12}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \equiv (p^3_{11}(w) + p^3_{22}(w))\omega_1|_w \wedge \pi_{12}|_w ,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w$$

$$\equiv (1 - p^3_{11}(w)p^3_{22}(w) + p^3_{12}(w)p^3_{21}(w))\omega_1|_w \wedge \pi_{12}|_w \,.$$

Then the defining functions of $\Sigma(R)$ are also independent.

(IV) On $U_{\omega_2\pi_{11}}$:

For $w \in U_{\omega_2\pi_{11}}$, by restricting ω_1, π_{12} to w, we introduce the inhomogeneous coordinate p^4_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^4_{11}(w)\omega_2|_w + p^4_{12}(w)\pi_{11}|_w$, $\pi_{12}|_w = p^4_{21}(w)\omega_2|_w + p^4_{22}(w)\pi_{11}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \equiv (p^4_{11}(w) + p^4_{22}(w))\omega_2|_w \wedge \pi_{11}|_w \,,$$

$$\begin{aligned} d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w \\ &\equiv (p^4_{11}(w)p^4_{22}(w) - p^4_{12}(w)p^4_{21}(w) - 1)\omega_2|_w \wedge \pi_{11}|_w \,. \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are also independent.

(V) On $U_{\omega_2\pi_{12}}$:

For $w \in U_{\omega_2\pi_{12}}$, by restricting ω_1, π_{11} to w, we introduce the inhomogeneous coordinate p^5_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^5_{11}(w)\omega_2|_w + p^5_{12}(w)\pi_{12}|_w$, $\pi_{11}|_w = p^5_{21}(w)\omega_2|_w + p^5_{22}(w)\pi_{12}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$\begin{aligned} d\varpi_1|_w &\equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \\ &\equiv (1 + p^5_{11}(w)p^5_{22}(w) - p^5_{12}(w)p^5_{21}(w))\omega_2|_w \wedge \pi_{12}|_w \,, \\ d\varpi_2|_w &\equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w \equiv (p^5_{11}(w) - p^5_{22}(w))\omega_2|_w \wedge \pi_{12}|_w \,. \end{aligned}$$

Then the defining functions of $\Sigma(R)$ are also independent.

(VI) On $U_{\pi_{11}\pi_{12}}$:

For $w \in U_{\pi_{11}\pi_{12}}$, by restricting ω_1, ω_2 to w, we introduce the inhomogeneous coordinate p^6_{ij} of fibers of $J(D,2)$ around w with $\omega_1|_w = p^6_{11}(w)\pi_{11}|_w + p^6_{12}(w)\pi_{12}|_w$, $\omega_2|_w = p^6_{21}(w)\pi_{11}|_w + p^6_{22}(w)\pi_{12}|_w$. Moreover, w satisfies $d\varpi_1|_w \equiv d\varpi_2|_w \equiv 0$:

$$d\varpi_1|_w \equiv \omega_1|_w \wedge \pi_{11}|_w + \omega_2|_w \wedge \pi_{12}|_w \equiv (-p^6_{12}(w) + p^6_{21}(w))\pi_{11}|_w \wedge \pi_{12}|_w \,,$$

$$d\varpi_2|_w \equiv \omega_1|_w \wedge \pi_{12}|_w - \omega_2|_w \wedge \pi_{11}|_w \equiv (p^6_{11}(w) + p^6_{22}(w))\pi_{11}|_w \wedge \pi_{12}|_w \,.$$

Then the defining functions of $\Sigma(R)$ are also independent.

Summarizing these discussions, the rank 2 prolongation $\Sigma(R)$ of a locally elliptic equation R is smooth, and it has the covering $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{11}} \cup P_{\omega_1\pi_{12}} \cup P_{\omega_2\pi_{11}} \cup P_{\omega_2\pi_{12}} \cup P_{\pi_{11}\pi_{12}}$, where $P_{\omega_1\omega_2} := p^{-1}(U) \cap U_{\omega_1\omega_2}$, $P_{\omega_1\pi_{11}} := p^{-1}(U) \cap U_{\omega_1\pi_{11}}$, $P_{\omega_1\pi_{12}} := p^{-1}(U) \cap U_{\omega_1\pi_{12}}$, $P_{\omega_2\pi_{11}} := p^{-1}(U) \cap U_{\omega_2\pi_{11}}$, $P_{\omega_2\pi_{12}} := p^{-1}(U) \cap U_{\omega_2\pi_{12}}$, and $P_{\pi_{11}\pi_{12}} := p^{-1}(U) \cap U_{\pi_{11}\pi_{12}}$. However, this covering is not essential in the following sense.

LEMMA 3. *Let (R, D) be a locally elliptic equation and $p : \Sigma(R) \to R$ be the rank 2 prolongation. Then, for any open set $U \subset R$, we have $p^{-1}(U) = P_{\omega_1\omega_2} \cup P_{\pi_{11}\pi_{12}}$.*

PROOF. It is sufficient to prove $P_{\omega_1\pi_{11}},\ P_{\omega_1\pi_{12}},\ P_{\omega_2\pi_{11}},\ P_{\omega_2\pi_{12}} \subset P_{\omega_1\omega_2}$. For the open set $P_{\omega_1\pi_{11}}$, we prove this property. Let w be a point in $P_{\omega_1\pi_{11}} \subset p^{-1}(U)$. Here, if $w \notin P_{\omega_1\omega_2}$, then the condition $\omega_1|_w \wedge \omega_2|_w = 0$ is satisfied. Hence, by $\omega_1|_w \wedge \omega_2|_w = p^2_{12}(w)\omega_1|_w \wedge \pi_{11}|_w$, we have $p^2_{12}(w) = 0$. However, w is an integral element. In terms of $f_1 = f_2 = 0$, we have $(p^2_{11})^2 = -1$. This is a contradiction. Thus, we have $P_{\omega_1\pi_{11}} \subset P_{\omega_1\omega_2}$. For other open sets, we also have the statement from the similar argument. □

THEOREM 3. *Let (R, D) be a locally elliptic equation. Then, the rank 2 prolongation $\Sigma(R)$ is a smooth submanifold of $J(D, 2)$, and it is an S^2-bundle over R.*

PROOF. By the above lemma, note that the fiber $p^{-1}(w)$ at $w \in R$ decompose to the disjoint union $p^{-1}(w) = \mathbf{R}^2 \cup \{\text{a point}\}$ as a set. Moreover, we obtain the statement from the same argument to the parabolic case. □

2.4. A characterization of equations by the fiber topology. We obtain one of the main results by summarizing theorems of the previous part of this section.

COROLLARY 1. *Let $R = \{F = 0\}$ be a second-order regular PDE and $\Sigma(R)$ be the its prolongation. Let $p : \Sigma(R) \to R$ be the natural projection. Then,*

(1) *$w \in R$ is hyperbolic $\iff$ $p^{-1}(w)$ is a topologically 2-dimensional torus T^2.*
(2) *$w \in R$ is parabolic $\iff$ $p^{-1}(w)$ is a topologically pinched 2-dimensional torus.*
(3) *$w \in R$ is elliptic $\iff$ $p^{-1}(w)$ is a topologically 2-dimensional sphere S^2.*

PROOF. Note that the fiber $p^{-1}(w)$ is defined by the structure equation of D at w as a subset in the fiber $J_w \cong Gr(2, 4)$ of the fibration $\pi : J(D, 2) \to R$. From this point of view, the topology of the fiber $p^{-1}(w)$ depends only on the pointwise structure equations (4), (6) and (8). □

3. Structures of the canonical systems on the rank 2 prolongations

In this section, we study the geometric structures of the rank 2 prolongations $(\Sigma(R), \hat{D})$ for each class of equations. We first recall Tanaka theory of weakly regular differential systems in this section. For more details, we refer to [12], [13], [14] and [17].

3.1. Derived system, Weak derived system. Let D be a differential system on a manifold R. We denote by $\mathcal{D} = \Gamma(D)$ the sheaf of sections to D. The *derived system* ∂D of a differential system D is defined, in terms of sections, by $\partial\mathcal{D} := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$. In general, ∂D is obtained as a subsheaf of the tangent sheaf of R. Moreover, *higher derived systems* $\partial^k D$ are defined successively by $\partial^k\mathcal{D} := \partial(\partial^{k-1}\mathcal{D})$, where we set $\partial^0 D = D$ by convention. On the other hand, k-th *weak derived systems* $\partial^{(k)} D$ of D are defined inductively by $\partial^{(k)}\mathcal{D} := \partial^{(k-1)}\mathcal{D} + [\mathcal{D}, \partial^{(k-1)}\mathcal{D}]$.

DEFINITION 4. A differential system D is called *regular* (respectively, *weakly regular*), if $\partial^k D$ (respectively, $\partial^{(k)} D$) is a subbundle for each k.

These derived systems are also interpreted by using annihilators as follows ([1], [9]): Let $D = \{\varpi_1 = \cdots = \varpi_s = 0\}$ be a differential system on a manifold R. We denote by $D^\perp$ the annihilator subbundle of D in T^*R, namely,

$$D^\perp = \bigcup_{x \in R} D^\perp(x)$$

where

$$D^\perp(x) = \{\omega \in T_x^*R \mid \omega(X) = 0 \text{ for any } X \in D(x)\} \subset T_x^*R\,.$$

Then the annihilator $(\partial D)^\perp$ of the first derived system of D is given by

$$(\partial D)^\perp = \{\varpi \in D^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ D^\perp)\}\,.$$

Moreover, the annihilator $(\partial^{(k+1)}D)^\perp$ of the $(k+1)$-th weak derived system of D is given by

$$\begin{aligned}(\partial^{(k+1)}D)^\perp = \{\varpi \in (\partial^{(k)}D)^\perp \mid d\varpi \equiv 0 \ (\mathrm{mod}\ &(\partial^{(k)}D)^\perp\,,\\ &(\partial^{(p)}D)^\perp \wedge (\partial^{(q)}D)^\perp,\ 2 \le p, q \le k-1)\}\,.\end{aligned}$$

We set $D^{-1} := D$, $D^{-k} := \partial^{(k-1)}D$ $(k \ge 2)$, for a weakly regular differential system D. Then we have ([12, Proposition 1.1]):

(T1) There exists a unique positive integer μ such that

$$D^{-1} \subset D^{-2} \subset \cdots \subset D^{-k} \subset \cdots \subset D^{-(\mu-1)} \subset D^{-\mu} = D^{-(\mu+1)} = \cdots$$

(T2) $[\mathcal{D}^p, \mathcal{D}^q] \subset \mathcal{D}^{p+q}$ for all $p,\ q < 0$.

3.2. Symbol algebra of differential system. Let (R, D) be a weakly regular differential system such that $TR = D^{-\mu} \supset D^{-(\mu-1)} \supset \cdots \supset D^{-1} =: D$. For all $x \in R$, we put $\mathfrak{g}_{-1}(x) := D^{-1}(x) = D(x)$, $\mathfrak{g}_p(x) := D^p(x)/D^{p+1}(x)$, $(p = -2, -3, \ldots, -\mu)$ and $\mathfrak{m}(x) := \bigoplus_{p=-1}^{-\mu} \mathfrak{g}_p(x)$. Then, $\dim \mathfrak{m}(x) = \dim R$. We set $\mathfrak{g}_p(x) = \{0\}$ when $p \le -\mu - 1$. For $X \in \mathfrak{g}_p(x)$, $Y \in \mathfrak{g}_q(x)$, the Lie bracket $[X, Y] \in \mathfrak{g}_{p+q}(x)$ is defined in the following way: Let $\varpi_p : D^p(x) \to \mathfrak{g}_p(x)$ be the projection of $D^p(x)$ onto $\mathfrak{g}_p(x)$ and $\tilde{X} \in \mathcal{D}^p$, $\tilde{Y} \in \mathcal{D}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{D}^{p+q}$, and we set $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{g}_{p+q}(x)$. It does not depend on the choice of the extensions because of the equation

$$[f\tilde{X}, g\tilde{Y}] = fg[\tilde{X}, \tilde{Y}] + f(\tilde{X}g)\tilde{Y} - g(\tilde{Y}f)\tilde{X} \quad (f, g \in C^\infty(R))\,.$$

The Lie algebra $\mathfrak{m}(x)$ is a nilpotent graded Lie algebra. we call $(\mathfrak{m}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, D) at x. Note that the symbol algebra $(\mathfrak{m}(x),\ [\ ,\ ])$ satisfies the generating conditions $[\mathfrak{g}_p, \mathfrak{g}_{-1}] = \mathfrak{g}_{p-1}$ $(p < 0)$.

Later, Morimoto [6] introduced the notion of a filtered manifold as generalization of the weakly regular differential system. We define a *filtered manifold* (R, F) by a pair of a manifold R and a tangential filtration F. Here, a tangential filtration F on R is a sequence $\{F^p\}_{p<0}$ of subbundles of the tangent bundle TR such that the following conditions are satisfied:

(M1) $TR = F^k = \cdots = F^{-\mu} \supset \cdots \supset F^p \supset F^{p+1} \supset \cdots \supset F^0 = \{0\}$,

(M2) $[\mathcal{F}^p, \mathcal{F}^q] \subset \mathcal{F}^{p+q}$ for all $p,\ q < 0$,

where $\mathcal{F}^p = \Gamma(F^p)$ is the set of sections of F^p.

Let (R, F) be a filtered manifold, for $x \in R$, we set $\mathfrak{f}_p(x) := F^p(x)/F^{p+1}(x)$, and $\mathfrak{f}(x) := \bigoplus_{p<0} \mathfrak{f}_p(x)$. For $X \in \mathfrak{f}_p(x),\ Y \in \mathfrak{f}_q(x)$, Lie bracket $[X, Y] \in \mathfrak{f}_{p+q}(x)$ is defined as follows: Let $\varpi_p : F^p(x) \to \mathfrak{f}_p(x)$ be the projection of $F^p(x)$ onto $\mathfrak{f}_p(x)$, $\tilde{X} \in \mathcal{F}^p,\ \tilde{Y} \in \mathcal{F}^q$ be any extensions such that $\varpi_p(\tilde{X}_x) = X$ and $\varpi_q(\tilde{Y}_x) = Y$. Then $[\tilde{X}, \tilde{Y}] \in \mathcal{F}^{p+q}$, and we set $[X, Y] := \varpi_{p+q}([\tilde{X}, \tilde{Y}]_x) \in \mathfrak{f}_{p+q}(x)$. It does not depend on the choice of the extensions. The Lie algebra $\mathfrak{f}(x)$ is also a nilpotent graded Lie algebra. We call $(\mathfrak{f}(x),\ [\ ,\ ])$ the *symbol algebra* of (R, F) at x. In general $(\mathfrak{f}(x),\ [\ ,\ ])$ does not satisfy the generating conditions.

3.3. Structures of rank 2 prolongations for hyperbolic equations. Let (R, D) be a locally hyperbolic equation, and $(\Sigma(R), \hat{D})$ the rank 2 prolongation. We first explain the geometric meaning of the open covering $P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}} \cup P_{\omega_2\pi_{11}} \cup P_{\pi_{11}\pi_{22}}$ in the proof of Theorem 1. The set $\Sigma(R)$ has a geometric decomposition:

$$\Sigma(R) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2 \quad \text{(disjoint union)}\,. \tag{10}$$

where $\Sigma_i = \{w \in \Sigma(R) \mid \dim\,(w \cap \text{fiber}) = i\}$, $i = 0, 1, 2$, and "fiber" means that the fiber of $TR \supset D \to TJ^1$. Then, locally, we have $\Sigma_0|_{p^{-1}(U)} = P_{\omega_1\omega_2},\ \Sigma_1|_{p^{-1}(U)} = (P_{\omega_1\pi_{22}} \cup P_{\omega_2\pi_{11}}) \backslash P_{\omega_1\omega_2},\ \Sigma_2|_{p^{-1}(U)} = P_{\pi_{11}\pi_{22}} \backslash (P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}} \cup P_{\omega_2\pi_{11}})$. The set Σ_0 is an open subset in $\Sigma(R)$, and is an $\mathbf{R}^2$–bundle over R. The set Σ_1 is a codimension 1 submanifold in $\Sigma(R)$, and is a $(\mathbf{R} \cup \mathbf{R})$-bundle over R. The set Σ_2 is a codimension 2 submanifold in $\Sigma(R)$, and is an section of $\Sigma(R) \to R$.

PROPOSITION 1. *The differential system $\hat{D}$ on $\Sigma(R)$ is regular, but is not weakly regular. More precisely, we obtain that $\hat{D} \subset \partial\hat{D} \subset \partial^2\hat{D} \subset \partial^3\hat{D} = T\Sigma(R)$. Moreover, we have $\partial^2\hat{D} = \partial^{(2)}\hat{D}$, $\partial^{(3)}\hat{D} = T\Sigma(R)$ on $\Sigma_0 \cup \Sigma_1$, and $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$ on Σ_2.*

PROOF. On each component Σ_i in the decomposition (10), we calculate the structure equation of $\hat{D}$. First, we consider it on Σ_0. The canonical system $\hat{D}$ on $U_{\omega_1\omega_2}$ is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\pi_{11}} = \varpi_{\pi_{22}} = 0\}$, where $\varpi_{\pi_{11}} := \pi_{11} - p^1_{11}\omega_1,\ \varpi_{\pi_{22}} := \pi_{22} - p^1_{22}\omega_2$. The structure equation of $\hat{D}$ on Σ_0 is given by

$$\begin{aligned} d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,, \\ d\varpi_{\pi_{11}} &\equiv \omega_1 \wedge (dp^1_{11} + f\omega_2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,, \\ d\varpi_{\pi_{22}} &\equiv \omega_2 \wedge (dp^1_{22} + g\omega_1) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,, \end{aligned}$$

using by appropriate functions f and g since $\pi_{11}, \pi_{22}, \omega_1, \omega_2$ are 1-forms on the base manifold R. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of

$\partial \hat{D}$ is written as

$$\begin{aligned} d\varpi_0 &\equiv 0 && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2\,, \\ d\varpi_1 &\equiv \omega_1 \wedge \varpi_{\pi_{11}} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,, \\ d\varpi_2 &\equiv \omega_2 \wedge \varpi_{\pi_{22}} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,. \end{aligned}$$

Hence we have $\partial^2 \hat{D} = \partial^{(2)} \hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2 \hat{D}$ is described by

$$\begin{aligned} d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \bmod\ & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\pi_{11}},\ \varpi_1 \wedge \varpi_{\pi_{22}}\,, \\ & \varpi_2 \wedge \varpi_{\pi_{11}}, \varpi_2 \wedge \varpi_{\pi_{22}}, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,. \end{aligned}$$

Therefore, we have $\partial^{(3)} \hat{D} = T\Sigma(R)$. Next, we consider on Σ_1. It is sufficient to prove on $U_{\omega_1\pi_{22}}$ because the differential system $\hat{D}$ on $U_{\omega_1\pi_{22}}$ is contact equivalent to the differential system $\hat{D}$ on $U_{\omega_2\pi_{11}}$. The canonical system $\hat{D}$ on $U_{\omega_1\pi_{22}}$ is given by $\hat{D} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_2} = \varpi_{\pi_{11}} = 0\right\}$, where $\varpi_{\omega_2} := \omega_2 - p^3_{12}\pi_{22},\ \varpi_{\pi_{11}} := \pi_{11} - p^3_{21}\omega_1$. For a point $w \in U_{\omega_1\pi_{22}}$, $w \in \Sigma_1$ if and only if $p^3_{12}(w) = 0$. Therefore, it is enough to consider at w in the hypersurface $\left\{p^3_{12} = 0\right\} \subset \Sigma(R)$. The structure equation at a point on $\left\{p^3_{12} = 0\right\}$ is given by

$$\begin{aligned} d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \\ d\varpi_{\omega_2} &\equiv \pi_{22} \wedge (dp^3_{12} + f\omega_1) && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \\ d\varpi_{\pi_{11}} &\equiv \omega_1 \wedge (dp^3_{21} + g\pi_{22}) && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial \hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial \hat{D}$ at a point on $\left\{p^3_{12} = 0\right\}$ is expressed as

$$\begin{aligned} d\varpi_0 &\equiv 0 && \bmod \varpi_0,\ \varpi_1,\ \varpi_2\,, \\ d\varpi_1 &\equiv \omega_1 \wedge \varpi_{\pi_{11}} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,, \\ d\varpi_2 &\equiv \varpi_{\omega_2} \wedge \pi_{22} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,. \end{aligned}$$

Hence we have $\partial^2 \hat{D} = \partial^{(2)} \hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2 \hat{D}$ at a point on $\left\{p^3_{12} = 0\right\}$ is described by

$$\begin{aligned} d\varpi_0 &\equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \\ &\equiv \omega_1 \wedge \varpi_1 + (\varpi_{\omega_2} + p^3_{12}\pi_{22}) \wedge \varpi_2 \\ &\equiv \omega_1 \wedge \varpi_1 + \varpi_{\omega_2} \wedge \varpi_2 \\ &\equiv \omega_1 \wedge \varpi_1 \qquad \bmod\ \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_2},\ \varpi_1 \wedge \varpi_{\pi_{11}}\,, \\ &\qquad\qquad\qquad\qquad \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_2 \wedge \varpi_{\pi_{11}},\ \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,. \end{aligned}$$

Thus, we have $\partial^{(3)}\hat{D} = T\Sigma(R)$. Finally, we consider on Σ_2. The canonical system $\hat{D}$ on $U_{\pi_{11}\pi_{22}}$ is given by $\hat{D} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_1} = \varpi_{\omega_2} = 0\right\}$, where $\varpi_{\omega_1} := \omega_1 - p^6_{11}\pi_{11}$, $\varpi_{\omega_2} := \omega_2 - p^6_{22}\pi_{22}$. For a point $w \in U_{\pi_{11}\pi_{22}}$, $w \in \Sigma_2$ if and only if $p^6_{11}(w) = p^6_{22}(w) = 0$. Therefore, we calculate the structure equation of $\hat{D}$ at a point in codimension 2 submanifold $\left\{p^6_{11} = p^6_{22} = 0\right\} \subset \Sigma(R)$. The structure equation is given by

$$\begin{aligned}
d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
d\varpi_{\omega_1} &\equiv \pi_{11} \wedge (dp^6_{11} + f\pi_{22}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
d\varpi_{\omega_2} &\equiv \pi_{22} \wedge (dp^6_{22} + g\pi_{11}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,
\end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ at a point on $\left\{p^6_{11} = p^6_{22} = 0\right\}$ is written as

$$\begin{aligned}
d\varpi_0 &\equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,,\\
d\varpi_1 &\equiv \varpi_{\omega_1} \wedge \pi_{11} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,\\
d\varpi_2 &\equiv \varpi_{\omega_2} \wedge \pi_{22} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,.
\end{aligned}$$

Hence, we have $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ at a point on $\left\{p^6_{11} = p^6_{22} = 0\right\}$ is described by

$$\begin{aligned}
d\varpi_0 \equiv 0 \quad \mod\ & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2}\,,\\
& \varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,.
\end{aligned}$$

Therefore we obtain $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$. □

From the above proposition, $(\Sigma(R), \hat{D})$ is locally weakly regular around $w \in \Sigma_0 \cup \Sigma_1$. So we can define the symbol algebra at w in the sense of Tanaka and the following holds:

PROPOSITION 2. *For $w \in \Sigma_0$, the symbol algebra $\mathfrak{m}_0(w)$ is isomorphic to $\mathfrak{m}_0$, where $\mathfrak{m}_0 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, whose bracket relations are given by*

$$[X_{p^1_{11}},\ X_{\omega_1}] = X_{\pi_{11}}\,, \quad [X_{p^1_{22}},\ X_{\omega_2}] = X_{\pi_{22}}\,, \quad [X_{\pi_{11}},\ X_{\omega_1}] = X_1\,,$$

$$[X_{\pi_{22}},\ X_{\omega_2}] = X_2\,, \quad [X_1,\ X_{\omega_1}] = [X_2,\ X_{\omega_2}] = X_0\,,$$

and the other brackets are trivial.
Here $\left\{X_0,\ X_1,\ X_2,\ X_{p^1_{11}},\ X_{p^1_{22}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{22}}\right\}$ is a basis of $\mathfrak{m}_0$ and

$$\mathfrak{g}_{-1} = \left\{X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{11}},\ X_{p^1_{22}}\right\},\ \mathfrak{g}_{-2} = \left\{X_{\pi_{11}},\ X_{\pi_{22}}\right\},$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\},\ \mathfrak{g}_{-4} = \left\{X_0\right\}.$$

For $w \in \Sigma_1$, the symbol algebra $\mathfrak{m}_1(w)$ is isomorphic to $\mathfrak{m}_1$, where $\mathfrak{m}_1 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ whose bracket relations are given by

$$[X_{p^3_{12}},\ X_{\pi_{22}}] = X_{\omega_2}\,, \quad [X_{p^3_{21}},\ X_{\omega_1}] = X_{\pi_{11}}\,, \quad [X_{\pi_{11}},\ X_{\omega_1}] = X_1\,,$$

$$[X_{\pi_{22}},\ X_{\omega_2}] = X_2\,, \quad [X_1,\ X_{\omega_1}] = X_0\,,$$

and the other brackets are trivial.
Here $\{X_0,\ X_1,\ X_2,\ X_{p^3_{12}},\ X_{p^3_{21}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{22}}\}$ is a basis of $\mathfrak{m}_1$ and

$$\mathfrak{g}_{-1} = \{X_{\omega_1},\ X_{\pi_{22}},\ X_{p^3_{12}},\ X_{p^3_{21}}\}, \quad \mathfrak{g}_{-2} = \{X_{\omega_2}\ X_{\pi_{11}}\}\,,$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,,\ \mathfrak{g}_{-4} = \{X_0\}\ .$$

PROOF. We first show that $\mathfrak{m}_0(w) \cong \mathfrak{m}_0$. On $U_{\omega_1\omega_2}$ in the proof of Proposition 1, we set $\varpi_{p^1_{11}} := dp^1_{11} + f\omega_2,\ \varpi_{p^1_{22}} := dp^1_{22} + g\omega_1$ and take a coframe:
$\{\varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}},\ \omega_1,\ \omega_2,\ \varpi_{p^1_{11}},\ \varpi_{p^1_{22}}\}$, then the structure equations are given by

$$\begin{aligned}
d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,,\\
d\varpi_{\pi_{11}} &\equiv \omega_1 \wedge \varpi_{p^1_{11}} && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,,\\
d\varpi_{\pi_{22}} &\equiv \omega_2 \wedge \varpi_{p^1_{22}} && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{22}}\,,
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 &\equiv 0 && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2\,,\\
d\varpi_1 &\equiv \omega_1 \wedge \varpi_{\pi_{11}} && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,,\\
d\varpi_2 &\equiv \omega_2 \wedge \varpi_{\pi_{22}} && \text{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,.
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2 \quad \text{mod}\ \ &\varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\pi_{11}},\ \varpi_1 \wedge \varpi_{\pi_{22}}\,,\\
&\varpi_2 \wedge \varpi_{\pi_{11}}, \varpi_2 \wedge \varpi_{\pi_{22}}, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{22}}\,.
\end{aligned}$$

We take the dual frame $\{X_0,\ X_1,\ X_2,\ X_{\pi_{11}},\ X_{\pi_{22}},\ X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{11}},\ X_{p^1_{22}}\}$, and set $[X_{\omega_1}, X_{p^1_{11}}] = A_{11}X_{\pi_{11}} + A_{22}X_{\pi_{22}},\ (A_{ii} \in \mathbf{R})$. Then we have

$$\begin{aligned}
d\varpi_{\pi_{11}}(X_{\omega_1}, X_{p^1_{11}}) &= X_{\omega_1}\varpi_{\pi_{11}}(X_{p^1_{11}}) - X_{p^1_{11}}\varpi_{\pi_{11}}(X_{\omega_1}) - \varpi_{\pi_{11}}([X_{\omega_1}, X_{p^1_{11}}])\,,\\
&= -\varpi_{\pi_{11}}([X_{\omega_1}, X_{p^1_{11}}]) = -A_{11}\,.
\end{aligned}$$

On the other hand, we have

$$d\varpi_{\pi_{11}}(X_{\omega_1}, X_{p^1_{11}}) = \omega_1(X_{\omega_1})\varpi_{p^1_{11}}(X_{p^1_{11}}) - \varpi_{p^1_{11}}(X_{\omega_1})\omega_1(X_{p^1_{11}}) = 1\,.$$

Therefore $A_{11} = -1$. From the same argument for $d\varpi_{\pi_{22}}$, we get $A_{22} = 0$. Hence we have $[X_{\omega_1}, X_{p^1_{11}}] = -X_{\pi_{11}}$. The other brackets are left to reader. Hence its dual frame satisfies the relation with respect to the algebra $\mathfrak{m}_0$.

Next, we show that the isomorphism $\mathfrak{m}_1(w) \cong \mathfrak{m}_1$. On $U_{\omega_1\pi_{22}}$ in the proof of Proposition 1, we set $\varpi_{p^3_{12}} := dp^3_{12} + f\omega_1$, $\varpi_{p^3_{21}} := dp^3_{21} + g\pi_{22}$, and take a coframe and its dual frame $\{\varpi_0, \varpi_1, \varpi_2, \varpi_{\omega_2}, \varpi_{\pi_{11}}, \omega_1, \pi_{22}, \varpi_{p^3_{12}}, \varpi_{p^3_{21}}\}$, $\{X_0, X_1, X_2, X_{\omega_2}, X_{\pi_{11}}, X_{\omega_1}, X_{\pi_{22}}, X_{p^3_{12}}, X_{p^3_{21}}\}$. From the proof of Proposition 1, the structure equations at a point on $\{p^3_{12} = 0\}$ are

$$\begin{aligned} d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \\ d\varpi_{\omega_2} &\equiv \pi_{22} \wedge \varpi_{p^3_{12}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \\ d\varpi_{\pi_{11}} &\equiv \omega_1 \wedge \varpi_{p^3_{21}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_2},\ \varpi_{\pi_{11}}\,, \end{aligned}$$

$$\begin{aligned} d\varpi_0 &\equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,, \\ d\varpi_1 &\equiv \omega_1 \wedge \varpi_{\pi_{11}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,, \\ d\varpi_2 &\equiv \varpi_{\omega_2} \wedge \pi_{22} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,. \end{aligned}$$

$$\begin{aligned} d\varpi_0 \equiv \omega_1 \wedge \varpi_1 \quad \mod\ & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_2},\ \varpi_1 \wedge \varpi_{\pi_{11}}\,, \\ & \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_2 \wedge \varpi_{\pi_{11}},\ \varpi_{\omega_2} \wedge \varpi_{\pi_{11}}\,. \end{aligned}$$

Thus we obtain the statement for $\mathfrak{m}_1$ from the same argument of the proof of $\mathfrak{m}_0$. □

In the rest of this hyperbolic case, we calculate the symbol algebra at a point w in Σ_2. From Proposition 1, D is not weakly regular around $w \in \Sigma_2$. Hence, at the point w, we can not define the symbol algebra in the sense of Tanaka. However, by taking the following filtration F on $\Sigma(R)$, we can define the symbol algebra $\mathfrak{m}_2(w)$ of $(\Sigma(R), F)$ at $w \in \Sigma_2$. We set $F^{-4}(w) = T_w(\Sigma(R))$, $F^{-3}(w) = \partial^{(2)}D(w)$, $F^{-2}(w) = \partial D(w)$, $F^{-1}(w) = D(w)$, where $w \in \Sigma(R)$. Then, $\{F^p\}$ defines the filtration on $\Sigma(R)$. For $w \in \Sigma_2$, we set $\mathfrak{g}_{-1}(w) := F^{-1}(w) = D(w)$, $\mathfrak{g}_{-2}(w) := F^{-2}(w)/F^{-1}(w)$, $\mathfrak{g}_{-3}(w) := F^{-3}(w)/F^{-2}(w)$, $\mathfrak{g}_{-4}(w) := T_w(\Sigma(R))/F^{-3}(w)$, and

$$\mathfrak{m}_2(w) = \mathfrak{g}_{-1}(w) \oplus \mathfrak{g}_{-2}(w) \oplus \mathfrak{g}_{-3}(w) \oplus \mathfrak{g}_{-4}(w)\,.$$

The way of the definition of the above symbol algebra in the sense of Morimoto coincides with the usual symbol algebra except for $[\mathfrak{g}_{-1}, \mathfrak{g}_{-3}]$.

PROPOSITION 3. *For $w \in \Sigma_2$, the symbol algebra $\mathfrak{m}_2(w)$ is isomorphic to $\mathfrak{m}_2$, where $\mathfrak{m}_2 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, whose bracket relations are given by*

$$[X_{p^6_{11}},\ X_{\pi_{11}}] = X_{\omega_1},\ [X_{p^6_{22}},\ X_{\pi_{22}}] = X_{\omega_2},\ [X_{\pi_{11}},\ X_{\omega_1}] = X_1,\ [X_{\pi_{22}},\ X_{\omega_2}] = X_2\,,$$

and the other brackets are trivial.
Here $\{X_0,\ X_1,\ X_2,\ X_{p^6_{11}},\ X_{p^6_{22}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{22}}\}$ is a basis of $\mathfrak{m}_2$ and

$$\mathfrak{g}_{-1} = \{X_{\pi_{11}},\ X_{\pi_{22}},\ X_{p^6_{11}},\ X_{p^6_{22}}\},\ \mathfrak{g}_{-2} = \{X_{\omega_1},\ X_{\omega_2}\}\,,$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,,\ \mathfrak{g}_{-4} = \{X_0\}\ .$$

PROOF. On $U_{\pi_{11}\pi_{22}}$ in the proof of Proposition 1, we set $\varpi_{p^6_{11}} := dp^6_{11} + f\pi_{22},\ \varpi_{p^6_{22}} := dp^6_{22} + g\pi_{11}$ and take a coframe:
$\{\varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},\ \pi_{11},\ \pi_{22},\ \varpi_{\pi^6_{11}},\ \varpi_{\pi^6_{22}}\}$ and its dual frame:
$\{X_0,\ X_1,\ X_2,\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{22}},\ X_{p^6_{11}},\ X_{p^6_{22}}\}$. From the proof of Proposition 1, the structure equations at a point on $\{p^6_{11} = p^6_{22} = 0\}$ are

$$\begin{aligned}
d\varpi_i &\equiv 0 \qquad (i = 0, 1, 2) && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
d\varpi_{\omega_1} &\equiv \pi_{11} \wedge \varpi_{p^6_{11}} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
d\varpi_{\omega_2} &\equiv \pi_{22} \wedge \varpi_{p^6_{22}} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,.
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 &\equiv 0 && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2\,,\\
d\varpi_1 &\equiv \varpi_{\omega_1} \wedge \pi_{11} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,\\
d\varpi_2 &\equiv \varpi_{\omega_2} \wedge \pi_{22} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,.
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 \equiv 0 \quad \text{mod } & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2},\\
& \varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}.
\end{aligned}$$

Thus we have the assertion by the same argument in the proof of Proposition 2. □

3.4. Structures of rank 2 prolongations for parabolic equations. Let (R, D) be a locally parabolic equation, and $(\Sigma(R), \hat{D})$ be the rank 2 prolongation. We use the geometric decomposition (10) of $\Sigma(R)$ which is similar to the hyperbolic case. From Lemma 2, locally, we have $\Sigma_0|_{p^{-1}(U)} = P_{\omega_1\omega_2},\ \Sigma_1|_{p^{-1}(U)} = P_{\omega_1\pi_{22}} \backslash P_{\omega_1\omega_2}$, and $\Sigma_2|_{p^{-1}(U)} = P_{\pi_{12}\pi_{22}} \backslash (P_{\omega_1\omega_2} \cup P_{\omega_1\pi_{22}})$, where p is the projection of the fibration $\Sigma(R) \to R$. The set Σ_0 is an open set in $\Sigma(R)$, and is an $\mathbf{R}^2$–bundle over R. The set Σ_1 is a submanifold in $J(D, 2)$ and contains singular points of $\Sigma(R)$ in $J(D, 2)$ and is an $\mathbf{R}$-bundle over R. The set Σ_2 is codimension 2 submanifold in $\Sigma(R)$, and is a section of $\Sigma(R) \to R$. We investigate the geometric structures of $(\Sigma(R), \hat{D})$ on a domain except for singular points in Σ_1.

PROPOSITION 4. *The differential system $\hat{D}$ on $\Sigma(R)$ is regular, but is not weakly regular. More precisely, we obtain that $\hat{D} \subset \partial\hat{D} \subset \partial^2\hat{D} \subset \partial^3\hat{D} = T\Sigma(R)$. Moreover, we have $\partial^2\hat{D} = \partial^{(2)}\hat{D}$, $\partial^{(3)}\hat{D} = T\Sigma(R)$ on $\Sigma_0 \cup \Sigma_1$, and $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$ on Σ_2.*

PROOF. On each component Σ_i in the decomposition, we calculate the structure equation of $\hat{D}$. First, we consider it on Σ_0. The canonical system $\hat{D}$ on $U_{\omega_1\omega_2}$ is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\pi_{12}} = \varpi_{\pi_{22}} = 0\}$, where $\varpi_{\pi_{12}} := \pi_{12} - p^1_{12}\omega_2,\quad \varpi_{\pi_{22}} :=$

$\pi_{22} - p^1_{12}\omega_1 - p^1_{22}\omega_2$. The structure equation of $\hat{D}$ on Σ_0 is written as

$$\begin{aligned} d\varpi_i &\equiv 0 \quad (i=0,1,2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}}\,, \\ d\varpi_{\pi_{12}} &\equiv \omega_2 \wedge (dp^1_{12} + f\omega_1) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}}\,, \\ d\varpi_{\pi_{22}} &\equiv g\omega_1 \wedge \omega_2 - dp^1_{12} \wedge \omega_1 - dp^1_{22} \wedge \omega_2 && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}}\,. \\ &\equiv -(dp^1_{12} + f\omega_1) \wedge \omega_1 - (dp^1_{22} - g\omega_1) \wedge \omega_2\,, \end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is expressed as

$$\begin{aligned} d\varpi_0 &\equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,, \\ d\varpi_1 &\equiv \omega_2 \wedge \varpi_{\pi_{12}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,, \\ d\varpi_2 &\equiv \omega_1 \wedge \varpi_{\pi_{12}} + \omega_2 \wedge \varpi_{\pi_{22}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,. \end{aligned}$$

Hence we have $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ is described by

$$\begin{aligned} d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2, \quad \mod\ & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\pi_{12}},\ \varpi_1 \wedge \varpi_{\pi_{22}}\,, \\ & \varpi_2 \wedge \varpi_{\pi_{12}},\ \varpi_2 \wedge \varpi_{\pi_{22}},\ \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,. \end{aligned}$$

Therefore, we obtain $\partial^{(3)}\hat{D} = T\Sigma(R)$. Next, we consider on Σ_1. It is enough to work on $U_{\pi_{12}\pi_{22}}$ since $\Sigma_1 \backslash \{\text{singular points}\}$ is covered by $U_{\pi_{12}\pi_{22}}$. The canonical system $\hat{D}$ on $U_{\pi_{12}\pi_{22}}$ is given by $\hat{D} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_1} = \varpi_{\omega_2} = 0\right\}$, where $\varpi_{\omega_1} := \omega_1 - p^6_{11}\pi_{12} - p^6_{12}\pi_{22}$, $\varpi_{\omega_2} := \omega_2 - p^6_{12}\pi_{12}$. For $w \in U_{\pi_{12}\pi_{22}}$, $w \in \Sigma_1$ if and only if $p^6_{11}(w) \neq 0, p^6_{12}(w) = 0$. Because, $w \in \Sigma_2$ is given by the coordinate $p^6_{11}(w) = 0, p^6_{12}(w) = 0$, and $w \in \Sigma_1 \backslash \Sigma_0$ is given by $p^6_{12}(w) = 0$. Therefore, we calculate the structure equation at w in the hypersurface $\left\{p^6_{11} \neq 0, p^6_{12} = 0\right\} \subset \Sigma(R)$. The structure equation at a point on $\left\{p^6_{11} \neq 0, p^6_{12} = 0\right\}$ is

$$\begin{aligned} d\varpi_i &\equiv 0 \quad (i=0,1,2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,, \\ d\varpi_{\omega_1} &\equiv \pi_{12} \wedge (dp^6_{11} + f\pi_{22}) + \pi_{22} \wedge (dp^6_{12} + g\pi_{22}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,, \\ d\varpi_{\omega_2} &\equiv \pi_{12} \wedge (dp^6_{12} + g\pi_{22}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,, \end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is given by

$$\begin{aligned} d\varpi_0 &\equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,, \\ d\varpi_1 &\equiv \varpi_{\omega_2} \wedge \pi_{12} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,, \\ d\varpi_2 &\equiv \varpi_{\omega_1} \wedge \pi_{12} + \varpi_{\omega_2} \wedge \pi_{22} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,. \end{aligned}$$

Hence we have $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ is written as

$$d\varpi_0 \equiv p^6_{11}\pi_{12}\wedge\varpi_1 \quad \text{mod } \varpi_0,\ \varpi_1\wedge\varpi_2,\ \varpi_1\wedge\varpi_{\omega_1},\ \varpi_1\wedge\varpi_{\omega_2},\ \varpi_2\wedge\varpi_{\omega_1},\ \varpi_2\wedge\varpi_{\omega_2},\ \varpi_{\omega_1}\wedge\varpi_{\omega_2}.$$

Here, if we set $\varpi'_0 := \varpi_0/p^6_{11}$, then structure equation is rewritten in the form:

$$d\varpi'_0 \equiv \pi_{12}\wedge\varpi_1 \quad \text{mod } \varpi'_0,\ \varpi_1\wedge\varpi_2,\ \varpi_1\wedge\varpi_{\omega_1},\ \varpi_1\wedge\varpi_{\omega_2},\ \varpi_2\wedge\varpi_{\omega_1},\ \varpi_2\wedge\varpi_{\omega_2},\ \varpi_{\omega_1}\wedge\varpi_{\omega_2}.$$

Hence we have $\partial^{(3)}\hat{D} = T\Sigma(R)$. Finally, we consider on Σ_2. We use the coordinate on $U_{\pi_{12}\pi_{22}}$. For $w \in U_{\pi_{12}\pi_{22}}$, $w \in \Sigma_2$ if and only if $p^6_{11}(w) = p^6_{12}(w) = 0$. Therefore, we calculate the structure equation at w in the codimension 2 submanifold $\{p^6_{11} = p^6_{12} = 0\} \subset \Sigma(R)$. The structure equation at a point on $\{p^6_{11} = p^6_{12} = 0\}$ is described by

$$\begin{aligned}
&d\varpi_i \equiv 0 \quad (i = 0, 1, 2) && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},\\
&d\varpi_{\omega_1} \equiv \pi_{12}\wedge(dp^6_{11} + f\pi_{22}) + \pi_{22}\wedge(dp^6_{12} + g\pi_{22}) && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},\\
&d\varpi_{\omega_2} \equiv \pi_{12}\wedge(dp^6_{12} + g\pi_{22}) && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},
\end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is given by

$$\begin{aligned}
&d\varpi_0 \equiv 0 && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\\
&d\varpi_1 \equiv \varpi_{\omega_2}\wedge\pi_{12} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1}\wedge\varpi_{\omega_2},\\
&d\varpi_2 \equiv \varpi_{\omega_1}\wedge\pi_{12} + \varpi_{\omega_2}\wedge\pi_{22} && \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1}\wedge\varpi_{\omega_2}.
\end{aligned}$$

Therefore, we get $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ is expressed as

$$d\varpi_0 \equiv 0 \quad \text{mod } \varpi_0,\ \varpi_1\wedge\varpi_2,\ \varpi_1\wedge\varpi_{\omega_1},\ \varpi_1\wedge\varpi_{\omega_2},\ \varpi_2\wedge\varpi_{\omega_1},\ \varpi_2\wedge\varpi_{\omega_2},\ \varpi_{\omega_1}\wedge\varpi_{\omega_2}.$$

Hence we have $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$. □

From the above proposition, $(\Sigma(R), \hat{D})$ is locally weakly regular around $w \in \Sigma_0 \cup \Sigma_1$. So we define the symbol algebra at w. On the other hand, for a point w on Σ_2, $(\Sigma(R), \hat{D})$ is not weakly regular around w. However, by taking the filtration on $\Sigma(R)$ which is same to the hyperbolic case, we can define the symbol algebra at w. Each structure of symbol algebras is given in the following.

PROPOSITION 5. *For $w \in \Sigma_0$, the symbol algebra $\mathfrak{m}_0(w)$ is isomorphic to $\mathfrak{m}_0$, where $\mathfrak{m}_0 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, whose bracket relations are given by*

$$[X_{p^1_{12}}, X_{\omega_2}] = X_{\pi_{12}}, \quad [X_{p^1_{12}}, X_{\omega_1}] = [X_{p^1_{22}}, X_{\omega_2}] = X_{\pi_{22}}, \quad [X_{\pi_{12}}, X_{\omega_2}] = X_1,$$

$$[X_{\pi_{12}},\ X_{\omega_1}] = [X_{\pi_{22}},\ X_{\omega_2}] = X_2\,, \quad [X_1,\ X_{\omega_1}] = [X_2,\ X_{\omega_2}] = X_0\,,$$

and the other brackets are trivial.
Here $\left\{X_0,\ X_1,\ X_2,\ X_{p^1_{12}},\ X_{p^1_{22}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{12}},\ X_{\pi_{22}}\right\}$ *is a basis of* $\mathfrak{m}_0$ *and*

$$\mathfrak{g}_{-1} = \left\{X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{12}},\ X_{p^1_{22}}\right\}, \quad \mathfrak{g}_{-2} = \left\{X_{\pi_{12}},\ X_{\pi_{22}}\right\}\,,$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,, \mathfrak{g}_{-4} = \{X_0\}\ .$$

For $w \in \Sigma_1$, *the symbol algebra* $\mathfrak{m}_1(w)$ *is isomorphic to* $\mathfrak{m}_1$, *where* $\mathfrak{m}_1 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, *whose bracket relations are given by*

$$[X_{p^6_{11}},\ X_{\pi_{12}}] = [X_{p^6_{12}},\ X_{\pi_{22}}] = X_{\omega_1}\,, \quad [X_{p^6_{12}},\ X_{\pi_{12}}] = X_{\omega_2}\,,$$

$$[X_{\pi_{12}},\ X_{\omega_2}] = X_1\,, \quad [X_{\pi_{12}},\ X_{\omega_1}] = [X_{\pi_{22}},\ X_{\omega_2}] = X_2\,, \quad [X_1, X_{\pi_{12}}] = X_0\,,$$

and the other brackets are trivial.
Here $\left\{X_0,\ X_1,\ X_2,\ X_{p^6_{11}},\ X_{p^6_{12}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{12}},\ X_{\pi_{22}}\right\}$ *is a basis of* $\mathfrak{m}_1$ *and*

$$\mathfrak{g}_{-1} = \left\{X_{\pi_{12}},\ X_{\pi_{22}},\ X_{p^6_{11}},\ X_{p^6_{12}}\right\},\ \mathfrak{g}_{-2} = \left\{X_{\omega_1},\ X_{\omega_2}\right\},$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,, \quad \mathfrak{g}_{-4} = \{X_0\}\ .$$

For $w \in \Sigma_2$, *the symbol algebra* $\mathfrak{m}_2(w)$ *is isomorphic to* $\mathfrak{m}_2$, *where* $\mathfrak{m}_2 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, *whose bracket relations are given by*

$$[X_{p^6_{11}},\ X_{\pi_{12}}] = [X_{p^6_{12}},\ X_{\pi_{22}}] = X_{\omega_1}, \quad [X_{p^6_{12}},\ X_{\pi_{12}}] = X_{\omega_2}\,,$$

$$[X_{\pi_{12}},\ X_{\omega_2}] = X_1\,, \quad [X_{\pi_{12}},\ X_{\omega_1}] = [X_{\pi_{22}},\ X_{\omega_2}] = X_2\,,$$

and the other brackets are trivial.
Here $\left\{X_0,\ X_1,\ X_2,\ X_{p^6_{11}},\ X_{p^6_{12}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{12}},\ X_{\pi_{22}}\right\}$ *is a basis of* $\mathfrak{m}_2$ *and*

$$\mathfrak{g}_{-1} = \left\{X_{\pi_{12}},\ X_{\pi_{22}},\ X_{p^6_{11}},\ X_{p^6_{12}}\right\},\ \mathfrak{g}_{-2} = \left\{X_{\omega_1},\ X_{\omega_2}\right\}\,,$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,, \quad \mathfrak{g}_{-4} = \{X_0\}\ .$$

PROOF. We first show that $\mathfrak{m}_0(w) \cong \mathfrak{m}_0$ for $w \in \Sigma_0$. On $U_{\omega_1\omega_2}$ in the proof of Proposition 4, we set $\varpi_{p^1_{12}} := dp^1_{12} + f\omega_1,\ \varpi_{p^1_{22}} := dp^1_{22} - g\omega_2$, and take a coframe: $\left\{\varpi_0, \varpi_1, \varpi_2, \varpi_{\pi_{12}}, \varpi_{\pi_{22}}, \omega_1, \omega_2, \varpi_{p^1_{12}}, \varpi_{p^1_{22}}\right\}$, then the structure equations are given by

$$d\varpi_i \equiv 0 \quad (i = 0, 1, 2) \qquad \mod \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}},$$

$$d\varpi_{\pi_{12}} \equiv \omega_2 \wedge \varpi_{p^1_{12}} \qquad \mod \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}}\,,$$

$$d\varpi_{\pi_{22}} \equiv -\varpi_{p^1_{12}} \wedge \omega_1 - \varpi_{p^1_{22}} \wedge \omega_2 \qquad \mod \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{12}},\ \varpi_{\pi_{22}}\ .$$

$$d\varpi_0 \equiv 0 \qquad \mod \ \varpi_0,\ \varpi_1,\ \varpi_2\,,$$

$$d\varpi_1 \equiv \omega_2 \wedge \varpi_{\pi_{12}} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,,$$
$$d\varpi_2 \equiv \omega_1 \wedge \varpi_{\pi_{12}} + \omega_2 \wedge \varpi_{\pi_{22}} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,,$$

$$d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2, \quad \text{mod } \varpi_0,\ \varpi_1 \wedge \varpi_2\,,\quad \varpi_1 \wedge \varpi_{\pi_{12}},\ \varpi_1 \wedge \varpi_{\pi_{22}}\,,$$
$$\varpi_2 \wedge \varpi_{\pi_{12}},\ \varpi_2 \wedge \varpi_{\pi_{22}},\ \varpi_{\pi_{12}} \wedge \varpi_{\pi_{22}}\,.$$

We take the dual frame: $\{X_0,\ X_1,\ X_2,\ X_{\pi_{12}},\ X_{\pi_{22}},\ X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{12}},\ X_{p^1_{22}}\}$. Then, by using the same argument to the hyperbolic case, we have the bracket relations of $\mathfrak{m}_0$.

Next, we show that the isomorphism $\mathfrak{m}_1(w) \cong \mathfrak{m}_1$ for a point on Σ_1. On $U_{\pi_{12}\pi_{22}}$ in the proof of Proposition 4, we set $\varpi_{p^6_{11}} := dp^6_{11} + f\pi_{22}$, $\varpi_{p^6_{12}} := dp^6_{12} + g\pi_{22}$, and take a coframe $\{\varpi'_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},\ \pi_{12},\ \pi_{22},\ \varpi_{p^6_{11}},\ \varpi_{p^6_{12}}\}$. and its dual frame $\{X_0,\ X_1,\ X_2,\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{12}},\ X_{\pi_{22}},\ X_{p^6_{11}},\ X_{p^6_{12}}\}$. From the proof of Proposition 4, the structure equations at a point on $\{p^6_{11} \neq 0,\ p^6_{12} = 0\}$ are described by

$$d\varpi'_0 \equiv 0 \qquad \text{mod } \varpi'_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,$$
$$d\varpi_i \equiv 0 \quad (i = 1, 2) \qquad \text{mod } \varpi'_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,$$
$$d\varpi_{\omega_1} \equiv \pi_{12} \wedge \varpi_{p^6_{11}} + \pi_{22} \wedge \varpi_{p^6_{12}} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,$$
$$d\varpi_{\omega_2} \equiv \pi_{12} \wedge \varpi_{p^6_{12}} \qquad \text{mod } \varpi'_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,.$$

$$d\varpi'_0 \equiv 0 \qquad \text{mod } \varpi'_0,\ \varpi_1,\ \varpi_2\,,$$
$$d\varpi_1 \equiv \varpi_{\omega_2} \wedge \pi_{12} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,$$
$$d\varpi_2 \equiv \varpi_{\omega_1} \wedge \pi_{12} + \varpi_{\omega_2} \wedge \pi_{22} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,$$

$d\varpi'_0 \equiv \pi_{12} \wedge \varpi_1$ mod $\varpi'_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2},\ \varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}$.

Then, by using the same argument to the hyperbolic case, we have the bracket relations of $\mathfrak{m}_1$.

Finally, we prove the statement for $\mathfrak{m}_2$. We use the coordinate on $U_{\pi_{12}\pi_{22}}$ which is same to the case of Σ_1. From the proof of Proposition 4, the structure equations at a point on $\{p^6_{11} = p^6_{12} = 0\}$ are expressed as

$$d\varpi_i \equiv 0 \quad (i = 0, 1, 2) \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,$$
$$d\varpi_{\omega_1} \equiv \pi_{12} \wedge \varpi_{p^6_{11}} + \pi_{22} \wedge \varpi_{p^6_{12}} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,$$
$$d\varpi_{\omega_2} \equiv \pi_{12} \wedge \varpi_{p^6_{12}} \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,.$$

$$d\varpi_0 \equiv 0 \qquad \text{mod } \varpi_0,\ \varpi_1,\ \varpi_2,$$

$$d\varpi_1 \equiv \varpi_{\omega_2} \wedge \pi_{12} \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,$$

$$d\varpi_2 \equiv \varpi_{\omega_1} \wedge \pi_{12} + \varpi_{\omega_2} \wedge \pi_{22} \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,$$

$$\begin{aligned} d\varpi_0 \equiv 0 \quad \mathrm{mod}\ \ &\varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2}, \\ &\varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,. \end{aligned}$$

Thus we have the statement for $\mathfrak{m}_2(w)$ from the same argument. □

3.5. Structures of rank 2 prolongations for elliptic equations. Let (R, D) be a locally elliptic equation and $(\Sigma(R), \hat{D})$ the rank 2 prolongation. We use the geometric decomposition (10) of $\Sigma(R)$ which is similar to the hyperbolic case. From Lemma 3, locally, we have $\Sigma 0|_{p^{-1}(U)} = P_{\omega_1\omega_2}$, $\quad \Sigma 2|_{p^{-1}(U)} = P_{\pi_{11}\pi_{12}} \backslash P_{\omega_1\omega_2}$, where p is the projection of the fibration $\Sigma(R) \to R$. The set Σ_0 is an open set in $\Sigma(R)$, and is an $\mathbf{R}^2$–bundle over R. The set Σ_2 is a codimension 2 submanifold of $\Sigma(R)$ and is a section of $\Sigma(R) \to R$.

PROPOSITION 6. *The differential system $\hat{D}$ on $\Sigma(R)$ is regular, but is not weakly regular. More precisely, we obtain that $\hat{D} \subset \partial\hat{D} \subset \partial^2\hat{D} \subset \partial^3\hat{D} = T\Sigma(R)$. Moreover, we have $\partial^2\hat{D} = \partial^{(2)}\hat{D}$, $\partial^{(3)}\hat{D} = T\Sigma(R)$ on Σ_0, and $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$ on Σ_2.*

PROOF. On each component Σ_i in the decomposition, we calculate the structure equation of $\hat{D}$. First, we consider it on Σ_0. The canonical system $\hat{D}$ on $U_{\omega_1\omega_2}$ is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\pi_{11}} = \varpi_{\pi_{12}} = 0\}$, where $\varpi_{\pi_{11}} := \pi_{11} - p^1_{11}\omega_1 - p^1_{12}\omega_2\,$, $\quad \varpi_{\pi_{12}} := \pi_{12} - p^1_{12}\omega_1 + p^1_{11}\omega_2$. The structure equation of $\hat{D}$ on Σ_0 is given by

$$d\varpi_i \equiv 0 \quad (i = 0, 1, 2) \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}},$$

$$d\varpi_{\pi_{11}} \equiv \omega_1 \wedge (dp^1_{11} + f\omega_2) + \omega_2 \wedge (dp^1_{12} + g\omega_2) \quad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}}\,,$$

$$d\varpi_{\pi_{12}} \equiv \omega_1 \wedge (dp^1_{12} + g\omega_2) - \omega_2 \wedge (dp^1_{11} + f\omega_2) \quad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}}\,,$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is written as

$$d\varpi_0 \equiv 0 \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2\,,$$

$$d\varpi_1 \equiv \omega_1 \wedge \varpi_{\pi_{11}} + \omega_2 \wedge \varpi_{\pi_{12}} \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,,$$

$$d\varpi_2 \equiv \omega_1 \wedge \varpi_{\pi_{12}} - \omega_2 \wedge \varpi_{\pi_{11}} \qquad \mathrm{mod}\ \ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,.$$

Hence we have $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ is expressed as

$$\begin{aligned} d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2,\ \mathrm{mod}\ &\varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\pi_{11}},\ \varpi_1 \wedge \varpi_{\pi_{12}}\,, \\ &\varpi_2 \wedge \varpi_{\pi_{11}},\ \varpi_2 \wedge \varpi_{\pi_{12}},\ \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,. \end{aligned}$$

Hence, we have $\partial^{(3)}\hat{D} = T\Sigma(R)$. Next we consider on Σ_2. The canonical system $\hat{D}$ on $U_{\pi_{11}\pi_{12}}$ is given by $\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_{\omega_1} = \varpi_{\omega_2} = 0\}$, where $\varpi_{\omega_1} := \omega_1 -$

$p^6_{11}\pi_{11} - p^6_{12}\pi_{12}$, $\varpi_{\omega_2} := \omega_2 - p^6_{12}\pi_{11} + p^6_{11}\pi_{12}$. For $w \in U_{\pi_{11}\pi_{12}}$, $w \in \Sigma_2$ if and only if $p^6_{11}(w) = p^6_{12}(w) = 0$. Therefore, we calculate the structure equation at w in the codimension 2 submanifold $\{p^6_{11} = p^6_{12} = 0\} \subset \Sigma(R)$. The structure equation at a point on $\{p^6_{11} = p^6_{12} = 0\}$ is described by

$$\begin{aligned}
&d\varpi_i \equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
&d\varpi_{\omega_1} \equiv \pi_{11} \wedge (dp^6_{11} + f\pi_{12}) + \pi_{12} \wedge (dp^6_{12} + g\pi_{12}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
&d\varpi_{\omega_2} \equiv \pi_{11} \wedge (dp^6_{12} + g\pi_{12}) - \pi_{12} \wedge (dp^6_{11} + f\pi_{12}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,
\end{aligned}$$

where f and g are appropriate functions. Hence we have $\partial\hat{D} = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\} = p_*^{-1}(D)$. The structure equation of $\partial\hat{D}$ is

$$\begin{aligned}
&d\varpi_0 \equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,,\\
&d\varpi_1 \equiv \varpi_{\omega_1} \wedge \pi_{11} + \varpi_{\omega_2} \wedge \pi_{12} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,\\
&d\varpi_2 \equiv \varpi_{\omega_1} \wedge \pi_{12} - \varpi_{\omega_2} \wedge \pi_{11} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,.
\end{aligned}$$

Hence we have $\partial^2\hat{D} = \partial^{(2)}\hat{D} = \{\varpi_0 = 0\}$. The structure equation of $\partial^2\hat{D}$ is given by

$$\begin{aligned}
d\varpi_0 \equiv 0 \quad \mod\ & \varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2}\,,\\
& \varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,.
\end{aligned}$$

Thus, we have $\partial^{(3)}\hat{D} = \partial^{(2)}\hat{D}$. □

From the above proposition, $(\Sigma(R), \hat{D})$ is locally weakly regular around $w \in \Sigma_0$. So we can define the symbol algebra at w in the sense of Tanaka. On the other hand, for a point w on Σ_2, $(\Sigma(R), \hat{D})$ is not weakly regular around w. However, by taking the filtration on $\Sigma(R)$ which is same to the hyperbolic case, we can define the symbol algebra at w. Each structure of symbol algebras is given in the following.

PROPOSITION 7. *For $w \in \Sigma_0$, the symbol algebra $\mathfrak{m}_0(w)$ is isomorphic to $\mathfrak{m}_0$, where $\mathfrak{m}_0 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, whose bracket relations are given by*

$$[X_{p^1_{11}},\ X_{\omega_1}] = [X_{p^1_{12}},\ X_{\omega_2}] = X_{\pi_{11}}\,, \quad [X_{p^1_{12}},\ X_{\omega_1}] = [X_{\omega_2},\ X_{p^1_{11}}] = X_{\pi_{12}}\,,$$

$$[X_{\pi_{11}},\ X_{\omega_1}] = [X_{\pi_{12}},\ X_{\omega_2}] = X_1\,, \quad [X_{\pi_{12}},\ X_{\omega_1}] = [X_{\omega_2},\ X_{\pi_{11}}] = X_2\,,$$

$[X_1,\ X_{\omega_1}] = [X_2,\ X_{\omega_2}] = X_0$, *and the other brackets are trivial.*
Here $\{X_0,\ X_1,\ X_2,\ X_{p^1_{11}},\ X_{p^1_{12}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{12}}\}$ is a basis of $\mathfrak{m}_0$ and

$$\mathfrak{g}_{-1} = \{X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{11}},\ X_{p^1_{12}}\},\ \mathfrak{g}_{-2} = \{X_{\pi_{11}},\ X_{\pi_{12}}\}\,,$$

$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,,\ \mathfrak{g}_{-4} = \{X_0\}\,.$$

For $w \in \Sigma_2$, the symbol algebra $\mathfrak{m}_2(w)$ is isomorphic to $\mathfrak{m}_2$, where $\mathfrak{m}_2 = \mathfrak{g}_{-4} \oplus \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, whose bracket relations are given by

$$[X_{p^6_{11}},\ X_{\pi_{11}}] = [X_{p^6_{12}},\ X_{\pi_{12}}] = X_{\omega_1}\,, \quad [X_{p^6_{12}},\ X_{\pi_{11}}] = [X_{\pi_{12}},\ X_{p^6_{11}}] = X_{\omega_2}\,,$$

$$[X_{\pi_{11}},\ X_{\omega_1}] = [X_{\pi_{12}},\ X_{\omega_2}] = X_1\,, \quad [X_{\pi_{12}},\ X_{\omega_1}] = [X_{\omega_2},\ X_{\pi_{11}}] = X_2\,,$$

and the other brackets are trivial.
Here $\{X_0,\ X_1,\ X_2,\ X_{p^6_{11}},\ X_{p^6_{12}},\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{12}}\}$ is a basis of $\mathfrak{m}_2$ and

$$\mathfrak{g}_{-1} = \{X_{\pi_{11}},\ X_{\pi_{12}},\ X_{p^6_{11}},\ X_{p^6_{12}}\},\ \mathfrak{g}_{-2} = \{X_{\omega_1},\ X_{\omega_2}\}\,,$$
$$\mathfrak{g}_{-3} = \{X_1,\ X_2\}\,,\ \mathfrak{g}_{-4} = \{X_0\}\,.$$

PROOF. We first show that $\mathfrak{m}_0(m) \cong \mathfrak{m}_0$. On $U_{\omega_1\omega_2}$ in the proof of Proposition 6, if we set $\varpi_{p^1_{11}} := dp^1_{11} + f\omega_2,\ \varpi_{p^1_{12}} := dp^1_{12} + g\omega_2$ and take a coframe:

$$\{\varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}},\ \omega_1,\ \omega_2,\ \varpi_{p^1_{11}},\ \varpi_{p^1_{12}}\},$$

then the structure equations are written as

$$\begin{aligned}
d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}}\,,\\
d\varpi_{\pi_{11}} &\equiv \omega_1 \wedge \varpi_{p^1_{11}} + \omega_2 \wedge \varpi_{p^1_{12}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}}\,,\\
d\varpi_{\pi_{12}} &\equiv \omega_1 \wedge \varpi_{p^1_{12}} - \omega_2 \wedge \varpi_{p^1_{11}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\pi_{11}},\ \varpi_{\pi_{12}}\,.
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 &\equiv 0 && \mod \varpi_0,\ \varpi_1,\ \varpi_2\,,\\
d\varpi_1 &\equiv \omega_1 \wedge \varpi_{\pi_{11}} + \omega_2 \wedge \varpi_{\pi_{12}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,,\\
d\varpi_2 &\equiv \omega_1 \wedge \varpi_{\pi_{12}} - \omega_2 \wedge \varpi_{\pi_{11}} && \mod \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,,
\end{aligned}$$

$$\begin{aligned}
d\varpi_0 \equiv \omega_1 \wedge \varpi_1 + \omega_2 \wedge \varpi_2, \quad \mod\ &\varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\pi_{11}},\ \varpi_1 \wedge \varpi_{\pi_{12}}\,,\\
&\varpi_2 \wedge \varpi_{\pi_{11}},\ \varpi_2 \wedge \varpi_{\pi_{12}},\ \varpi_{\pi_{11}} \wedge \varpi_{\pi_{12}}\,.
\end{aligned}$$

We take the dual frame $\{X_0,\ X_1,\ X_2,\ X_{\pi_{11}},\ X_{\pi_{12}},\ X_{\omega_1},\ X_{\omega_2},\ X_{p^1_{11}},\ X_{p^1_{12}}\}$. Then, by the same argument to the hyperbolic case, we have the bracket relations of $\mathfrak{m}_0$.

Next, we prove the statement for the algebra $\mathfrak{m}_2$. On $U_{\pi_{11}\pi_{12}}$ in the proof of Proposition 6, we set $\varpi_{p^6_{11}} := dp^6_{11} + f\pi_{12},\ \varpi_{p^6_{12}} := dp^6_{12} + g\pi_{12}$ and take a coframe: $\{\varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2},\ \pi_{11},\ \pi_{12},\ \varpi_{p^6_{11}},\ \varpi_{p^6_{12}}\}$, then the structure equations at a point on Σ_2 are given by

$$\begin{aligned}
d\varpi_i &\equiv 0 \quad (i = 0, 1, 2) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,\\
d\varpi_{\omega_1} &\equiv \pi_{11} \wedge (dp^6_{11} + f\pi_{12}) + \pi_{12} \wedge (dp^6_{12} + g\pi_{12}) && \mod \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,,
\end{aligned}$$

$$d\varpi_{\omega_2} \equiv \pi_{11} \wedge (dp^6_{12} + g\pi_{12}) - \pi_{12} \wedge (dp^6_{11} + f\pi_{12}) \qquad \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2,\ \varpi_{\omega_1},\ \varpi_{\omega_2}\,.$$

$$\begin{aligned} d\varpi_0 &\equiv 0 && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2\,,\\ d\varpi_1 &\equiv \varpi_{\omega_1} \wedge \pi_{11} + \varpi_{\omega_2} \wedge \pi_{12} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,,\\ d\varpi_2 &\equiv \varpi_{\omega_1} \wedge \pi_{12} - \varpi_{\omega_2} \wedge \pi_{11} && \bmod\ \varpi_0,\ \varpi_1,\ \varpi_2, \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,, \end{aligned}$$

$$\begin{aligned} d\varpi_0 \equiv 0 \quad \bmod\ &\varpi_0,\ \varpi_1 \wedge \varpi_2,\ \varpi_1 \wedge \varpi_{\omega_1},\ \varpi_1 \wedge \varpi_{\omega_2}\,,\\ &\varpi_2 \wedge \varpi_{\omega_1},\ \varpi_2 \wedge \varpi_{\omega_2},\ \varpi_{\omega_1} \wedge \varpi_{\omega_2}\,. \end{aligned}$$

Let $\{X_0,\ X_1,\ X_2,\ X_{\omega_1},\ X_{\omega_2},\ X_{\pi_{11}},\ X_{\pi_{12}},\ X_{p^6_{11}},\ X_{p^6_{12}}\}$ be the dual frame. Then, by using the same argument to the hyperbolic case, we have the bracket relations of $\mathfrak{m}_2$. □

4. Construction of singular solutions and the theory of submanifold of the rank 2 prolongation of the Second Jet space

In sections 2 and 3, we studied various properties of the rank 2 prolongations $(\Sigma(R), \hat{D})$ of single equations (R, D). Under these prolongations, we mention the strategy of the construction of the geometric singular solutions for each class of equations (R, D). Moreover, we construct singular solutions for model equations belonging to each class. For this purpose, we first consider the rank 2 prolongation $\Sigma(J^2)$ of the second jet space $J^2(\mathbf{R}^2, \mathbf{R})$. For the 2-jet space $J^2(\mathbf{R}^2, \mathbf{R})$, we denote the rank 2 prolongation of $J^2(\mathbf{R}^2, \mathbf{R})$ by $(\Sigma(J^2), \hat{C}^2)$. This space $\Sigma(J^2)$ is a submanifold of the Grassmann bundle $J(C^2, 2)$. The geometry of $(\Sigma(J^2), \hat{C}^2)$ in $J(C^2, 2)$ is studied in [9]. From now on, we refer to [9] for the obtained results. For an open set $V \subset J^2(\mathbf{R}^2, \mathbf{R})$, ${\Pi^2_1}^{-1}(V)$ is covered by 6 open sets:

$${\Pi^2_1}^{-1}(V) = V_{xy} \cup V_{xt} \cup V_{yr} \cup V_{rs} \cup V_{rt} \cup V_{st}\,,$$

where $\Pi^2_1 : \Sigma(J^2) \to J^2$ is the projection and each open set is given by

$$\begin{aligned} V_{xy} &:= \{w \in {\Pi^2_1}^{-1}(V) \mid dx \wedge dy|_w \neq 0\},\ V_{xt} := \{w \in {\Pi^2_1}^{-1}(V) \mid dx \wedge dt|_w \neq 0\},\\ V_{yr} &:= \{w \in {\Pi^2_1}^{-1}(V) \mid dy \wedge dr|_w \neq 0\},\ V_{rs} := \{w \in {\Pi^2_1}^{-1}(V) \mid dr \wedge ds|_w \neq 0\},\\ V_{rt} &:= \{w \in {\Pi^2_1}^{-1}(V) \mid dr \wedge dt|_w \neq 0\},\ V_{st} := \{w \in {\Pi^2_1}^{-1}(V) \mid ds \wedge dt|_w \neq 0\}. \end{aligned}$$

The prolongation $\Sigma(J^2)$ has the similar geometric decomposition: $\Sigma(J^2) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$, where $\Sigma_i = \{w \in \Sigma(J^2) \mid \dim(w \cap \text{fiber}) = i\}$ $(i = 0, 1, 2)$, and “fiber” means that the fiber of $T(J^2) \supset C^2 \to T(J^1)$. Then, locally,

$$\begin{aligned} &\Sigma_0|_{{\Pi^2_1}^{-1}(V)} = V_{xy}|_{{\Pi^2_1}^{-1}(V)}, \quad \Sigma_1|_{{\Pi^2_1}^{-1}(V)} = \{(V_{xt} \cup V_{yr}) \backslash V_{xy}\}\,|_{{\Pi^2_1}^{-1}(V)}\,,\\ &\Sigma_2|_{{\Pi^2_1}^{-1}(V)} = \{(V_{rs} \cup V_{rt} \cup V_{st}) \backslash (V_{xy} \cup V_{xt} \cup V_{yr})\}\,|_{{\Pi^2_1}^{-1}(V)}\,, \end{aligned}$$

The set $\Sigma_0 = J^3$ is an open set in $\Sigma(J^2)$ and is an $\mathbf{R}^4$-bundle over J^2. The set Σ_1 is a codimension 1 submanifold in $\Sigma(J^2)$. The set Σ_2 is a codimension 2 submanifold in $\Sigma(J^2)$ and is a $\mathbf{P}^2$-bundle over J^2. In the following, we give the description of the canonical system $(\Sigma(J^2), \hat{C}^2)$ on each coordinate.

(A) $V_{xy} \cong J^3,\ (x, y, z, p, q, r, s, t, p_{111}, p_{112}, p_{122}, p_{222})$:
$\hat{C}^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_r = \varpi_s = \varpi_t = 0\}$, where $\varpi_r = dr - p_{111}dx - p_{112}dy,\ \varpi_s = ds - p_{112}dx - p_{122}dy,\ \varpi_t = dt - p_{122}dx - p_{222}dy.$

(B) $V_{xt},\ (x, y, z, p, q, r, s, t, a, B, c, e)$:
$\hat{C}^2 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = \varpi_s = 0\right\}$, where $\varpi_y = dy - adx - Bdt,\ \varpi_r = dr - cdx - (a^2 + eB)dt,\ \varpi_s = ds - edx - adt.$

(C) $V_{yr},\ (x, y, z, p, q, r, s, t, a, B, c, e)$:
$\hat{C}^2 = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_s = \varpi_t = 0\}$, where $\varpi_x = dx - ady - Bdr,\ \varpi_s = ds - cdy + adr,\ \varpi_t = dt - edy - (a^2 + Bc)dr.$

(D) $V_{rs},\ (x, y, z, p, q, r, s, t, B, D, E, F)$:
$\hat{C}^2 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_t = 0\right\}$, where $\varpi_x = dx - (DE - BF)dr - Bds,\ \varpi_y = dy - Bdr - Dds,\ \varpi_t = dt - Edr - Fds.$

(E) $V_{rt},\ (x, y, z, p, q, r, s, t, A, D, E, F)$:
$\hat{C}^2 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_s = 0\right\}$, where $\varpi_x = dx - Adr + (DE - CF)dt,\ \varpi_y = dy + (AF - (DE - CF)E)dr - Ddt,$
$\varpi_s = ds - Edr - Fdt.$

(F) $V_{st},\ (x, y, z, p, q, r, s, t, A, B, E, F)$:
$\hat{C}^2 = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = \varpi_r = 0\right\}$, where $\varpi_x = dx - Ads - Bdt,\ \varpi_y = dy - Bds + (BE - AF)dt,\ \varpi_r = dr - Eds - Fdt.$

The reason we introduced $\Sigma(J^2)$ is that $\Sigma(R)$ is regarded as the subset in $\Sigma(J^2)$. More precisely, we need to construct the equivariant embedding $\iota : \Sigma(R) \hookrightarrow \Sigma(J^2)$ which give the following commutative diagram:

$$\begin{array}{ccc} \Sigma(R) & \hookrightarrow & \Sigma(J^2) \\ \downarrow & & \downarrow \\ R & \hookrightarrow & J^2(\mathbf{R}^2, \mathbf{R}) \,. \end{array} \tag{11}$$

Here, the correspondences except for ι are already given. This diagram is an extension of the following commutative diagram.

$$\begin{array}{ccc} R^{(1)} & \hookrightarrow & J^3(\mathbf{R}^2, \mathbf{R}) \\ \downarrow & & \downarrow \\ R & \hookrightarrow & J^2(\mathbf{R}^2, \mathbf{R}) \,. \end{array} \tag{12}$$

where $R^{(1)}$ is the prolongation of (R, D) with independence condition. In general, for given second order PDE $R = \{F = 0\}$ with independent variables x, y, this prolongation $R^{(1)}$ corresponds to a third order PDE system which is obtained by partial derivation of $F = 0$ for the two variables x, y. Hence, $R^{(1)}$ can be regarded naturally as a submanifold in J^3 which is also the prolongation of J^2 with the independence condition.

Let us return to the diagram (11). If we can construct the equivariant embedding $\iota : \Sigma(R) \hookrightarrow \Sigma(J^2)$, then we can obtain singular solutions L by the following strategy:

Find an integral manifold L of $(\Sigma(R), \hat{D}) \subset (\Sigma(J^2), \hat{C}^2)$ passing through the $\Sigma_1 \cup \Sigma_2$.

Indeed, in the rest of this section, we construct all singular solutions for model equations belonging to the each class. However, we do not discuss the construction of the singular solutions passing through singular points of $\Sigma(R)$ (see Lemma 1).

4.1. Singular solutions of a hyperbolic equation. We consider the wave equation $R = \{s = 0\}$ as a model equation. The differential system $D = \{ \varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by $\varpi_0 = dz - pdx - qdy,\ \varpi_1 = dp - rdx,\ \varpi_2 = dq - tdy$. The structure equation of D is written as

$$d\varpi_0 = -dp \wedge dx - dq \wedge dy\,, \quad d\varpi_1 = -dr \wedge dx\,, \quad d\varpi_2 = -dt \wedge dy\,.$$

For an open set U in R, we have the covering $p^{-1}(U) = P_{xy} \cup P_{xt} \cup P_{yr} \cup P_{rt}$ of the fibration $p : \Sigma(R) \to R$ followed by Theorem 1, where

$$U_{xy} := \left\{v \in \pi^{-1}(U) \mid dx|_v \wedge dy|_v \neq 0\right\}, \quad U_{xt} := \left\{v \in \pi^{-1}(U) \mid dx|_v \wedge dt|_v \neq 0\right\},$$
$$U_{yr} := \left\{v \in \pi^{-1}(U) \mid dy|_v \wedge dr|_v \neq 0\right\}, \quad U_{rt} := \left\{v \in \pi^{-1}(U) \mid dr|_v \wedge dt|_v \neq 0\right\},$$
$$P_{xy} := p^{-1}(U) \cap U_{xy}\,, \quad P_{xt} := p^{-1}(U) \cap U_{xt}\,,$$
$$P_{yr} := p^{-1}(U) \cap U_{yr}\,, \quad P_{rt} := p^{-1}(U) \cap U_{rt}\,.$$

The geometric decomposition $\Sigma(R) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is given by $\Sigma_0|_{p^{-1}(U)} = P_{xy},\ \Sigma_1|_{p^{-1}(U)} = (P_{xt} \cup P_{yr}) \backslash P_{xy},\ \Sigma_2|_{p^{-1}(U)} = P_{rt} \backslash (P_{xy} \cup P_{xt} \cup P_{yr})$. Now, by using this decomposition, we consider embeddings from $\Sigma(R)$ into $\Sigma(J^2)$.

(i) On the open set $V_{xy} = J^3 \subset \Sigma(J^2)$.

On V_{xy}, we consider the submanifold $\overline{\Sigma}_{xy} = \{s = p_{112} = p_{122} = 0\}$. On $\overline{\Sigma}_{xy}$, we have the induced differential system $D_{\overline{\Sigma}_{xy}} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_r = \varpi_t = 0\}$, where $\varpi_r = dr - p_{111}dx,\ \varpi_t = dt - p_{222}dx$. Clearly, this system $(\overline{\Sigma}_{xy}, D_{\overline{\Sigma}_{xy}})$ is isomorphic to $(P_{xy}, \hat{D}) \subset (\Sigma(R), \hat{D})$. Indeed, this system is equal to the third order PDE which is obtained by partial derivation of the original equation $s = 0$ for the independent variables x, y. The projection to R of these integral manifolds are regular solutions of the wave equation $s = 0$.

(ii) On the open set $V_{xt} \subset \Sigma(J^2)$.

We consider singular solutions of corank 1 which are the projections of integral manifolds of $\Sigma(J^2)$ passing through Σ_1. On V_{xt}, we consider the submanifold $\overline{\Sigma}_{xt} = \{s = a = e = 0\}$. On $\overline{\Sigma}_{xt}$, we have the differential system $D_{\overline{\Sigma}_{xt}} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_y = \varpi_r = 0\right\}$, where $\varpi_y = dy - Bdt$, $\varpi_r = dr - cdx$. Note that $w \in \Sigma_1 \iff B(w) = 0$. Clearly, this system $(\overline{\Sigma}_{xt}, D_{\overline{\Sigma}_{xt}})$ is isomorphic to $(P_{xt}, \hat{D}) \subset (\Sigma(R), \hat{D})$. We construct integral manifolds of this system in the following. Let $\iota : S \hookrightarrow \overline{\Sigma}_{xt} \subset \Sigma(J^2)$ be a graph defined by

$$(x,\ y(x,t),\ z(x,t),\ p(x,t),\ q(x,t),\ r(x,t),\ t,\ B(x,t),\ c(x,t)) \text{ around } (x_0, t_0)\,.$$

If S is an integral submanifold of $(\overline{\Sigma}_{xt}, D_{\overline{\Sigma}_{xt}})$, then the following conditions are satisfied:

$$\iota^*\varpi_0 = \iota^*(dz - pdx - qdy) = (z_x - p - qy_x)dx + (z_t - qy_t)dt = 0\,, \tag{13}$$

$$\iota^*\varpi_1 = \iota^*(dp - rdx) = (p_x - r)dx + p_t dt = 0\,, \tag{14}$$

$$\iota^*\varpi_2 = \iota^*(dq - tdy) = (q_x - ty_x)dx + (q_t - ty_t)dt = 0\,, \tag{15}$$

$$\iota^*\varpi_y = \iota^*(dy - Bdt) = y_x dx + (y_t - B)dt = 0\,, \tag{16}$$

$$\iota^*\varpi_r = \iota^*(dr - cdx) = (r_x - c)dx + r_t dt = 0\,. \tag{17}$$

We have $y(x,t) = y(t)$, $B(x,t) = y'(t)$ from (16), and note that the condition passing through Σ_1 is $B(t_0) = y'(t_0) = 0$. From (15), we have $q = \int ty_t dt = ty - Y$ where $Y := \int ydt$. From (13), $z = \int (ty - Y)y_t dt + z_0(x) = \frac{ty^2}{2} + \frac{1}{2}\int y^2 dt - Yy + z_0(x)$ where $z_0(x)$ is a function on S depending only x, and $p = z_x = z_0'(x)$. For (14), the function p satisfies $p_t = 0$ and we have $r = z_0''(x)$. For (17), the function r satisfies $r_t = 0$ and we have $c = z_0'''(x)$. Therefore, we obtain the solution of $s = 0$ around (x_0, t_0) given by

$$\begin{aligned}
&(x,\ y(x,t),\ z(x,t),\ p(x,t),\ q(x,t),\ r(x,t),\ t,\ B(x,t),\ c(x,t)) \\
&= \biggl(x,\ y(t),\ \frac{ty^2}{2} + \frac{1}{2}\int y^2 dt - y\int ydt + z_0(x),\ z_0'(x),\ ty \\
&\qquad - \int ydt,\ z_0''(x),\ t, y',\ z_0'''(x)\biggr)\,.
\end{aligned}$$

for arbitrary functions $y(t)$ and $z_0(x)$. These integral surfaces with the condition $y'(t_0) = 0$ are geometric singular solutions of corank 1.

(iii) On the open set $V_{yr} \subset \Sigma(J^2)$.

We omit this case since V_{yr} is isomorphic to V_{xt} by the symmetry for x and y.

(iv) On the open set $V_{rt} \subset \Sigma(J^2)$.

We will consider singular solutions of corank 2 which are the projections of integral manifolds of $\Sigma(J^2)$ passing through Σ_2. On V_{rt}, we consider the submanifold $\overline{\Sigma}_{rt} =$

$\{s = E = F = 0\}$. On $\overline{\Sigma}_{rt}$, we have the induced differential system:

$$D_{\overline{\Sigma}_{rt}} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = 0\right\},$$

where $\varpi_x = dx - Adr,\ \varpi_y = dy - Ddt$. Note that $w \in \Sigma_2 \iff A(w) = D(w) = 0$. This system $(\overline{\Sigma}_{rt}, D_{\overline{\Sigma}_{rt}})$ is isomorphic to $(P_{rt}, \hat{D}) \subset (\Sigma(R), \hat{D})$. We construct integral manifolds of this system in the following. Let $\iota : S \hookrightarrow \overline{\Sigma}_{rt} \subset \Sigma(J^2)$ be a graph defined by

$$(x(r,t),\ y(r,t),\ z(r,t),\ p(r,t),\ q(r,t),\ r,\ t,\ A(r,t),\ D(r,t)) \text{ around } (r_0, t_0)\,.$$

If S is an integral submanifold of $(\overline{\Sigma}_{rt}, D_{\overline{\Sigma}_{rt}})$, then the following conditions are satisfied:

$$\iota^*\varpi_0 = \iota^*(dz - pdx - qdy) = (z_r - px_r - qy_r)dr + (z_t - px_t - qy_t)dt = 0\,, \tag{18}$$

$$\iota^*\varpi_1 = \iota^*(dp - rdx) = (p_r - rx_r)dr + (p_t - rx_t)dt = 0\,, \tag{19}$$

$$\iota^*\varpi_2 = \iota^*(dq - tdy) = (q_r - ty_r)dr + (q_t - ty_t)dt = 0\,, \tag{20}$$

$$\iota^*\varpi_y = \iota^*(dx - Adr) = (x_r - A)dr + x_t dt = 0\,, \tag{21}$$

$$\iota^*\varpi_r = \iota^*(dy - Ddt) = y_r dr + (y_t - D)dt = 0\,. \tag{22}$$

From (22), we have $y(r,t) = y(t),\ D(x,t) = y'(t)$. From (21), we have $x(r,t) = x(r),\ A(x,t) = x'(r)$. From (20), $q = \int ty'dt = ty - Y$ where $Y := \int ydt$. From (19), $p = \int rx'dr = rx - X$ where $X := \int xdr$. From (18), $z = \frac{1}{2}\left(rx^2 + ty^2 + \int x^2dr + \int y^2dt\right) - \left(x\int xdr + y\int ydt\right)$. Hence, we get the solution of $s = 0$ around (x_0, t_0) on U_{rt} given by

$$\begin{aligned}
&(x(r,t),\ y(r,t),\ z(r,t),\ p(r,t),\ q(r,t),\ r,\ t,\ A(r,t),\ D(r,t)) \\
&= \Bigg(x(r),\ y(t),\ \frac{1}{2}\left(rx^2 + ty^2 + \int x^2dr + \int y^2dt\right) - \left(x\int xdr + y\int ydt\right), \\
&\qquad rx - \int xdr,\ ty - \int ydt,\ r,\ t,\ x'(r),\ y'(t)\Bigg)\,.
\end{aligned}$$

for arbitrary functions $x(r)$ and $y(t)$. These integral surfaces with the condition $x'(r_0) = y'(t_0) = 0$ are geometric singular solutions of corank 2.

4.2. Singular solutions of a parabolic equation. We consider the equation $R = \{r = 0\}$. The differential system $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by $\varpi_0 = dz - pdx - qdy,\ \varpi_1 = dp - sdy,\ \varpi_2 = dq - sdx - tdy$. The structure equation of D is written as

$$d\varpi_0 = -dp \wedge dx - dq \wedge dy\,, \quad d\varpi_1 = -ds \wedge dy\,, \quad d\varpi_2 = -ds \wedge dx - dt \wedge dy\,.$$

Let U be an open set in R. We have the covering $p^{-1}(U) = P_{xy} \cup P_{xt} \cup P_{st}$ of the fibration $p : \Sigma(R) \to R$ from Lemma 2, where $U_{xy} := \left\{v \in \pi^{-1}(U) \mid dx|_v \wedge dy|_v \neq 0\right\}$, $U_{xt} := \left\{v \in \pi^{-1}(U) \mid dx|_v \wedge dt|_v \neq 0\right\}$, $U_{st} := \left\{v \in \pi^{-1}(U) \mid ds|_v \wedge dt|_v \neq 0\right\}$, $P_{xy} := p^{-1}(U) \cap U_{xy},\ P_{xt} := p^{-1}(U) \cap U_{xt},\ P_{st} := p^{-1}(U) \cap U_{st}$. The geometric decomposition

$\Sigma(R) = \Sigma_0 \cup \Sigma_1 \cup \Sigma_2$ is given by $\Sigma_0|_{p^{-1}(U)} = P_{xy}$, $\Sigma_1|_{p^{-1}(U)} = P_{xt} \backslash P_{xy}$, $\Sigma_2|_{p^{-1}(U)} = P_{st} \backslash (P_{xy} \cup P_{xt})$. This prolongation $\Sigma(R)$ is realized as a submanifold of $\Sigma(J^2)$ as follows:

(i) On the open set $J^3 = V_{xy} \subset \Sigma(J^2)$.

On V_{xy} in $\Sigma(J^2)$, we consider the submanifold given by $\overline{\Sigma}_{xy} = \{r = p_{111} = p_{112} = 0\}$. We have the induced differential system $D_{\overline{\Sigma}_{xy}} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_s = \varpi_t = 0\}$ on $\overline{\Sigma}_{xy}$, where $\varpi_s = ds - p_{122}dy$, $\varpi_t = dt - p_{122}dx - p_{222}dy$. This system $(\overline{\Sigma}_{xy}, D_{\overline{\Sigma}_{xy}})$ is isomorphic to $(P_{xy}, \hat{D}) \subset (\Sigma(R), \hat{D})$. Indeed, this system is equal to the third order PDE which is obtained by partial derivation of the original equation $r = 0$ for the independent variables x, y. The projection to R of these integral manifolds are regular solutions of the equation $r = 0$.

(ii) On the open set $V_{st} \subset \Sigma(J^2)$.

We will consider singular solutions of corank 1 and 2 which are obtained by the projections of integral manifolds of $\Sigma(J^2)$ passing through smooth points in $\Sigma(R)$. Recall that $\Sigma_1 \backslash \{\text{singular points}\} \subset \Sigma(R)$ is covered by P_{st}. Hence, we work on V_{st} and consider the submanifold given by $\overline{\Sigma}_{st} = \{r = E = F = 0\}$. We have the induced differential system $D_{\overline{\Sigma}_{st}} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = 0\right\}$ on $\overline{\Sigma}_{st}$, where $\varpi_x = dx - Ads - Bdt$, $\varpi_y = dy - Bds$. Note that

$$
\begin{aligned}
w \in \Sigma_1 \backslash \{\text{singular points}\} &\iff A(w) \neq 0, B(w) = 0 \\
w \in \Sigma_2 \qquad\qquad\qquad &\iff A(w) = B(w) = 0\,.
\end{aligned}
$$

This system $(\overline{\Sigma}_{st}, D_{\overline{\Sigma}_{st}})$ is isomorphic to $(P_{st}, \hat{D}) \subset (\Sigma(R), \hat{D})$. We construct integral manifolds of this system. Let $\iota : S \hookrightarrow \overline{\Sigma}_{st} \subset \Sigma(J^2)$ be a graph defined by

$$(x(s,t),\ y(s,t),\ z(s,t),\ p(s,t),\ q(s,t),\ s,\ t,\ A(s,t),\ B(x,t)) \text{ around } (s_0, t_0)\,.$$

If S is an integral manifold of $D_{\overline{\Sigma}_{st}}$, then the following conditions are satisfied:

$$\iota^*\varpi_0 := (z_s - px_s - qy_s)ds + (z_t - px_t - qy_t)dt = 0\,, \tag{23}$$

$$\iota^*\varpi_1 := (p_s - sy_s)ds + (p_t - sy_t)dt = 0\,, \tag{24}$$

$$\iota^*\varpi_2 := (q_s - sx_s - ty_s)ds + (q_t - sx_t - ty_t)dt = 0\,, \tag{25}$$

$$\iota^*\varpi_x := (x_s - A)ds + (x_t - B)dt = 0\,, \tag{26}$$

$$\iota^*\varpi_y := (y_s - B)ds + y_t dt = 0\,. \tag{27}$$

We have $y(s,t) = y(s)$, $B(s,t) = y'(s)$ from (27). From (26), $x = ty'(s) + x_0(s)$, where $x_0(s)$ is a function on S depending only s, and $A = x_s = ty''(s) + x_0'(s)$. From (24), $p = \int sy_s ds = sy - Y$ where $Y := \int y ds$. From (25), we also have $q = tsy' + sx_0 - \int x_0 ds$. Similarly, from (23), $z = t(sy - Y)y' + syx_0 + \int (yx_0)ds - x_0 \int y ds - y \int x_0 ds$. Hence we have solutions of $r = 0$ given by

$$\begin{aligned}&(x(s,t),\ y(s,t),\ z(s,t),\ p(s,t),\ q(s,t),\ s,\ t,\ A(s,t),\ B(s,t))\\ &= (ty'(s)+x_0(s),\ y(s),\ t(sy-\int yds)y'+syx_0+\int(yx_0)ds-x_0\int yds-y\int x_0ds\,,\\ &\qquad sy-\int yds,\ tsy'+sx_0-\int x_0ds,\ s,\ t,\ ty''+x_0',\ y')\,.\end{aligned}$$

for arbitrary functions $y(s)$ and $x_0(s)$. These integral surfaces which satisfy the condition

$$A(s_0,t_0) = t_0y''(s_0)+x_0'(s_0) \neq 0\ ,\ B(s_0) = y'(s_0) = 0$$

are geometric singular solutions of corank 1. On the other hand, these integral with the condition

$$A(s_0,t_0) = t_0y''(s_0)+x_0'(s_0) = 0\ ,\ B(s_0) = y'(s_0) = 0$$

are geometric singular solutions of corank 2.

4.3. Singular solutions of an elliptic equation. We consider the Laplace equation $R = \{r+t=0\}$. The differential system $D = \{\varpi_0 = \varpi_1 = \varpi_2 = 0\}$ is given by $\varpi_0 = dz - pdx - qdy,\ \varpi_1 = dp - rdx - sdy,\ \varpi_2 = dq - sdx + rdy$. The structure equation of D is expressed as

$$\begin{aligned}d\varpi_0 &= -dp\wedge dx - dq\wedge dy\,, \quad d\varpi_1 = -dr\wedge dx - ds\wedge dy\,,\\ d\varpi_2 &= -ds\wedge dx + dr\wedge dy\,.\end{aligned}$$

Then, for an open set $U \subset R$, we have the covering $p^{-1}(U) = P_{xy}\cup P_{rs}$ of the fibration $p : \Sigma(R) \to R$, where

$$\begin{aligned}&U_{xy} := \left\{v\in\pi^{-1}(U)\mid dx|_v\wedge dy|_v\neq 0\right\}, \quad U_{rs} := \left\{v\in\pi^{-1}(U)\mid dr|_v\wedge ds|_v\neq 0\right\},\\ &P_{xy} := p^{-1}(U)\cap U_{xy}\,, \quad P_{rs} := p^{-1}(U)\cap U_{rs}\,.\end{aligned}$$

The geometric decomposition $\Sigma(R) = \Sigma_0\cup\Sigma_2$ is given by $\Sigma_0|_{p^{-1}(U)} = P_{xy},\ \Sigma_2|_{p^{-1}(U)} = P_{rs}\backslash P_{xy}$. This prolongation $\Sigma(R)$ is realized as a submanifold of $\Sigma(J^2)$ as follows:

(i) $J^3 = V_{xy} \subset \Sigma(J^2)$.

On V_{xy}, we consider the submanifold given by $\overline{\Sigma}_{xy} = \{r+t=0,\ p_{111} = -p_{122},\ p_{112} = -p_{222}\}$. We have the induced differential system $D_{\overline{\Sigma}_{xy}} = \{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_r = \varpi_s = 0\}$ on $\overline{\Sigma}_{xy}$, where $\varpi_r = dr - p_{111}dx - p_{112}dy,\ \varpi_s = ds - p_{112}dx + p_{111}dy$. This system $(\overline{\Sigma}_{xy}, D_{\overline{\Sigma}_{xy}})$ is isomorphic to $(P_{xy}, \hat{D}) \subset (\Sigma(R), \hat{D})$. Indeed, this system is equal to the third order PDE which is obtained by partial derivation of the original equation $r+t=0$ for the independent variables x, y. The projection to R of integral manifolds are regular solutions of the wave equation $r+t=0$.

(ii) On $V_{rs} \subset \Sigma(J^2)$.

We will consider singular solutions of corank 2 which are the projections of integral manifolds of $\Sigma(J^2)$ passing through Σ_2. On V_{rs}, we consider the submanifold given by $\overline{\Sigma}_{rs} = \{r+t=0, E=-1, F=0\}$. We have the induced differential system $D_{\overline{\Sigma}_{rs}} = \left\{\varpi_0 = \varpi_1 = \varpi_2 = \varpi_x = \varpi_y = 0\right\}$ on $\overline{\Sigma}_{rs}$, where $\varpi_x = dx + Ddr - Bds$, $\varpi_y = dy - Bdr - Dds$. Recall that $w \in \Sigma_2 \iff B(w) = D(w) = 0$. This system $(\overline{\Sigma}_{rs}, D_{\overline{\Sigma}_{rs}})$ is isomorphic to $(P_{rs}, \hat{D}) \subset (\Sigma(R), \hat{D})$. We construct integral manifolds of this system. Let $\iota : S \hookrightarrow \overline{\Sigma}_{rs} \subset \Sigma(J^2)$ be a graph defined by

$$(x(r,s), y(r,s), z(r,s), p(r,s), q(r,s), r, s, B(r,s), D(r,s)) \text{ around } (r_0, s_0)\,.$$

If S is an integral manifold of $D_{\overline{\Sigma}_{rs}}$, then the following conditions are satisfied:

$$\iota^*\varpi_0 :=(z_r - px_r - qy_r)dr + (z_s - px_s - qy_s)ds = 0\,, \tag{28}$$

$$\iota^*\varpi_1 :=(p_r - rx_r - sy_r)dr + (p_s - rx_s - sy_s)ds = 0\,, \tag{29}$$

$$\iota^*\varpi_2 :=(q_r - sx_r + ry_r)dr + (q_s - sx_s + ry_s)ds = 0\,, \tag{30}$$

$$\iota^*\varpi_x :=(x_r + D)dr + (x_s - B)ds = 0\,, \tag{31}$$

$$\iota^*\varpi_y :=(y_r - B)dr + (y_s - D)ds = 0\,. \tag{32}$$

From (31) and (32), a complex function $f(z) := y(r,s) + ix(r,s)$ $(z := r + is)$ must be a holomorphic function. From (29) and (30), $p(r,s), q(r,s)$ are considered as solutions of a differential equation

$$q_s = sx_s - ry_s\,, \quad p_r = rx_r + sy_r \tag{33}$$

$$q_r = sx_r - ry_r\,, \quad p_s = rx_s + sy_s. \tag{34}$$

for given functions $x_r = -y_s$, $y_r = x_s$. Then, we also get Cauchy-Riemann equation $q_r = -p_s$, $q_s = p_r$ from Cauchy-Riemann equation for $y(r,s), x(r,s)$. Hence a complex function $g(z) := p(r,s) + iq(r,s)$ $(z := r + is)$ is also holomorphic. From (28), $z(r,s)$ is considered as a solution of a differential equation

$$z_r = px_r + qy_r\,, \quad z_s = px_s + qy_s \tag{35}$$

for given functions $p, q, x_r = -y_s, y_r = x_s$.

Conversely, for a given holomorphic function $f(z) = y(r,s) + ix(r,s)$ $(z := r + is)$ we consider the differential equation (33), (34) for p, q where x, y are given functions. Then, the differential equation is Frobenius since $y(r,s)$ and $x(r,s)$ satisfy Cauchy-Riemann equation. Therefore, the existence of the solution of (33), (34) is guaranteed and $g(z) := p(r,s) + iq(r,s)$ $(z := r + is)$ is holomorphic. Next, we consider the differential equation (35) for z where x, y, p, q are given. Then, this differential equation is Frobenius since $f(z)$ and $g(z)$ are holomorphic functions and have solutions. Finally, let $f(z) = y(r,s) + ix(r,s)$ $(z := r + is)$ be a holomorphic function and $p(r,s), q(r,s), z(r,s)$ be the functions obtained by the above construction. Then,

$$(x(r,s),\ y(r,s),\ z(r,s),\ p(r,s),\ q(r,s),\ r,\ s,\ y_r(s,t),\ y_s(s,t))$$

is an integral surface. These integral surfaces which satisfy the condition $y_r(s_0,t_0) = y_s(s_0,t_0) = 0$ are geometric singular solutions of corank 2.

5. Tower constructions of special rank 4 distributions

In sections 2 and 3, we studied geometric structures of rank 2 prolongations for each class of equations. In this section, we define special rank 4 distributions which are generalization of distributions induced by PDEs and construct tower structures of these distributions by successive rank 2 prolongations.

DEFINITION 5. Let R be a $k+6$ dimensional manifold ($k \geq 0$), and D be a differential system of rank 4 on R. Then,

(i) (R, D) is *hyperbolic type* at $w \in R$ if there exists a local coframe $\{\varpi_i, \theta_j, \omega_j, \pi_j\}$ ($i = 1, \ldots, k,\ j = 1, 2$) around $w \in R$ such that $D = \{\varpi_i = \theta_j = 0\}$ around $w \in R$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_i &\equiv 0 && \mod \varpi_i, \theta_j \\ d\theta_1 &\equiv \omega_1 \wedge \pi_1 && \mod \varpi_i, \theta_j\,, \\ d\theta_2 &\equiv \omega_2 \wedge \pi_2 && \mod \varpi_i, \theta_j\,. \end{aligned} \tag{36}$$

(ii) (R, D) is *parabolic type* at $w \in R$ if there exists a local coframe $\{\varpi_i, \theta_j, \omega_j, \pi_j\}$ ($i = 1, \ldots, k,\ j = 1, 2$) around $w \in R$ such that $D = \{\varpi_i = \theta_j = 0\}$ around $w \in R$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_i &\equiv 0 && \mod \varpi_i, \theta_j \\ d\theta_1 &\equiv \qquad\qquad \omega_2 \wedge \pi_1 && \mod \varpi_i, \theta_j\,, \\ d\theta_2 &\equiv \omega_1 \wedge \pi_1 + \omega_2 \wedge \pi_2 && \mod \varpi_i, \theta_j\,. \end{aligned} \tag{37}$$

(iii) (R, D) is *elliptic type* at $w \in R$ if there exists a local coframe $\{\varpi_i, \theta_j, \omega_j, \pi_j\}$ ($i = 1, \ldots, k,\ j = 1, 2$) around $w \in R$ such that $D = \{\varpi_i = \theta_j = 0\}$ around $w \in R$ and the following structure equation holds at w:

$$\begin{aligned} d\varpi_i &\equiv 0 && \mod \varpi_i, \theta_j \\ d\theta_1 &\equiv \omega_1 \wedge \pi_1 + \omega_2 \wedge \pi_2 && \mod \varpi_i, \theta_j\,, \\ d\theta_2 &\equiv \omega_1 \wedge \pi_2 - \omega_2 \wedge \pi_1 && \mod \varpi_i, \theta_j\,. \end{aligned} \tag{38}$$

PROPOSITION 8. *Let (R, D) be a hyperbolic type, parabolic type or elliptic type. Then the first derived system ∂D of D is a subbundle of rank 6 and the Cauchy characteristic system $Ch(D)$ of D is trivial, that is $Ch(D) = \{0\}$.*

Here, the Cauchy characteristic system $Ch(D)$ of a differential system (R, D) is defined by

$$Ch(D)(x) = \{X \in D(x) \mid X \rfloor d\omega_i \equiv 0 \quad (\mathrm{mod}\ \ \omega_1, \ldots, \omega_s) \quad \text{for } i = 1, \ldots, s\},$$

where $D = \{\, \omega_1 = \cdots = \omega_s = 0 \,\}$ is defined locally by defining 1-forms $\{\omega_1, \ldots, \omega_s\}$.

PROOF. This statement is obtained by the very definitions. □

REMARK 2. In fact, the converse of the above proposition also holds. Namely, let D be a differential system of rank 4 on a $k+6$ dimensional manifold R with rank $\partial D = 6$, $Ch(D) = \{0\}$. Then, for any $w \in R$, (R, D) is a hyperbolic type, parabolic type or elliptic type at w ([10]).

PROPOSITION 9.

(i) *If (R, D) is locally hyperbolic, then the rank 2 prolongation $(\Sigma(R), \hat{D})$ of (R, D) is also hyperbolic at any point in $\Sigma(R)$. Moreover, $\Sigma(R)$ is a T^2-bundle over R.*

(ii) *If (R, D) is locally parabolic, then $(\Sigma(R)\backslash \{singular\ points\}, \hat{D})$ is also parabolic at any point in $\Sigma(R)\backslash \{singular\ points\}$. Moreover, $\Sigma(R)\backslash \{singular\ points\}$ is an $S^1 \times \mathbf{R}$-bundle over R.*

(iii) *If (R, D) is locally elliptic, then the rank 2 prolongation $(\Sigma(R), \hat{D})$ of (R, D) is also elliptic at any point in $\Sigma(R)$. Moreover, $\Sigma(R)$ is an S^2-bundle over R.*

PROOF. These statements are obtained by the same arguments of the proof of Theorem 1, Proposition 2, 3 for the hyperbolic case, Theorem 2, Proposition 5 for the parabolic case and Theorem 3, Proposition 7 for the elliptic case. □

For the locally hyperbolic, locally parabolic or locally elliptic type distribution (R, D), we can define k-th rank 2 prolongation $(\Sigma^k(R), \hat{D}^k)$ of (R, D) by the above Proposition, successively. For hyperbolic and elliptic type (R, D), we define

$$(\Sigma^k(R), \hat{D}^k) := (\Sigma(\Sigma^{k-1}(R)), \hat{\hat{D}}^{k-1}) \quad (k = 1, 2, \ldots),$$

where $(\Sigma^0(R), \hat{D}^0) := (R, D)$. For parabolic type (R, D), we define

$$(\Sigma^k(R), \hat{D}^k) := (\Sigma(\Sigma^{k-1}(R))\backslash \{singular\ points\}, \hat{\hat{D}}^{k-1}) \qquad (k = 1, 2, \ldots)$$

where $(\Sigma^0(R), \hat{D}^0) := (R, D)$.

THEOREM 4. *If (R, D) is locally hyperbolic, locally parabolic or locally elliptic then the k-th rank 2 prolongation $(\Sigma^k(R), \hat{D}^k)$ of (R, D) is also hyperbolic, parabolic or elliptic at any point in $\Sigma^k(R)$, respectively.*

PROOF. This theorem is obtained from the successive applications of Proposition 9. □

REMARK 3. For the hyperbolic case, Bryant, Griffiths and Hsu proved the above theorem for the exterior differential systems in [2]. By our argument, for parabolic and elliptic cases, one can show that Theorem 4 have the similar extension for the exterior differential system ([10]).

ACKNOWLEDGMENT. We would like to express our special thanks to Professor Keizo Yamaguchi for many useful discussions on the subject. Moreover, the first author is supported by Osaka City University Advanced Mathematical Institute.

References

[1] R. BRYANT, S. S. CHERN, R. GARDNER, H. GOLDSCMIDT and P. GRIFFITHS, *Exterior Differential Systems*, MSRI Publ. vol. **18**, Springer Verlag, Berlin (1991).

[2] R. BRYANT, P. GRIFFITHS and L. HSU, Hyperbolic Exterior Differential Systems and their Conservation Laws, Part I, *Selecta Math*, New Series, Vol. **1**, No. 1 (1995).

[3] B. KRUGLIKOV and V. LYCHAGIN, Geometry of differential equations, *Handbook of global analysis*, **1214**, Elsevier Sci. B. V., Amsterdam, (2008), 725–771.

[4] V. LYCHAGIN, Geometric theory of singularities of solutions of nonlinear differential equations (Russian), Translated in J. Soviet Math. **51** (1990), no 6, 2735–2757.

[5] V. LYCHAGIN, Differential equations on two-dimensional manifolds (Russian), Izv. Vyssh. Uchebn. Zaved. Mat. 1992, no. 5, 25–35.

[6] T. MORIMOTO, Geometric structures on filtered manifolds, Hokkaido Math. J. **22** (1993), 263–347.

[7] R. MONTGOMERY and M. ZHITOMIRSKII, Geometric approach to Goursat flags, Ann. Inst. H. Poincaré-AN **18** (2001), 459–493.

[8] T. NODA and K. SHIBUYA, Second order type-changing PDE for a scalar function on a plane, Osaka J. Math. **49** (2012), 101–124.

[9] K. SHIBUYA, On the prolongation of 2-jet space of 2 independent and 1 dependent variables, Hokkaido Math. J. **38** (2009), 587–626.

[10] K. SHIBUYA, Rank 4 distributions of type hyperbolic, parabolic and elliptic, in preparation.

[11] K. SHIBUYA and K. YAMAGUCHI, Drapeau theorem for differential systems, Diff. Geom. and its Appl. **27** (2009), 793–808.

[12] N. TANAKA, On differential systems, graded Lie algebras and pseudo-groups, J. Math. Kyoto Univ. **10** (1970), 1–82.

[13] N. TANAKA, On generalized graded Lie algebras and geometric structures I, J. Math. Soc. Japan **19** (1967), 215–254.

[14] N. TANAKA, On the equivalence problems associated with simple graded Lie algebras, Hokkaido Math. J. **8** (1979), no. 1, 23–84.

[15] K. YAMAGUCHI, Contact geometry of higher order, Japan. J. Math. **8** (1982), 109–176.

[16] K. YAMAGUCHI, Geometrization of jet bundles, Hokkaido Math. J. **12** (1983), 27–40.

[17] K. YAMAGUCHI, Differential systems associated with simple graded Lie algebras, Advanced Studies in Pure Math. **22** (1993), 413–494.

[18] K. YAMAGUCHI, Contact geometry of second order I, *Differential Equations -Geometry, Symmetries and Integrability*, The Abel symposium 2008, Abel symposia **5**, 2009, 335–386.

Present Addresses:
TAKAHIRO NODA
GRADUATE SCHOOL OF MATHEMATICS,
NAGOYA UNIVERSITY,
CHIKUSA-KU, NAGOYA, 464–8602 JAPAN.
e-mail: m04031x@math.nagoya-u.ac.jp

KAZUHIRO SHIBUYA
GRADUATE SCHOOL OF SCIENCE,
HIROSHIMA UNIVERSITY,
HIGASHI-HIROSHIMA, 739–8521 JAPAN.
e-mail: shibuya@hiroshima-u.ac.jp