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Some Inequalities for Power Series of Two Operators in Hilbert Spaces

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Abstract. Some inequalities for functions defined by power series concerning two operators in both the non-commutative and commutative case are given. Natural examples for fundamental functions that can be represented by power series are presented as well.

1. Introduction

For power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ with complex coefficients we can naturally construct another power series which have as coefficients the absolute values of the coefficient of the original series, namely, $f_a(z) := \sum_{n=0}^{\infty} |a_n| z^n$. It is obvious that this new power series have the same radius of convergence as the original series, and that if all coefficients $a_n \geq 0$, then $f_a = f$.

With this notation S.S. Dragomir [4] (also see [5]) showed the following:

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset C$, $R > 0$. Let $T \in B(H)$, $\alpha, \beta \geq 0$ with $\alpha + \beta \geq 1$ and such that

$$\|T\|^{2\alpha}, \|T\|^{2\beta} < R.$$

Then

$$\begin{aligned} &\left|\left\langle Tf(|T|^{\alpha+\beta})|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2 \\ &\leq \left\langle f_a(|T|^{2\alpha})|T|^{2\alpha}x, x\right\rangle\left\langle f_a(|T^*|^{2\beta})|T^*|^{2\beta}y, y\right\rangle \end{aligned}$$

for any $x, y \in H$.

This is an extension of the following inequality for a bounded linear operator $T \in B(H)$ by Furuta [7] (also see [8]):

$$\left|\left\langle T|T|^{\alpha+\beta-1}x, y\right\rangle\right|^2 \leq \left\langle |T|^{2\alpha}x, x\right\rangle\left\langle |T^*|^{2\beta}y, y\right\rangle, \quad x, y \in H.$$

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Motivated by this result for one operator, we investigate in the current paper some inequalities for functions defined by power series concerning two operators in both the non-commutative and commutative case. In particular, for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ we show that

$$\left\| f(AB) \right\| \le f_a^{1/p}\left(\|A\|^p\right) f_a^{1/q}\left(\|B\|^q\right) .$$

Moreover we prove this inequality is also valid for every unitarily invariant norm.

The following is one among some examples given in this paper:

If $\|A\|^p, \|B\|^q < 1$, then

$$\left\|(1_H \pm AB)^{-1}\right\| \le \left(1 - \|A\|^p\right)^{-1/p}\left(1 - \|B\|^q\right)^{-1/q} .$$

2. Some General Norm Inequalities

The following result concerning norm inequalities for two bounded operators may be stated:

THEOREM 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(\neq 0)$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. If A and B are two bounded operators on the Hilbert space H and for $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$*

$$\|A\|^p, \|B\|^q < R, \tag{2.1}$$

then

$$\left\| f(AB)\right\| \le \min\left\{K_1(p,q), K_2(p,q)\right\} \tag{2.2}$$

where

$$K_1(p,q) := f_a^{1/p}\left(\|A\|^p\right) f_a^{1/q}\left(\|B\|^q\right), \tag{2.3}$$

and

$$K_2(p,q) := \frac{f_a\left(\|A\|^p\right) f_a\left(\|B\|^q\right)}{f_a\left(\|A\|^{p-1}\|B\|^{q-1}\right)}. \tag{2.4}$$

PROOF. By the properties of operator norm, observe that, for any $j \in \mathbf{N}$ we have

$$\left\|(AB)^j\right\| \le \|A\|^j \|B\|^j .$$

If we multiply with $|a_j|$ and use the generalized triangle inequality we have

$$\left\| \sum_{j=0}^{n} a_j (AB)^j \right\| \le \sum_{j=0}^{n} |a_j| \|A\|^j \|B\|^j \tag{2.5}$$

for any $n \in \mathbf{N}$.

Now, by Hölder's inequality we have

$$\sum_{j=0}^{n} |a_j| \, \|A\|^j \, \|B\|^j \le \left(\sum_{j=0}^{n} |a_j| \, \|A\|^{jp} \right)^{1/p} \left(\sum_{j=0}^{n} |a_j| \, \|B\|^{jq} \right)^{1/q} \tag{2.6}$$

for any $n \in \mathbf{N}$, and by (2.5) we get

$$\left\| \sum_{j=0}^{n} a_j \, (AB)^j \right\| \le \left(\sum_{j=0}^{n} |a_j| \, \|A\|^{jp} \right)^{1/p} \left(\sum_{j=0}^{n} |a_j| \, \|B\|^{jq} \right)^{1/q}. \tag{2.7}$$

Since the series whose partial sums are involved in (2.7) are convergent, then by taking $n \to \infty$ in (2.7) we deduce the first inequality in (2.3).

Further, by utilizing the following Hölder's type inequality obtained by Dragomir and Sándor in 1990 [6] (see also [2, Corollary 2.34]):

$$\sum_{k=0}^{n} m_k \, |x_k|^p \sum_{k=0}^{n} m_k \, |y_k|^q \ge \sum_{k=0}^{n} m_k \, |x_k y_k| \sum_{k=0}^{n} m_k \, |x_k|^{p-1} \, |y_k|^{q-1} \;, \tag{2.8}$$

that holds for nonnegative numbers m_k and complex numbers x_k, y_k where $k \in \{0, ..., n\}$, we observe that the convergence of the series $\sum_{k=0}^{\infty} m_k \, |x_k|^p$ and $\sum_{k=0}^{\infty} m_k \, |y_k|^q$ imply the convergence of the series $\sum_{k=0}^{\infty} m_k \, |x_k|^{p-1} \, |y_k|^{q-1}$.

Utilising (2.8) we then have

$$\sum_{j=0}^{n} |a_j| \, \|A\|^j \, \|B\|^j \le \frac{\sum_{j=0}^{n} |a_j| \, \|A\|^{jp} \sum_{j=0}^{n} |a_j| \, \|B\|^{jq}}{\sum_{j=0}^{n} |a_j| \, \|A\|^{j(p-1)} \, \|B\|^{j(q-1)}}$$

which together with (2.5) gives

$$\left\| \sum_{j=0}^{n} a_j \, (AB)^j \right\| \le \frac{\sum_{j=0}^{n} |a_j| \, \|A\|^{jp} \sum_{j=0}^{n} |a_j| \, \|B\|^{jp}}{\sum_{j=0}^{n} |a_j| \, \|A\|^{j(p-1)} \, \|B\|^{j(q-1)}}, \tag{2.9}$$

for any $n \in \mathbf{N}$.

Since all the series whose partial sums are involved in (2.9) are convergent, then by taking $n \to \infty$ in (2.9) we deduce the second inequality in (2.2). □

REMARK 1. The case $p = q = 2$ produces the Schwarz's type inequality

$$\left\| f(AB) \right\|^2 \le f_a\left(\|A\|^2\right) f_a\left(\|B\|^2\right), \tag{2.10}$$

provided $\|A\|^2, \|B\|^2 < R$.

The finite-dimensional case is as follows:

THEOREM 2. *Theorem* 1 *also holds for every unitarily invariant norm* $|||\cdot|||$ *on a finite matrix algebra. Moreover, we have the inequalities*

$$|||f(AB)||| \le \min\{L_1(p,q), L_2(p,q)\} \tag{2.11}$$

where

$$L_1(p,q) := f_a^{1/p}(|||\,|A|^p|||) f_a^{1/q}(|||\,|B|^q|||) \tag{2.12}$$

and

$$L_2(p,q) := \frac{f_a(|||\,|A|^p|||) f_a(|||\,|B|^q|||)}{f_a(|||\,|A|^p|||^{1/q}|||\,|B|^q|||^{1/p})}, \tag{2.13}$$

provided $|||\,|A|^p|||, |||\,|B|^q||| < R$.

PROOF. Since $|||AB||| \le |||A|||\cdot|||B|||$ and $|||AB||| \le |||\,|A|^p|||^{1/p}\cdot|||\,|B|^q|||^{1/q}$ where $p>1$, $\frac{1}{p}+\frac{1}{q}=1$ (see for instance [1, p. 95]), we have by the Hölder inequality that

$$\begin{aligned} \left|\left|\left|\sum_{j=0}^{n} a_j (AB)^j\right|\right|\right| &\le \sum_{j=0}^{n}|a_j|\,|||AB|||^j \le \sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^{j/p}\cdot|||\,|B|^q|||^{j/q} \\ &\le \left(\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^j\right)^{1/p}\left(\sum_{j=0}^{n}|a_j|\,|||\,|B|^q|||^j\right)^{1/q} \end{aligned} \tag{2.14}$$

for any $n\in\mathbf{N}$.

Since all the series whose partial sums are involved in (2.14) are convergent, then by taking $n\to\infty$ in (2.14) we deduce the first inequality in (2.11).

Utilising the inequality (2.8) we also have

$$\begin{aligned} &\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^{j/p}\cdot|||\,|B|^q|||^{j/q} \\ &\le \frac{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^j \sum_{j=0}^{n}|a_j|\,|||\,|B|^q|||^j}{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^{j\frac{p-1}{p}}\cdot|||\,|B|^q|||^{j\frac{q-1}{q}}} \\ &= \frac{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^j \sum_{j=0}^{n}|a_j|\,|||\,|B|^q|||^j}{\sum_{j=0}^{n}|a_j|\,|||\,|A|^p|||^{\frac{j}{q}}\cdot|||\,|B|^q|||^{\frac{j}{p}}} \end{aligned} \tag{2.15}$$

for any $n\in\mathbf{N}$.

Since all the series whose partial sums are involved in (2.15) are convergent, then by taking $n\to\infty$ in (2.15) we deduce the first inequality in (2.11). □

REMARK 2. The case $p=q=2$ produces the Schwarz's type inequality

$$|||f(AB)|||^2 \le f_a(|||\,|A|^2|||) f_a(|||\,|B|^2|||),$$

provided $\left\| |A|^2 \right\|, \left\| |B|^2 \right\| < R$.

A refinement of the inequality (2.10) may be found in the following theorem:

THEOREM 3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(\neq 0)$ be a function defined by power series with complex coefficients and convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. If A and B are two bounded operators on the Hilbert space H and*

$$\|A\|^2, \|B\|^2 < R, \tag{2.16}$$

then

$$\begin{aligned} \left\| f(AB) \right\|^2 &\le f_a\left(\|A\|^{1+\alpha} \|B\|^{1-\alpha}\right) f_a\left(\|A\|^{1-\alpha} \|B\|^{1+\alpha}\right) \\ &\le f_a\left(\|A\|^2\right) f_a\left(\|B\|^2\right), \end{aligned} \tag{2.17}$$

where $\alpha \in [0, 1]$.

If $\sum_{n=0}^{\infty} |a_n| < \infty$ and in addition to the condition (2.16) *we have $\|A\|, \|B\| < R$, then*

$$\left\| f(AB) \right\| \le f_a(1) \cdot \frac{f_a\left(\|A\|^2\right) f_a\left(\|B\|^2\right)}{f_a\left(\|A\|\right) f_a\left(\|B\|\right)}. \tag{2.18}$$

PROOF. We utilize the *Callebaut inequality* (see for instance [2, Remark 3.31])

$$\left(\sum_{j=1}^{n} p_j a_j b_j \right)^2 \le \sum_{j=1}^{n} p_j a_j^{1+\alpha} b_j^{1-\alpha} \sum_{j=1}^{n} p_j a_j^{1-\alpha} b_j^{1+\alpha} \le \sum_{j=1}^{n} p_j a_j^2 \sum_{j=1}^{n} p_j b_j^2$$

that holds for $\alpha \in [0, 1]$ and the nonnegative numbers a_j, b_j, p_j with $j \in \{1, \dots, n\}$.

Therefore

$$\begin{aligned} \left(\sum_{j=0}^{n} |a_j| \, \|A\|^j \, \|B\|^j \right)^2 &\le \sum_{j=0}^{n} |a_j| \, \|A\|^{(1+\alpha)j} \, \|B\|^{(1-\alpha)j} \sum_{j=0}^{n} |a_j| \, \|A\|^{(1-\alpha)j} \, \|B\|^{(1+\alpha)j} \\ &\le \sum_{j=0}^{n} |a_j| \, \|A\|^{2j} \sum_{j=0}^{n} |a_j| \, \|B\|^{2j} \end{aligned}$$

and by (2.5) we get

$$\begin{aligned} &\left\| \sum_{j=0}^{n} a_j (AB)^j \right\|^2 \\ &\quad \le \sum_{j=0}^{n} |a_j| \, \|A\|^{(1+\alpha)j} \, \|B\|^{(1-\alpha)j} \sum_{j=0}^{n} |a_j| \, \|A\|^{(1-\alpha)j} \, \|B\|^{(1+\alpha)j} \\ &\quad \le \sum_{j=0}^{n} |a_j| \, \|A\|^{2j} \sum_{j=0}^{n} |a_j| \, \|B\|^{2j}, \end{aligned} \tag{2.19}$$

for any $n \in \mathbf{N}$.

Since all the series whose partial sums are involved in (2.19) are convergent, then by taking $n \to \infty$ in (2.19) we deduce the inequality (2.17).

For the second part, we use the following inequality obtained by S.S. Dragomir in 1984 [3] (see also [2, Theorem 2.20]):

$$\frac{\sum_{j=1}^{n} p_j a_j b_j \sum_{j=1}^{n} p_j a_j \sum_{j=1}^{n} p_j b_j}{\sum_{j=1}^{n} p_j} \le \sum_{j=1}^{n} p_j a_j^2 \sum_{j=1}^{n} p_j b_j^2$$

that holds for the nonnegative numbers a_j, b_j, p_j with $j \in \{1, \ldots, n\}$ and $\sum_{j=1}^{n} p_j > 0$.

Utilising this inequality, we have

$$\sum_{j=0}^{n} |a_j| \, \|A\|^j \, \|B\|^j \le \sum_{j=0}^{n} |a_j| \cdot \frac{\sum_{j=0}^{n} |a_j| \, \|A\|^{2j} \sum_{j=0}^{n} |a_j| \, \|B\|^{2j}}{\sum_{j=0}^{n} |a_j| \, \|A\|^2 \sum_{j=0}^{n} |a_j| \, \|B\|^2}$$

which together with (2.5) produces

$$\left\| \sum_{j=0}^{n} a_j (AB)^j \right\| \le \sum_{j=0}^{n} |a_j| \cdot \frac{\sum_{j=0}^{n} |a_j| \, \|A\|^{2j} \sum_{j=0}^{n} |a_j| \, \|B\|^{2j}}{\sum_{j=0}^{n} |a_j| \, \|A\|^2 \sum_{j=0}^{n} |a_j| \, \|B\|^2}. \tag{2.20}$$

Since all the series whose partial sums are involved in (2.20) are convergent, then by taking $n \to \infty$ in (2.20) we deduce the inequality (2.18). □

REMARK 3. The condition $f_a(1) < \infty$ can be avoided if a complex parameter $|z| < R$ is introduced. Namely, we can obtain the following generalization of (2.18)

$$\| f(zAB) \| \le f_a(|z|) \cdot \frac{f_a\left(|z| \, \|A\|^2\right) f_a\left(|z| \, \|B\|^2\right)}{f_a\left(|z| \, \|A\|\right) f_a\left(|z| \, \|B\|\right)}, \tag{2.21}$$

provided $|z| \, \|A\|^2, |z| \, \|B\|^2, |z| \, \|A\|, |z| \, \|B\| < R$.

The finite-dimensional version of Theorem 3 is as follows:

THEOREM 4. *Theorem* 3 *also holds for every unitarily invariant norm* $|||\cdot|||$ *on a finite matrix algebra. Moreover, we have the inequalities*

$$\begin{aligned} & |||f(AB)|||^2 \\ & \le f_a\left(|||\,|A|^2|||^{\frac{1+\alpha}{2}} \, |||\,|B|^2|||^{\frac{1-\alpha}{2}}\right) f_a\left(|||\,|A|^2|||^{\frac{1-\alpha}{2}} \, |||\,|B|^2|||^{\frac{1+\alpha}{2}}\right) \\ & \le f_a\left(|||\,|A|^2|||\right) f_a\left(|||\,|B|^2|||\right), \end{aligned} \tag{2.22}$$

provided

$$|||\,|A|^2|||, \; |||\,|B|^2||| < R, \tag{2.23}$$

where $\alpha \in [0, 1]$.

If $\sum_{n=0}^{\infty} |a_n| < \infty$ *and in addition to the condition* (2.23) *we have*

$$|||\,|A|^2|||^{1/2},\ |||\,|B|^2|||^{1/2} < R\,,$$

then

$$|||\, f\,(AB)\,||| \le f_a\,(1) \cdot \frac{f_a\big(|||\,|A|^2|||\big)\, f_a\,\big(|||\,|B|^2|||\big)}{f_a\big(|||\,|A|^2|||^{1/2}\big)\, f_a\big(|||\,|B|^2|||^{1/2}\big)}\,. \tag{2.24}$$

The details of the proof are left to the reader.

3. Some Vector Inequalities for Normal Operators

The case of normal operators is as follows:

THEOREM 5. *Let* $f\,(z) = \sum_{n=0}^{\infty} a_n z^n$ *be a function defined by power series with complex coefficients and convergent on the open disk* $D\,(0, R) \subset \mathbf{C}$, $R > 0$. *If* A *and* B *are two commuting normal operators on the Hilbert space* H, $z \in \mathbf{C}$ *and*

$$|z|\,\|A\|^2\,,\ |z|\,\|B\|^2 < R\,, \tag{3.1}$$

then we have

$$\big|\langle f\,(zAB)\,x, y\rangle\big|^2 \le \big\langle f_a(|z|\,|A|^2)x, x\big\rangle\big\langle f_a(|z|\,|B|^2)y, y\big\rangle \tag{3.2}$$

for any $x, y \in H$.

PROOF. By utilizing Schwarz inequality we have for any $x, y \in H$ that

$$\big|\langle A^j x, (B^*)^j y\rangle\big|^2 \le \langle A^j x, A^j x\rangle\big\langle (B^*)^j y, (B^*)^j y\big\rangle$$

for any $j \in \mathbf{N}$, which in operator modulus notations is equivalent with

$$\big|\langle B^j A^j x, y\rangle\big|^2 \le \big\langle |A^j|^2 x, x\big\rangle\big\langle |(B^*)^j|^2 y, y\big\rangle\,. \tag{3.3}$$

Since A and B are normal operators, then

$$|A^j|^2 = |A|^{2j} \quad \text{and} \quad \big|(B^*)^j\big|^2 = |B|^{2j}$$

for any $j \in \mathbf{N}$.

By the commutativity of A with B we also have

$$B^j A^j = (AB)^j$$

for any $j \in \mathbf{N}$ and then by (3.3) we have

$$\big|\langle (AB)^j\, x, y\rangle\big|^2 \le \big\langle |A|^{2j} x, x\big\rangle\big\langle |B|^{2j} y, y\big\rangle \tag{3.4}$$

for any $x, y \in H$ and for any $j \in \mathbf{N}$.

If we multiply the inequality (3.3) with $\left|a_j\right| |z|^j$, sum over j from 0 to m and use the generalized triangle inequality and the Cauchy-Bunyakovsky-Schwarz weighted inequality, we have successively

$$\begin{aligned}
(3.5)\qquad & \left|\left\langle \sum_{j=0}^{m} a_j z^j \,(AB)^j\, x, y\right\rangle\right| \\
&\le \sum_{j=0}^{m} |a_j||z|^j \left|\left\langle (AB)^j\, x, y\right\rangle\right| \\
&\le \sum_{j=0}^{m} |a_j|\,|z|^j \left\langle |A|^{2j} x, x\right\rangle^{1/2} \left\langle |B|^{2j} y, y\right\rangle^{1/2} \\
&\le \left(\sum_{j=0}^{m} |a_j|\,|z|^j \left\langle\, |A|^{2j}\, x, x\right\rangle\right)^{1/2} \left(\sum_{j=0}^{m} |a_j||z|^j \left\langle\, |B|^{2j}\, y, y\right\rangle\right)^{1/2} \\
&= \left\langle \sum_{j=0}^{m} |a_j|\,|z|^j\, |A|^{2j}\, x, x\right\rangle^{1/2} \left\langle \sum_{j=0}^{m} |a_j|\,|z|^j\, |B|^{2j}\, y, y\right\rangle^{1/2}
\end{aligned}$$

for any $x, y \in H$ and for any $m \in \mathbf{N}$.

Since the series $\sum_{j=0}^{\infty} \left|a_j\right| |z|^j\, |A|^{2j}$, $\sum_{j=0}^{\infty} \left|a_j\right| |z|^j\, |B|^{2j}$ and $\sum_{j=0}^{\infty} a_j z^j\, (AB)^j$ are convergent, then by taking the limit over $m \to \infty$ in (3.5) we deduce the desired result (3.2). □

COROLLARY 1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function defined by power series with real coefficients and convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. If A and B are two commuting normal operators on the Hilbert space H satisfying the condition* (3.1) *then we have the norm inequality*

$$\left\| f(zAB) \right\|^2 \le \left\| f_a(|z||A|^2) \right\| \left\| f_a(|z|\,|B|^2) \right\| \tag{3.6}$$

and the numerical radius inequality

$$w\big[f(zAB) \big] \le \frac{1}{2} \left\| f_a(|z|\,|A|^2) + f_a(|z|\,|B|^2) \right\|. \tag{3.7}$$

PROOF. From (3.2) we also have the inequalities

$$\begin{aligned}
\left|\langle f(zAB)\, x, x\rangle\right| &\le \left\langle f_a(|z|\,|A|^2)x, x\right\rangle^{1/2} \left\langle f_a(|z|\,|B|^2)x, x\right\rangle^{1/2} \\
&\le \frac{1}{2}\left\langle [f_a(|z|\,|A|^2) + f_a(|z|\,|B|^2)]x, x\right\rangle
\end{aligned}$$

for any $x \in H$, which, by taking the supremum over $\|x\| = 1$, produces the desired result (3.7). □

REMARK 4. If A is a normal operator and $z \in \mathbf{C}$ with $|z| \|A\|^2, |z| < R$, then by taking $B = 1_H$ in (3.2) we get

$$\left|\langle f(zA)\, x, y\rangle\right|^2 \le f_a(|z|) \left\langle f_a(|z|\,|A|^2)x, x\right\rangle \|y\|^2 \tag{3.8}$$

for any $x, y \in H$.

If A is a normal operator and $z \in \mathbf{C}$ with $|z| \|A\|^2, |z| < R$, then by taking $B = A$ in (3.2) we get

$$\left|\langle f(zA^2)x, y\rangle\right|^2 \le \left\langle f_a(|z|\,|A|^2)x, x\right\rangle\left\langle f_a(|z|\,|A|^2)y, y\right\rangle \tag{3.9}$$

and by taking $B = A^*$ in (3.2) we also get

$$\left|\langle f(z\,|A|^2)x, y\rangle\right|^2 \le \left\langle f_a(|z||A|^2)x, x\right\rangle\left\langle f_a(|z||A|^2)y, y\right\rangle \tag{3.10}$$

for any $x, y \in H$.

Moreover, if U and V are two commuting unitary operators, then by taking $A = U$ and $B = V$ in (3.2) we get

$$\left|\langle f(zUV)x, y\rangle\right| \le f_a(|z|)\|x\|\|y\| \tag{3.11}$$

for any $x, y \in H$ and $z \in \mathbf{C}$ with $|z| < R$.

The following result for two power series can be stated as well:

THEOREM 6. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with complex coefficients and both of them convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. If A and B are two normal operators on the Hilbert space H, $z, u \in \mathbf{C}$ and*

$$|z| \|A\|, |u| \|B\| \le R \tag{3.12}$$

then we have

$$\begin{aligned} &\left|\langle f(zA)x, g(uB)y\rangle\right|^2 \\ &\quad \le f_a\left(|z|^2\right) g_a\left(|u|^2\right)\left\langle f_a(|A|^2)x, x\right\rangle\left\langle g_a(|B|^2)y, y\right\rangle \end{aligned} \tag{3.13}$$

for any $x, y \in H$.

PROOF. By Schwarz's inequality we also have the following inequality for normal operators

$$\left|\langle A^j x, B^k y\rangle\right| \le \left\langle |A|^{2j} x, x\right\rangle^{1/2}\left\langle |B|^{2k} y, y\right\rangle^{1/2} \tag{3.14}$$

for any $x, y \in H$ and $j, k \in \mathbf{N}$.

If we multiply (3.14) with $\left|a_j\right| |z|^j |b_k| |u|^k$, sum over j and k from 0 to m and use the generalized triangle inequality, then we have successively

$$
\begin{aligned}
(3.15)\qquad &\left|\left\langle \sum_{j=0}^{m} a_j z^j A^j x, \sum_{k=0}^{m} b_k u^k B^k y \right\rangle\right| \\
&\le \sum_{j=0}^{m}\sum_{k=0}^{m} |a_j|\,|z|^j\,|b_k|\,|u|^k\,\left|\langle A^j x, B^k y\rangle\right| \\
&\le \sum_{j=0}^{m}\sum_{k=0}^{m} |a_j|\,|z|^j\,|b_k|\,|u|^k\,\left\langle |A|^{2j} x, x\right\rangle^{1/2} \left\langle |B|^{2k} y, y\right\rangle^{1/2} \\
&= \sum_{j=0}^{m} |a_j|\,|z|^j\,\left\langle |A|^{2j} x, x\right\rangle^{1/2} \sum_{k=0}^{m} |b_k|\,|u|^k\,\left\langle |B|^{2k} y, y\right\rangle^{1/2}
\end{aligned}
$$

for any $x, y \in H$ and $m \in \mathbf{N}$.

Further, by the Cauchy-Bunyakovsky-Schwarz inequality we also have

$$\sum_{j=0}^{m} |a_j|\,|z|^j \left\langle |A|^{2j} x, x\right\rangle^{1/2} \le \left(\sum_{j=0}^{m} |a_j|\,|z|^{2j}\right)^{1/2} \left\langle \sum_{j=0}^{m} |a_j|\,|A|^{2j} x, x\right\rangle^{1/2}$$

and

$$\sum_{k=0}^{m} |b_k|\,|u|^k \left\langle |B|^{2k} y, y\right\rangle^{1/2} \le \left(\sum_{k=0}^{m} |b_k|\,|u|^{2k}\right)^{1/2} \left\langle \sum_{k=0}^{m} |b_k|\,|B|^{2k} y, y\right\rangle^{1/2}$$

for any $x, y \in H$ and $m \in \mathbf{N}$, which together with (3.15) provide

$$
\begin{aligned}
(3.16)\qquad &\left|\left\langle \sum_{j=0}^{m} a_j z^j A^j x, \sum_{k=0}^{m} b_k u^k B^k y \right\rangle\right| \\
&\le \left(\sum_{j=0}^{m} |a_j|\,|z|^{2j}\right)^{1/2} \left\langle \sum_{j=0}^{m} |a_j|\,|z|^j\,|A|^{2j} x, x\right\rangle^{1/2} \\
&\quad\times \left(\sum_{k=0}^{m} |b_k|\,|u|^{2k}\right)^{1/2} \left\langle \sum_{k=0}^{m} |b_k|\,|u|^k\,|B|^{2k} y, y\right\rangle^{1/2}
\end{aligned}
$$

for any $x, y \in H$ and $m \in \mathbf{N}$.

Since the series whose partial sums are involved in the inequality (3.16) are convergent, then taking the limit over $m \to \infty$ in (3.16) we deduce the desired result (3.13). □

COROLLARY 2. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and be $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be two functions defined by power series with real coefficients and both of them convergent on the open disk $D(0, R) \subset \mathbf{C}$, $R > 0$. If A and B are two normal operators on the Hilbert space H that* *satisfy condition* (3.12) *then we have*

$$\left\| g\left(\overline{u}B^{*}\right) f\left(zA\right)\right\|^{2} \leq f_{a}\left(|z|^{2}\right) g_{a}\left(|u|^{2}\right) \left\| f_{a}\left(|A|^{2}\right)\right\| \left\| g_{a}\left(|B|^{2}\right)\right\| \tag{3.17}$$

and

$$w\big(g(\overline{u}B^{*}) f(zA)\big) \leq \frac{1}{2} f_{a}\big(|z|^{2}\big) g_{a}\big(|u|^{2}\big) \left\| f_{a}\big(|A|^{2}\big) + g_{a}\big(|B|^{2}\big)\right\| . \tag{3.18}$$

4. Some Examples

As some natural examples that are useful for applications, we can point out that, if

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n}}{n} z^{n} = \ln \frac{1}{1+z}, \quad z \in D(0,1); \\ g(z) &= \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)!} z^{2n} = \cos z, \quad z \in \mathbf{C}; \\ h(z) &= \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)!} z^{2n+1} = \sin z, \quad z \in \mathbf{C}; \\ l(z) &= \sum_{n=0}^{\infty} (-1)^{n} z^{n} = \frac{1}{1+z}, \quad z \in D(0,1); \end{aligned} \tag{4.1}$$

then the corresponding functions constructed by the use of the absolute values of the coefficients are

$$\begin{aligned} f_{a}(z) &= \sum_{n=1}^{\infty} \frac{1}{n} z^{n} = \ln \frac{1}{1-z}, \quad z \in D(0,1); \\ g_{a}(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n} = \cosh z, \quad z \in \mathbf{C}; \\ h_{A}(z) &= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1} = \sinh z, \quad z \in \mathbf{C}; \\ l_{A}(z) &= \sum_{n=0}^{\infty} z^{n} = \frac{1}{1-z}, \quad z \in D(0,1). \end{aligned} \tag{4.2}$$

Other important examples of functions as power series representations with nonnegative coefficients are:

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{n} \quad z \in \mathbf{C}, \tag{4.3}$$

$$\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \quad z\in D(0,1);$$

$$\sin^{-1}(z)=\sum_{n=0}^{\infty}\frac{\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}\,(2n+1)\,n!}z^{2n+1}, \quad z\in D(0,1);$$

$$\tanh^{-1}(z)=\sum_{n=1}^{\infty}\frac{1}{2n-1}z^{2n-1}, \quad z\in D(0,1)$$

$$_2F_1(\alpha,\beta,\gamma,z)=\sum_{n=0}^{\infty}\frac{\Gamma(n+\alpha)\,\Gamma(n+\beta)\,\Gamma(\gamma)}{n!\Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(n+\gamma)}z^n, \alpha,\beta,\gamma>0,$$
$$z\in D(0,1);$$

where Γ is Gamma function.

On making use of Theorem 1, we can state some particular examples as follows:

EXAMPLE 1. a) If A and B are two bounded operators on the Hilbert space H and for $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\|A\|,\|B\|<1$, then

$$\left\|(1_H\pm AB)^{-1}\right\|\leq\min\left\{S_1(p,q),S_2(p,q)\right\} \tag{4.4}$$

where

$$S_1(p,q):=\left(1-\|A\|^p\right)^{-1/p}\left(1-\|B\|^q\right)^{-1/q},$$

and

$$S_2(p,q):=\frac{1-\|A\|^{p-1}\|B\|^{q-1}}{\left(1-\|A\|^p\right)\left(1-\|B\|^q\right)}.$$

We also have the following inequality for the logarithm

$$\left\|\ln(1_H\pm AB)^{-1}\right\|\leq\min\left\{T_1(p,q),T_2(p,q)\right\} \tag{4.5}$$

where

$$T_1(p,q):=\left[\ln(1-\|A\|^p)^{-1}\right]^{1/p}\left[\ln(1-\|B\|^q)^{-1}\right]^{1/q}$$

and

$$T_2(p,q):=\frac{\left[\ln(1-\|A\|^p)^{-1}\right]\left[\ln(1-\|B\|^q)^{-1}\right]}{\ln\left(1-\|A\|^{p-1}\|B\|^{q-1}\right)^{-1}}.$$

b) If A and B are two bounded operators on the Hilbert space H and $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, then

$$\left\|\exp(AB)\right\|\leq\min\left\{U_1(p,q),U_2(p,q)\right\} \tag{4.6}$$

where

$$U_1(p,q) := \exp\left(\frac{1}{p}\|A\|^p + \frac{1}{q}\|B\|^q\right),$$

and

$$U_2(p,q) := \exp\bigl(\|A\|^p + \|B\|^q - \|A\|^{p-1}\|B\|^{q-1}\bigr).$$

Theorem 2 provides the following results for unitarily invariant norm $|||\cdot|||$ on a finite matrix algebra.

EXAMPLE 2. a) Let $|||\cdot|||$ be a unitarily invariant norm on a finite matrix algebra. If $|||\,|A|^p|||$, $|||\,|B|^q||| < 1$, where $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then

$$|||(I \pm AB)^{-1}||| \le \min\left\{V_1(p,q), V_2(p,q)\right\} \tag{4.7}$$

where

$$V_1(p,q) := \bigl(1 - |||\,|A|^p|||\bigr)^{-1/p}\bigl(1 - |||\,|B|^q|||\bigr)^{-1/q},$$

and

$$V_2(p,q) := \frac{1 - |||\,|A|^p|||^{1/q}\,|||\,|B|^q|||^{1/p}}{\bigl(1 - |||\,|A|^p|||\bigr)\bigl(1 - |||\,|B|^q|||\bigr)},$$

and

$$\left|\left|\left|\ln(I \pm AB)^{-1}\right|\right|\right| \le \min\left\{W_1(p,q), W_2(p,q)\right\} \tag{4.8}$$

where

$$W_1(p,q) := \left[\ln(1 - |||\,|A|^p|||)^{-1}\right]^{1/p}\left[\ln(1 - |||\,|B|^q|||)^{-1}\right]^{1/q},$$

and

$$W_2(p,q) := \frac{\ln\bigl(1 - |||\,|A|^p|||\bigr)^{-1}\ln\bigl(1 - |||\,|B|^q|||\bigr)^{-1}}{\ln\bigl(1 - |||\,|A|^p|||^{1/q}\,|||\,|B|^q|||^{1/p}\bigr)^{-1}}.$$

b) For any two matrices we have

$$\left|\left|\left|\exp(AB)\right|\right|\right| \le \min\left\{Z_1(p,q), Z_2(p,q)\right\} \tag{4.9}$$

where

$$Z_1(p,q) := \exp\left(\frac{1}{p}|||\,|A|^p||| + \frac{1}{q}|||\,|B|^q|||\right),$$

and

$$Z_2(p,q) := \exp\left(|||\,|A|^p||| + |||\,|B|^q||| - |||\,|A|^p|||^{1/q}\,|||\,|B|^q|||^{1/p}\right).$$

Employing the inequalities from Theorem 3 and Remark 3 we can state:

EXAMPLE 3. a) If A and B are two bounded operators on the Hilbert space H and

$$\|A\|, \|B\| < 1\,,$$

then

$$\left\|(1_H \pm AB)^{-1}\right\|^2 \le \left(1-\|A\|^{1+\alpha}\|B\|^{1-\alpha}\right)^{-1}\left(1-\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right)^{-1} \le \left(1-\|A\|^2\right)^{-1}\left(1-\|B\|^2\right)^{-1}, \tag{4.10}$$

and

$$\left\|\ln(1_H \pm AB)^{-1}\right\|^2 \le \ln\left(1-\|A\|^{1+\alpha}\|B\|^{1-\alpha}\right)^{-1}\ln\left(1-\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right)^{-1} \le \ln\left(1-\|A\|^2\right)^{-1}\ln\left(1-\|B\|^2\right)^{-1}, \tag{4.11}$$

where $\alpha \in [0,1]$.

b) For any bounded linear operators A and B we have the inequalities

$$\|\exp(AB)\|^2 \le \exp\left(\|A\|^{1+\alpha}\|B\|^{1-\alpha}+\|A\|^{1-\alpha}\|B\|^{1+\alpha}\right) \le \exp\left(\|A\|^2+\|B\|^2\right), \tag{4.12}$$

where $\alpha \in [0,1]$, and

$$\|\exp(zAB)\| \le \exp\left(|z|\left(1+\|A\|^2+\|B\|^2-\|A\|-\|B\|\right)\right), \tag{4.13}$$

where $z \in \mathbf{C}$.

Finally, by the use of the result in Theorem 5 we also have:

EXAMPLE 4. a) If A and B are two commuting normal operators on the Hilbert space H with $\|A\|, \|B\| < 1$ and $z \in D(0,1)$ then we have the inequalities

$$\left|\left\langle (1_H \pm zAB)^{-1}x, y\right\rangle\right|^2 \le \left\langle (1_H - |z|\,|A|^2)^{-1}x, x\right\rangle\left\langle (1_H - |z|\,|B|^2)^{-1}y, y\right\rangle, \tag{4.14}$$

$$\left|\left\langle \ln(1_H \pm zAB)^{-1}x, y\right\rangle\right|^2 \le \left\langle \ln(1_H - |z|\,|A|^2)^{-1}x, x\right\rangle\left\langle \ln(1_H - |z|\,|B|^2)^{-1}y, y\right\rangle, \tag{4.15}$$

and

$$\left|\left\langle {}_2F_1(\alpha,\beta,\gamma,zAB)\,x, y\right\rangle\right|^2 \le \left\langle {}_2F_1(\alpha,\beta,\gamma,|z|\,|A|^2)x, x\right\rangle\left\langle {}_2F_1(\alpha,\beta,\gamma,|z|\,|B|^2)y, y\right\rangle \tag{4.16}$$

where $\alpha, \beta, \gamma > 0$, for any $x, y \in H$.

b) If A and B are two commuting normal operators on the Hilbert space H and $z \in \mathbf{C}$ then we have the inequalities

$$\left|\langle \sin(zAB)\,x,\,y\rangle\right|^2, \left|\langle \sinh(zAB)\,x,\,y\rangle\right|^2 \le \left\langle \sinh(|z||A|^2)x,\,x\right\rangle\left\langle \sinh(|z||B|^2)y,\,y\right\rangle, \tag{4.17}$$

$$\left|\langle \cos(zAB)\,x,\,y\rangle\right|^2, \left|\langle \cosh(zAB)\,x,\,y\rangle\right|^2 \le \left\langle \cosh(|z|\,|A|^2)x,\,x\right\rangle\left\langle \cosh(|z||B|^2)y,\,y\right\rangle \tag{4.18}$$

and

$$\left|\langle \exp(zAB)\,x,\,y\rangle\right|^2 \le \left\langle \exp(|z||A|^2)x,\,x\right\rangle\left\langle \exp(|z||B|^2)y,\,y\right\rangle \tag{4.19}$$

for any $x, y \in H$.

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