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The Best Constant of Three Kinds of Discrete Sobolev Inequalities on Regular Polyhedron

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Abstract. We consider three kinds of discrete Sobolev inequalities corresponding to a graph Laplacian $\boldsymbol{A}$ on regular M-hedron for $M = 4, 6, 8, 12, 20$. Discrete heat kernel $\boldsymbol{H}(t) = \exp(-t\boldsymbol{A})$, Green matrix $\boldsymbol{G}(a) = (\boldsymbol{A}+a\boldsymbol{I})^{-1}$ and pseudo Green matrix $\boldsymbol{G}_*$ are obtained and investigated in a detailed manner. The best constants of the inequalities are given by means of eigenvalues of $\boldsymbol{A}$.

1. Conclusion

We consider 5 kinds of regular M-hedron for $M = 4, 8, 6, 20, 12$.

Regular polyhedron	Surfaces M	Vertices N	Edges E
Tetrahedron	4	4	6
Octahedron	8	6	12
Hexahedron	6	8	12
Icosahedron	20	12	30
Dodecahedron	12	20	30

From Euler polyhedron theorem, $M + N = E + 2$ holds. Considering the symmetries of polyhedra, we have set the indices of vertices as Fig.1.1 $\sim$ Fig.1.5. We define the set $e = e(M)$, where each element $(i, j) = (j, i)$ represents edge connecting vertices i and j, as follows:

$e(4) = \{(0, 1), (1, 2), (2, 0), (0, 3), (1, 3), (2, 3)\}$.

$e(8) = \{(0, 1), (1, 2), (2, 0), (5, 4), (4, 3), (3, 5), (0, 5), (1, 4), (2, 3), (0, 3), (1, 5), (2, 4)\}$.

$e(6) = \{(0, 1), (1, 2), (2, 3), (3, 0), (4, 5), (5, 6), (6, 7), (7, 4), (0, 4), (1, 5), (2, 6), (3, 7)\}$.

$e(20) = \{(8, 0), (0, 4), (4, 8), (10, 6), (6, 2), (2, 10), (8, 9), (0, 1), (4, 5), (11, 10), (7, 6),$

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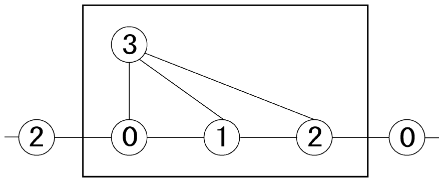


FIGURE 1.1. Tetrahedron

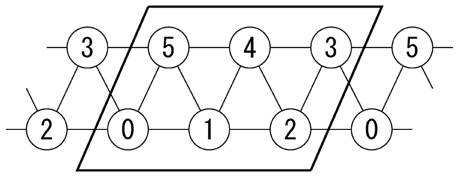


FIGURE 1.2. Octahedron

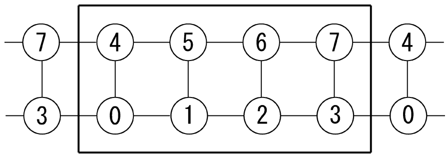


FIGURE 1.3. Hexahedron

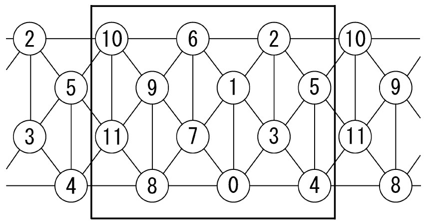


FIGURE 1.4. Icosahedron

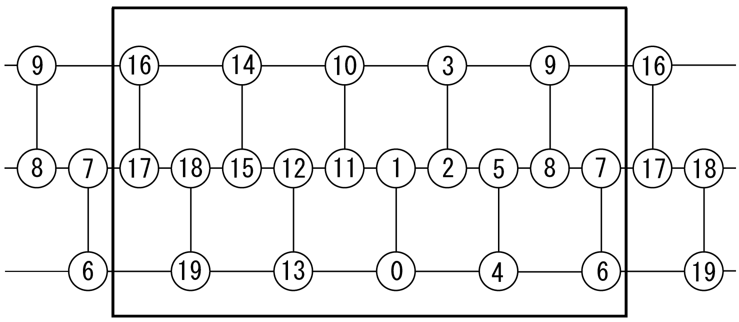


FIGURE 1.5. Dodecahedron

$$(3,2),\,(4,11),\,(11,9),\,(9,6),\,(8,7),\,(7,1),\,(1,2),\,(0,3),\,(3,5),\,(5,10),\,(8,11),\,(11,5),$$
$$(5,2),\,(0,7),\,(7,9),\,(9,10),\,(4,3),\,(3,1),\,(1,6)\Big\}.$$
$$e(12)=\Big\{(19,13),\,(13,0),\,(0,4),\,(4,6),\,(6,19),\,(16,14),\,(14,10),\,(10,3),\,(3,9),\,(9,16),$$
$$(19,18),\,(13,12),\,(0,1),\,(4,5),\,(6,7),\,(17,16),\,(15,14),\,(11,10),\,(2,3),\,(8,9),$$
$$(17,18),\,(18,15),\,(15,12),\,(12,11),\,(11,1),\,(1,2),\,(2,5),\,(5,8),\,(8,7),\,(7,17)\Big\}.$$

We introduce $\boldsymbol{A} = \boldsymbol{A}(M) = \big(a(M; i, j)\big)$ $(0 \le i, j \le N-1)$ on regular polyhedra.

$$\begin{array}{llllll} a(4; i, j) = & 3 \quad (i = j), & -1 & (i, j) \in e(4), & 0 & (\text{else}). \\ a(8; i, j) = & 4 \quad (i = j), & -1 & (i, j) \in e(8), & 0 & (\text{else}). \\ a(6; i, j) = & 3 \quad (i = j), & -1 & (i, j) \in e(6), & 0 & (\text{else}). \\ a(20; i, j) = & 5 \quad (i = j), & -1 & (i, j) \in e(20), & 0 & (\text{else}). \\ a(12; i, j) = & 3 \quad (i = j), & -1 & (i, j) \in e(12), & 0 & (\text{else}). \end{array}$$

We call $\boldsymbol{A}$ "graph Laplacian" or "discrete harmonic operator" in this paper. $\boldsymbol{A}$ is a $N \times N$ real symmetric non-negative definite matrix and has an eigenvalue 0, whose corresponding eigenvector is $\boldsymbol{1} = {}^t(1, 1, \dots, 1)$. Let $\lambda_0 = 0, \lambda_1 > 0, \dots, \lambda_{N-1} > 0$ be eigenvalues of $\boldsymbol{A}$. Jordan canonical form of $\boldsymbol{A}$ is given by $\widetilde{\boldsymbol{A}} = \widetilde{\boldsymbol{A}}(M) = \mathrm{diag}\{\lambda_0, \lambda_1, \dots, \lambda_{N-1}\}$. Distributions of the eigenvalues of $\boldsymbol{A}$ are shown in appendix. Concrete forms of $\widetilde{\boldsymbol{A}}(M)$ are given as

$$\widetilde{\boldsymbol{A}}(4) = \mathrm{diag}\{0, 4, 4, 4\}, \quad \widetilde{\boldsymbol{A}}(8) = \mathrm{diag}\{0, 4, 4, 4, 6, 6\}, \quad \widetilde{\boldsymbol{A}}(6) = \mathrm{diag}\{0, 2, 2, 2, 4, 4, 4, 6\},$$

$$\widetilde{\boldsymbol{A}}(20) = \mathrm{diag}\{0, 6, 6, 6, 6, 6, 5-p, 5-p, 5-p, 5+p, 5+p, 5+p\},$$

$$\widetilde{\boldsymbol{A}}(12) = \mathrm{diag}\{0, 2, 2, 2, 2, 2, 3, 3, 3, 3, 5, 5, 5, 5, 3-p, 3-p, 3-p, 3+p, 3+p, 3+p\},$$

where $p = \sqrt{5}$. For $M = 4, 8, 6, 20, 12$, $n = 1, 2, 3, \dots$ and $0 < a < \infty$, we introduce $C_0(n) = C_0(M; n)$, $C_0(n, a) = C_0(M; n, a)$ and $C_1(a) = C_1(M; a)$ as

$$C_0(n) = \frac{1}{N} \sum_{k=1}^{N-1} \lambda_k^{-n}, \quad C_0(n, a) = \frac{1}{N} \sum_{k=0}^{N-1} (\lambda_k + a)^{-n}, \quad C_1(a) = \frac{1}{2} C_0(1, a).$$

For any $\boldsymbol{u} = {}^t(u(0), u(1), \dots, u(N-1)) \in \mathbf{C}^N$, $\boldsymbol{u}(t) = {}^t(u(0, t), u(1, t), \dots, u(N-1, t)) \in \mathbf{C}^N$ on regular polyhedron, we define three kinds of Sobolev energy using $\boldsymbol{A}$ as

$$E(n, \boldsymbol{u}) = \boldsymbol{u}^* \boldsymbol{A}^n \boldsymbol{u}, \quad E(n, a, \boldsymbol{u}) = \boldsymbol{u}^* (\boldsymbol{A} + a\boldsymbol{I})^n \boldsymbol{u},$$

$$F(a, \boldsymbol{u}(t)) = \int_{-\infty}^{\infty} \left\| \left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I} \right) \boldsymbol{u}(t) \right\|^2 dt,$$

where $\|\boldsymbol{u}(t)\|^2 = \boldsymbol{u}^*(t)\boldsymbol{u}(t)$. In our previous work [2, §4], we obtained the concrete forms of

$$\text{Discrete heat kernel} \qquad \boldsymbol{H}(t) = \exp(-t\boldsymbol{A}), \tag{1.1}$$

$$\text{Green matrix} \qquad \boldsymbol{G}(a) = (a\boldsymbol{I} + \boldsymbol{A})^{-1} = \int_0^{\infty} e^{-at} \boldsymbol{H}(t) dt, \tag{1.2}$$

$$\text{Pseudo Green matrix} \qquad \boldsymbol{G}_* = \lim_{a \to +0} \big(\boldsymbol{G}(a) - a^{-1} \boldsymbol{E}_0 \big), \tag{1.3}$$

where $\boldsymbol{E}_0 = N^{-1} \boldsymbol{1}^t \boldsymbol{1}$ is projection matrix to eigenspace for eigenvalue 0 of $\boldsymbol{A}$. The element of $\boldsymbol{H}(t)$ in (1.1) are all positive value shown by [2, §5,6].

In this paper, we have obtained the best constants of three kinds of discrete Sobolev inequalities on regular polyhedron as following theorems.

THEOREM 1.1. *For any $\boldsymbol{u} \in \mathbf{C}^N$ with ${}^t\mathbf{1}\boldsymbol{u} = 0$, there exists a positive constant C which is independent of $\boldsymbol{u}$, such that the discrete Sobolev inequality*

$$\left(\max_{0 \le j \le N-1} |\, u(j) \,| \right)^2 \le C\, E(n, \boldsymbol{u}) \tag{1.4}$$

holds. Among such C, the best constant is $C_0(M; n)$, or equivalently

$$C_0(4; n) = \frac{3}{4^{n+1}}\,, \quad C_0(8; n) = \frac{3^{n+1} + 2^{n+1}}{6 \cdot 12^n}\,, \quad C_0(6; n) = \frac{3 \cdot 6^n + 3^{n+1} + 2^n}{8 \cdot 12^n}\,,$$

$$C_0(20; n) = \frac{5 \cdot 10^n + 3^{n+1}\left((5+p)^n + (5-p)^n\right)}{12 \cdot 60^n},$$

$$C_0(12; n) = \frac{1}{20 \cdot 60^n}\Big[5 \cdot 30^n + 4 \cdot 20^n + 4 \cdot 12^n + 3 \cdot 15^n \left((3+p)^n + (3-p)^n\right)\Big].$$

If one replaces C by $C_0(M; n)$ in the above inequality (1.4), *the equality holds iff $\boldsymbol{u}$ is parallel to one of column vectors of $\boldsymbol{G}_*^n$.*

THEOREM 1.2. *For any $\boldsymbol{u} \in \mathbf{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}$, such that the discrete Sobolev inequality*

$$\left(\max_{0 \le j \le N-1} |\, u(j) \,| \right)^2 \le C\, E(n, a, \boldsymbol{u}) \tag{1.5}$$

holds. Among such C, the best constant is $C_0(M; n, a)$. If one replaces C by $C_0(M; n, a)$ in the above inequality (1.5), *the equality holds iff $\boldsymbol{u}$ is parallel to one of column vectors of $\boldsymbol{G}(a)^n$.*

THEOREM 1.3. *For any bounded continuous function $\boldsymbol{u}(t) \in \mathbf{C}^N$, there exists a positive constant C which is independent of $\boldsymbol{u}(t)$, such that the discrete Sobolev-type inequality*

$$\left(\sup_{0 \le j \le N-1,\ -\infty < s < \infty} |\, u(j, s) \,| \right)^2 \le C\, F(a, \boldsymbol{u}(t)) \tag{1.6}$$

holds. Among such C, the best constant is $C_1(M; a) = \frac{1}{2} C_0(M; 1, a)$. If one replaces C by $C_1(M; a)$ in the above inequality (1.6), *the equality holds iff $\boldsymbol{u}$ is parallel to one of column vectors of*

$$\int_{|t|}^{\infty} \frac{1}{2} e^{-a\sigma} \boldsymbol{H}(\sigma) d\sigma \quad (-\infty < t < \infty)\,. \tag{1.7}$$

Here, we list the concrete form of $C_0(M;n)$ and $C_0(M;n,a)$ for small value of n.

n	$C_0(4;n)$	$C_0(8;n)$	$C_0(6;n)$	$C_0(20;n)$	$C_0(12;n)$
1	$\frac{3}{16}$	$\frac{13}{72}$	$\frac{29}{96}$	$\frac{7}{36}$	$\frac{137}{300}$
2	$\frac{3}{64}$	$\frac{35}{864}$	$\frac{139}{1152}$	$\frac{53}{1080}$	$\frac{3197}{9000}$
3	$\frac{3}{256}$	$\frac{97}{10368}$	$\frac{737}{13824}$	$\frac{187}{12960}$	$\frac{203989}{540000}$
4	$\frac{3}{1024}$	$\frac{275}{124416}$	$\frac{4147}{165888}$	$\frac{913}{194400}$	$\frac{3718217}{8100000}$
5	$\frac{3}{4096}$	$\frac{793}{1492992}$	$\frac{24089}{1990656}$	$\frac{377}{233280}$	$\frac{142218677}{243000000}$

$$C_0(4;1,a) = \frac{a+1}{a(a+4)}, \quad C_0(8;1,a) = \frac{a^2+6a+4}{a(a+4)(a+6)},$$
$$C_0(6;1,a) = \frac{a^3+9a^2+20a+6}{a(a+2)(a+4)(a+6)}, \quad C_0(20;1,a) = \frac{a^3+11a^2+30a+10}{a(a+6)(a^2+10a+20)},$$
$$C_0(12;1,a) = \frac{a^5+13a^4+59a^3+109a^2+70a+6}{a(a+2)(a+3)(a+5)(a^2+6a+4)}.$$

$$C_0(4;2,a) = \frac{a^2+2a+4}{a^2(a+4)^2}, \quad C_0(8;2,a) = \frac{a^4+12a^3+48a^2+80a+96}{a^2(a+4)^2(a+6)^2},$$
$$C_0(6;2,a) = \frac{a^6+18a^5+124a^4+408a^3+664a^2+528a+288}{a^2(a+2)^2(a+4)^2(a+6)^2},$$
$$C_0(20;2,a) = \frac{a^6+22a^5+186a^4+760a^3+1560a^2+1600a+1200}{a^2(a+6)^2(a^2+10a+20)^2},$$
$$C_0(12;2,a) = \frac{\gamma(a)}{a^2(a+2)^2(a+3)^2(a+5)^2(a^2+6a+4)^2},$$
$$\gamma(a) = a^{10}+26a^9+290a^8+1812a^7+6947a^6+16842a^5$$
$$+25718a^4+23960a^3+12808a^2+3648a+720.$$

Research on discrete Sobolev inequalities was performed in [1] on graphs and also in our previous papers [3] and [4] on periodic one-dimensional lattices. In [2], we investigated discrete Sobolev inequalities corresponding to discrete harmonic operator on regular polyhedra and obtained Theorem 1.1 and 1.2 in the case of $n = 1$. In this paper, we extend these conclusions into discrete polyharmonic operator $\boldsymbol{A}^n$ $(n = 2, 3, \dots)$.

This paper is organized as follows. In section 2, we start with difference equations and corresponding Green matrices. In section 3, we calculate diagonal values of the matrices, which is shown to be the best constants of discrete Sobolev inequalities later. In section 4, we derive reproducing relations. Finally in section 5, we prove the above theorems.

2. Difference equations and Green matrices

We explain three propositions concerning discrete heat kernel (1.1), Green matrix (1.2) and pseudo Green matrix (1.3) in this section. We assume that $0 < a < \infty$ and $n = 1, 2, 3, \ldots$.

PROPOSITION 2.1. *For arbitrary bounded continuous function $\boldsymbol{f}(t) \in \mathbf{C}^N$, the discrete heat equation*

$$\left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{u} = \boldsymbol{f}(t) \quad (-\infty < t < \infty) \tag{2.1}$$

has a unique solution given by

$$\boldsymbol{u}(t) = \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{f}(s)ds\,, \quad \boldsymbol{H}_*(t) = Y(t)e^{-at}\boldsymbol{H}(t) \quad (-\infty < t < \infty)\,, \tag{2.2}$$

where $Y(t) = 1\ (0 \le t < \infty),\ 0\ (-\infty < t < 0)$ is Heaviside step function.

PROOF OF PROPOSITION 2.1. Through Fourier transform

$$\boldsymbol{u}(t) \quad \overset{\frown}{\longrightarrow} \quad \widehat{\boldsymbol{u}}(\omega) = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t}\boldsymbol{u}(t)\,dt\,,$$

(2.1) is transformed into $\left(\sqrt{-1}\,\omega + \boldsymbol{A} + a\boldsymbol{I}\right)\widehat{\boldsymbol{u}}(\omega) = \widehat{\boldsymbol{f}}(\omega)\ (-\infty < \omega < \infty)$. Solving this we have $\widehat{\boldsymbol{u}}(\omega) = \widehat{\boldsymbol{H}}_*(\omega)\,\widehat{\boldsymbol{f}}(\omega)$, where

$$\widehat{\boldsymbol{H}}_*(\omega) = \left(\sqrt{-1}\,\omega + \boldsymbol{A} + a\boldsymbol{I}\right)^{-1} = \int_{-\infty}^{\infty} e^{-\sqrt{-1}\omega t}Y(t)e^{-at}\boldsymbol{H}(t)\,dt \quad (-\infty < \omega < \infty)\,.$$

From inverse Fourier transform, we have (2.2). This completes the proof of Proposition 2.1. ■

It should be noted that $\boldsymbol{H}_*(t)$ satisfies the relation,

$$\left(\frac{d}{dt} + \boldsymbol{A} + a\boldsymbol{I}\right)\boldsymbol{H}_* = \boldsymbol{O}\,, \quad \boldsymbol{H}_*(t-s)\Big|_{s=t-0} - \boldsymbol{H}_*(t-s)\Big|_{s=t+0} = \boldsymbol{I} \quad (-\infty < t < \infty)\,.$$

Let λ_k $(0 \le k \le N-1)$ be eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{q}_k \in \mathbf{C}^N$ $(0 \le k \le N-1)$ be corresponding eigenvectors. The eigenvectors $\boldsymbol{q}_k$ are chosen to satisfy the relation $\boldsymbol{q}_k^*\boldsymbol{q}_l = \delta(k-l)$, where $\delta(k) = 1\ (k=0),\ \ 0\ (k \ne 0)$. We introduce a unitary $N \times N$ matrix $\boldsymbol{Q} = (\boldsymbol{q}_0, \ldots, \boldsymbol{q}_{N-1})$ and orthogonal projection matrices $\boldsymbol{E}_k = \boldsymbol{q}_k\boldsymbol{q}_k^*$ $(0 \le k \le N-1)$. It is easy to see that the relations

$$\boldsymbol{Q}^*\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^* = \boldsymbol{I}\,, \quad \boldsymbol{E}_k\boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k\,, \quad \boldsymbol{E}_k^* = \boldsymbol{E}_k$$

holds. Using $\boldsymbol{E}_k$, we have the spectral decomposition of $\boldsymbol{I}$ and $\boldsymbol{A}$ as

$$\boldsymbol{I} = \boldsymbol{Q}\boldsymbol{Q}^* = \sum_{k=0}^{N-1} \boldsymbol{q}_k \boldsymbol{q}_k^* = \sum_{k=0}^{N-1} \boldsymbol{E}_k, \tag{2.3}$$

$$\boldsymbol{A} = \boldsymbol{Q}\widetilde{\boldsymbol{A}}\boldsymbol{Q}^* = \sum_{k=0}^{N-1} \lambda_k \boldsymbol{q}_k \boldsymbol{q}_k^* = \sum_{k=0}^{N-1} \lambda_k \boldsymbol{E}_k = \sum_{k=1}^{N-1} \lambda_k \boldsymbol{E}_k \,. \tag{2.4}$$

For (2.4), using $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we have

$$\boldsymbol{A}^n = \sum_{k=1}^{N-1} \lambda_k^n \boldsymbol{E}_k \,, \quad (\boldsymbol{A} + a\boldsymbol{I})^n = \sum_{k=0}^{N-1} (\lambda_k + a)^n \boldsymbol{E}_k \,. \tag{2.5}$$

PROPOSITION 2.2. *For arbitrary $\boldsymbol{f} \in \mathbf{C}^N$, the problem $(\boldsymbol{A}+a\boldsymbol{I})^n \boldsymbol{u} = \boldsymbol{f}$ has a unique solution is given by $\boldsymbol{u} = \boldsymbol{G}^n \boldsymbol{f}$. $\boldsymbol{G}^n = \boldsymbol{G}(a)^n$ is Green matrix expressed as*

$$\boldsymbol{G}^n = \sum_{k=0}^{N-1} (\lambda_k + a)^{-n} \boldsymbol{E}_k \,. \tag{2.6}$$

PROOF OF PROPOSITION 2.2. From (2.3) and the second formula of (2.5), we have

$$\sum_{k=0}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = (\boldsymbol{A} + a\boldsymbol{I})^n \boldsymbol{u} = \sum_{k=0}^{N-1} (\lambda_k + a)^n \boldsymbol{E}_k \boldsymbol{u} \,.$$

Operating $\boldsymbol{E}_l$ from the left on both sides of the above relation and using the relation $\boldsymbol{E}_k \boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l \boldsymbol{u} = (\lambda_l + a)^{-n} \boldsymbol{E}_l \boldsymbol{f}$ $(0 \le l \le N-1)$. Then, we have (2.6) as

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{N-1} \boldsymbol{E}_l \boldsymbol{u} = \sum_{l=0}^{N-1} (\lambda_l + a)^{-n} \boldsymbol{E}_l \boldsymbol{f} = \left(\sum_{l=0}^{N-1} (\lambda_l + a)^{-1} \boldsymbol{E}_l \right)^n \boldsymbol{f} = \boldsymbol{G}^n \boldsymbol{f} \,.$$

This completes the proof of Proposition 2.2. ■

PROPOSITION 2.3. *For arbitrary $\boldsymbol{f} \in \mathbf{C}^N$ satisfying the solvability condition ${}^t\mathbf{1}\boldsymbol{f} = 0$, the problem $\boldsymbol{A}^n \boldsymbol{u} = \boldsymbol{f}$ with the orthogonality condition ${}^t\mathbf{1}\boldsymbol{u} = 0$ has a unique solution is given by $\boldsymbol{u} = \boldsymbol{G}_*^n \boldsymbol{f}$. $\boldsymbol{G}_*^n$ is a pseudo Green matrix expressed as*

$$\boldsymbol{G}_*^n = \sum_{k=1}^{N-1} \lambda_k^{-n} \boldsymbol{E}_k \,. \tag{2.7}$$

$\boldsymbol{G}_*^n$ satisfies $\boldsymbol{A}^n \boldsymbol{G}_*^n = \boldsymbol{G}_*^n \boldsymbol{A}^n = \boldsymbol{I} - \boldsymbol{E}_0,\ \boldsymbol{G}_*^n \boldsymbol{E}_0 = \boldsymbol{E}_0 \boldsymbol{G}_*^n = \boldsymbol{O}$.

PROOF OF PROPOSITION 2.3. From (2.3) and the first formula of (2.5) and $\boldsymbol{E}_0 \boldsymbol{f} = N^{-1}\mathbf{1}{}^t\mathbf{1}\boldsymbol{f} = \mathbf{0}$, we have

$$\sum_{k=1}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \sum_{k=0}^{N-1} \boldsymbol{E}_k \boldsymbol{f} = \boldsymbol{I}\boldsymbol{f} = \boldsymbol{f} = \boldsymbol{A}^n \boldsymbol{u} = \sum_{k=1}^{N-1} \lambda_k^n \boldsymbol{E}_k \boldsymbol{u} \,.$$

Operating $\boldsymbol{E}_l$ from the left on both sides of the above relation and using the relation $\boldsymbol{E}_k\boldsymbol{E}_l = \delta(k-l)\boldsymbol{E}_k$, we obtain $\boldsymbol{E}_l\boldsymbol{u} = \lambda_l^{-n}\boldsymbol{E}_l\boldsymbol{f}$ $(1 \le l \le N-1)$. Then, using $\boldsymbol{E}_0\boldsymbol{u} = N^{-1}\boldsymbol{1}^t\boldsymbol{1}\boldsymbol{u} = \boldsymbol{0}$, we see that

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{l=0}^{N-1}\boldsymbol{E}_l\boldsymbol{u} = \sum_{l=1}^{N-1}\boldsymbol{E}_l\boldsymbol{u} = \sum_{l=1}^{N-1}\lambda_l^{-n}\boldsymbol{E}_l\boldsymbol{f} = \left(\sum_{l=1}^{N-1}\lambda_l^{-1}\boldsymbol{E}_l\right)^n\boldsymbol{f} = \boldsymbol{G}_*^n\boldsymbol{f}\,,$$

which gives (2.7). In fact, $\boldsymbol{G}_*^n$ satisfies

$$\boldsymbol{A}^n\boldsymbol{G}_*^n = \sum_{k=0}^{N-1}\sum_{l=1}^{N-1}\lambda_k^n\lambda_l^{-n}\boldsymbol{E}_k\boldsymbol{E}_l = \sum_{k=1}^{N-1}\boldsymbol{E}_k = \boldsymbol{I} - \boldsymbol{E}_0\,, \quad \boldsymbol{G}_*^n\boldsymbol{E}_0 = \sum_{k=1}^{N-1}\lambda_k^{-n}\boldsymbol{E}_k\boldsymbol{E}_0 = \boldsymbol{O}\,.$$

We see that $\boldsymbol{G}_*^n$ is a Penrose-Moore generalized inverse matrix of $\boldsymbol{A}^n$. This completes the proof of Proposition 2.3. ■

3. Best constant

We introduce N-dimensional vector

$$\boldsymbol{\delta}_j \;=\; {}^t(0,\ldots,0,\overset{j}{\smile}{1},0,\ldots,0)\,.$$

First, we show that the diagonal value of $\boldsymbol{G}^n$ is equal to a harmonic mean of all eigenvalues of $(\boldsymbol{A}+a\boldsymbol{I})^n$. We see that this value is equal to the best constant of (1.5) in Theorem 1.2.

LEMMA 3.1. *For $n = 1, 2, 3, \ldots$ and any fixed j $(0 \le j \le N-1)$, we have*

$$ {}^t\boldsymbol{\delta}_j\boldsymbol{G}^n\boldsymbol{\delta}_j = \frac{1}{N}\sum_{k=0}^{N-1}(\lambda_k + a)^{-n} = C_0(n,a)\,. \tag{3.1}$$

PROOF OF LEMMA 3.1. Since the diagonal values ${}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_j$ do not depend on j as we have seen in [2, §4], we have (3.1) in the case of $n = 1$ as

$$\begin{aligned} {}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_j &= \frac{1}{N}\sum_{j=0}^{N-1}{}^t\boldsymbol{\delta}_j\boldsymbol{G}\boldsymbol{\delta}_j = \frac{1}{N}\sum_{j=0}^{N-1}{}^t\boldsymbol{\delta}_j\sum_{k=0}^{N-1}(\lambda_k+a)^{-1}\,\boldsymbol{E}_k\boldsymbol{\delta}_j \\ &= \frac{1}{N}\sum_{k=0}^{N-1}(\lambda_k+a)^{-1}\sum_{j=0}^{N-1}{}^t\boldsymbol{\delta}_j\boldsymbol{E}_k\boldsymbol{\delta}_j = \frac{1}{N}\sum_{k=0}^{N-1}(\lambda_k+a)^{-1}\sum_{j=0}^{N-1}(\boldsymbol{q}_k^*\boldsymbol{\delta}_j)^2 \\ &= \frac{1}{N}\sum_{k=0}^{N-1}(\lambda_k+a)^{-1}\,. \end{aligned} \tag{3.2}$$

Using (1.2), we have

$$\boldsymbol{G}^2 = \int_0^\infty e^{-at}\boldsymbol{H}(t)dt\int_0^\infty e^{-as}\boldsymbol{H}(s)ds = \int_0^\infty\int_0^\infty e^{-a(t+s)}\boldsymbol{H}(t+s)dtds$$

$$= \frac{1}{2} \int_0^\infty \int_{-\tau}^{\tau} d\sigma e^{-a\tau} \boldsymbol{H}(\tau) d\tau = \int_0^\infty \tau e^{-a\tau} \boldsymbol{H}(\tau) d\tau$$
$$= -\partial_a \int_0^\infty e^{-a\tau} \boldsymbol{H}(\tau) d\tau = -\partial_a \boldsymbol{G} ,$$

that is, $\boldsymbol{G}^2 = -\partial_a \boldsymbol{G}$. So we have

$$\boldsymbol{G}^n = \boldsymbol{G}^{n-2}\boldsymbol{G}^2 = -\boldsymbol{G}^{n-2}\partial_a \boldsymbol{G} = \frac{-1}{n-1}\partial_a \boldsymbol{G}^{n-1} = \cdots = \frac{(-1)^{n-1}}{(n-1)!}\partial_a^{n-1}\boldsymbol{G} .$$

Then, taking the diagonal value $\boldsymbol{G}^n$ and using (3.2), we have

$$^t\boldsymbol{\delta}_j \boldsymbol{G}^n \boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_j \frac{(-1)^{n-1}}{(n-1)!}\partial_a^{n-1} \boldsymbol{G}\boldsymbol{\delta}_j = \frac{(-1)^{n-1}}{(n-1)!}\partial_a^{n-1}\left({}^t\boldsymbol{\delta}_j \boldsymbol{G}\boldsymbol{\delta}_j\right)$$
$$= \frac{(-1)^{n-1}}{(n-1)!}\partial_a^{n-1}\left(\frac{1}{N}\sum_{j=0}^{N-1}\left(\lambda_j + a\right)^{-1}\right) = \frac{1}{N}\sum_{j=0}^{N-1}\left(\lambda_j + a\right)^{-n} .$$

This completes the proof of Lemma 3.1. ■

Next, we show that the diagonal value of $\boldsymbol{G}_*^n$ is equal to a harmonic mean of positive eigenvalues of $\boldsymbol{A}^n$. We see that this value is equal to the best constant of (1.4) in Theorem 1.1.

LEMMA 3.2. *For $n = 1, 2, 3, \ldots$ and any fixed j $(0 \le j \le N-1)$, we have*

$$^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{\delta}_j = \frac{1}{N}\sum_{k=1}^{N-1}\lambda_k^{-n} = C_0(n) . \tag{3.3}$$

PROOF OF LEMMA 3.2. Using the relation

$$\boldsymbol{G}\boldsymbol{E}_0 = \sum_{j=0}^{N-1}\left(\lambda_j + a\right)^{-1}\boldsymbol{E}_j \boldsymbol{E}_0 = a^{-1}\boldsymbol{E}_0 , \quad \boldsymbol{G}^k \boldsymbol{E}_0 = a^{-k}\boldsymbol{E}_0 ,$$
$$\boldsymbol{E}_0^k = \boldsymbol{E}_0 \quad (k = 1, 2, 3, \ldots),$$

we have

$$\left(\boldsymbol{G} - a^{-1}\boldsymbol{E}_0\right)^n = \sum_{k=0}^{n}\binom{n}{k}\boldsymbol{G}^{n-k}\left(-a^{-1}\boldsymbol{E}_0\right)^k = \binom{n}{0}\boldsymbol{G}^n + \sum_{k=1}^{n}\binom{n}{k}(-1)^k a^{-k}\boldsymbol{G}^{n-k}\boldsymbol{E}_0^k$$
$$= \boldsymbol{G}^n + a^{-n}\boldsymbol{E}_0 \sum_{k=1}^{n}\binom{n}{k}(-1)^k = \boldsymbol{G}^n - a^{-n}\boldsymbol{E}_0 .$$

For the diagonal value of the above relation, using (3.1), we have

$$^t\boldsymbol{\delta}_j\left(\boldsymbol{G} - a^{-1}\boldsymbol{E}_0\right)^n\boldsymbol{\delta}_j = {}^t\boldsymbol{\delta}_j\left(\boldsymbol{G}^n - a^{-n}\boldsymbol{E}_0\right)\boldsymbol{\delta}_j = \frac{1}{N}\sum_{k=1}^{N-1}(\lambda_k + a)^{-n} .$$

Taking the limit as $a \to +0$ of both sides and using (1.3), we have (3.3). This completes the proof of Lemma 3.2. ■

Finally, we show that L^2 norm of one of column vectors of $\boldsymbol{H}_*(t)$ is equal to a half of harmonic mean of positive eigenvalues of $\boldsymbol{A}$. We see that this value is equal to the best constant of (1.6) in Theorem 1.3.

LEMMA 3.3. *For any fixed* j $(0 \le j \le N-1)$, *we have*

$$\int_{-\infty}^{\infty} \left\| \boldsymbol{H}_*(t)\boldsymbol{\delta}_j \right\|^2 dt = C_1(a)\,. \tag{3.4}$$

PROOF OF LEMMA 3.3. Noting ${}^t\boldsymbol{H} = \boldsymbol{H}$ and using (3.1), we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left\| \boldsymbol{H}_*(t)\boldsymbol{\delta}_j \right\|^2 dt &= \int_{-\infty}^{\infty} \left\| Y(t)e^{-at}\boldsymbol{H}(t)\boldsymbol{\delta}_j \right\|^2 dt \\
&= \int_{-\infty}^{\infty} {}^t\big(Y(t)e^{-at}\boldsymbol{H}(t)\boldsymbol{\delta}_j\big)\big(Y(t)e^{-at}\boldsymbol{H}(t)\boldsymbol{\delta}_j\big)dt \\
&= {}^t\boldsymbol{\delta}_j \int_0^{\infty} e^{-2at}\boldsymbol{H}(2t)dt\boldsymbol{\delta}_j = \frac{1}{2}{}^t\boldsymbol{\delta}_j \int_0^{\infty} e^{-a\tau}\boldsymbol{H}(\tau)d\tau\boldsymbol{\delta}_j \\
&= \frac{1}{2}{}^t\boldsymbol{\delta}_j\boldsymbol{G}(a)\boldsymbol{\delta}_j = \frac{1}{2}C_0(1,a) = C_1(a)\,.
\end{aligned}$$

So we have (3.4). This completes the proof of Lemma 3.3. ■

4. Reproducing relation

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^N$, we introduce inner product

$$(\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{v}^*\boldsymbol{u}\,, \quad \|\,\boldsymbol{u}\,\|^2 = (\boldsymbol{u}, \boldsymbol{u})\,,$$

$$(\boldsymbol{u},\ \boldsymbol{v})_H = ((\boldsymbol{A}+a\boldsymbol{I})^n\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{v}^*(\boldsymbol{A}+a\boldsymbol{I})^n\boldsymbol{u}\,, \quad \|\,\boldsymbol{u}\,\|_H^2 = (\boldsymbol{u}, \boldsymbol{u})_H = E(n, a, \boldsymbol{u})\,.$$

For $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}_0^N := \{\, \boldsymbol{x} \mid \boldsymbol{x} \in \mathbf{C}^N \ and\ {}^t\mathbf{1}\boldsymbol{x} = 0 \,\}$, we introduce inner product

$$(\boldsymbol{u},\ \boldsymbol{v})_A = (\boldsymbol{A}^n\boldsymbol{u},\ \boldsymbol{v}) = \boldsymbol{v}^*\boldsymbol{A}^n\boldsymbol{u}\,, \quad \|\,\boldsymbol{u}\,\|_A^2 = (\boldsymbol{u}, \boldsymbol{u})_A = E(n, \boldsymbol{u})\,.$$

First, we show the positive definiteness of Sobolev inner product $(\cdot,\cdot)_A$ and $(\cdot,\cdot)_H$.

LEMMA 4.1.

(1) *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}_0^N$, $(\boldsymbol{u}, \boldsymbol{v})_A$ *is an inner product.*

(2) *For* $\boldsymbol{u}, \boldsymbol{v} \in \mathbf{C}^N$, $(\boldsymbol{u}, \boldsymbol{v})_H$ *is an inner product.*

PROOF OF LEMMA 4.1. (2) is obvious since $(\boldsymbol{A}+a\boldsymbol{I})^n$ is positive definite. We show only (1). For $\boldsymbol{u} \in \mathbf{C}_0^N$, we have

$$\boldsymbol{u} = \boldsymbol{I}\boldsymbol{u} = \sum_{k=0}^{N-1} \boldsymbol{E}_k\boldsymbol{u} = \sum_{k=1}^{N-1} \boldsymbol{E}_k\boldsymbol{u}\,, \quad \|\boldsymbol{u}\|^2 = \sum_{k,l=0}^{N-1} \boldsymbol{u}^*\boldsymbol{E}_l^*\boldsymbol{E}_k\boldsymbol{u} = \sum_{k=1}^{N-1} \|\boldsymbol{E}_k\boldsymbol{u}\|^2\,.$$

From the relation $\boldsymbol{E}_k = \boldsymbol{E}_k \boldsymbol{E}_k = \boldsymbol{E}_k^* \boldsymbol{E}_k$, we have

$$\|\boldsymbol{u}\|_A^2 = \boldsymbol{u}^* \boldsymbol{A}^n \boldsymbol{u} = \boldsymbol{u}^* \sum_{k=0}^{N-1} \lambda_k^n \boldsymbol{E}_k \boldsymbol{u} = \sum_{k=1}^{N-1} \lambda_k^n \boldsymbol{u}^* \boldsymbol{E}_k^* \boldsymbol{E}_k \boldsymbol{u} = \sum_{k=1}^{N-1} \lambda_k^n \|\boldsymbol{E}_k \boldsymbol{u}\|^2$$

$$\geq \Big(\min_{1 \leq k \leq N-1} \lambda_k^n\Big) \sum_{k=1}^{N-1} \|\boldsymbol{E}_k \boldsymbol{u}\|^2 = \Big(\min_{1 \leq k \leq N-1} \lambda_k^n\Big) \|\boldsymbol{u}\|^2 .$$

Since $\lambda_k > 0$ $(1 \leq k \leq N-1)$, we have $\|\boldsymbol{u}\|_A^2 \geq 0$ and $\|\boldsymbol{u}\|_A^2 = 0$ holds iff $\boldsymbol{u} = \boldsymbol{0}$. This completes the proof of Lemma 4.1. ■

Next, we show that $\boldsymbol{G}^n$ and $\boldsymbol{G}_*^n$ are reproducing matrix for Sobolev inner product $(\cdot, \cdot)_H$ and $(\cdot, \cdot)_A$, respectively. Applying Schwarz inequality to these reproducing relations, we can prove Theorem 1.1 and 1.2 in next section.

LEMMA 4.2. *For any* $\boldsymbol{u} \in \mathbf{C}_0^N$ *and fixed* j $(0 \leq j \leq N-1)$, *we have the following reproducing relations.*

(1) $u(j) = (\boldsymbol{u},\ \boldsymbol{G}_*^n \boldsymbol{\delta}_j)_A$.

(2) $C_0(n) = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{\delta}_j = \|\ \boldsymbol{G}_*^n \boldsymbol{\delta}_j\ \|_A^2 = E(n, \boldsymbol{G}_*^n \boldsymbol{\delta}_j)$.

PROOF OF LEMMA 4.2. Note that $\boldsymbol{G}_*^{n*} = \boldsymbol{G}_*^n$. For any $\boldsymbol{u} \in \mathbf{C}_0^N$ and fixed j $(0 \leq j \leq N-1)$, we have (1) as follows:

$$(\boldsymbol{u},\ \boldsymbol{G}_*^n \boldsymbol{\delta}_j)_A = (\boldsymbol{A}^n \boldsymbol{u},\ \boldsymbol{G}_*^n \boldsymbol{\delta}_j) = \boldsymbol{\delta}_j^* \boldsymbol{G}_*^{n*} \boldsymbol{A}^n \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{A}^n \boldsymbol{u}$$

$$= {}^t\boldsymbol{\delta}_j (\boldsymbol{I} - \boldsymbol{E}_0) \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{I} \boldsymbol{u} - {}^t\boldsymbol{\delta}_j \boldsymbol{E}_0 \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{u} - \frac{1}{N} \boldsymbol{1}{}^t\boldsymbol{1} \boldsymbol{u} = u(j) .$$

Putting $\boldsymbol{u} = \boldsymbol{G}_*^n \boldsymbol{\delta}_j$ in (1) and using (3.3), we obtain (2). ■

LEMMA 4.3. *For any* $\boldsymbol{u} \in \mathbf{C}^N$ *and fixed* j $(0 \leq j \leq N-1)$, *we have the following reproducing relations.*

(1) $u(j) = (\boldsymbol{u},\ \boldsymbol{G}^n \boldsymbol{\delta}_j)_H$.

(2) $C_0(n, a) = {}^t\boldsymbol{\delta}_j \boldsymbol{G}^n \boldsymbol{\delta}_j = \|\ \boldsymbol{G}^n \boldsymbol{\delta}_j\ \|_H^2 = E(n, a, \boldsymbol{G}^n \boldsymbol{\delta}_j)$.

PROOF OF LEMMA 4.3. Note that $\boldsymbol{G}^{n*} = \boldsymbol{G}^n$. For any $\boldsymbol{u} \in \mathbf{C}^N$ and fixed j $(0 \leq j \leq N-1)$, we have (1) as follows:

$$(\boldsymbol{u},\ \boldsymbol{G}^n \boldsymbol{\delta}_j)_H = ((\boldsymbol{A} + a\boldsymbol{I})^n \boldsymbol{u},\ \boldsymbol{G}^n \boldsymbol{\delta}_j) = {}^t\boldsymbol{\delta}_j \boldsymbol{G}^n\ (\boldsymbol{A} + a\boldsymbol{I})^n \boldsymbol{u} = {}^t\boldsymbol{\delta}_j \boldsymbol{I} \boldsymbol{u} = u(j) .$$

Putting $\boldsymbol{u} = \boldsymbol{G}^n \boldsymbol{\delta}_j$ in (1) and using (3.1), we obtain (2). ■

5. Proof of Theorems

This section is devoted to the proof of main theorems.

PROOF OF THEOREM 1.1. Applying Schwarz inequality to Lemma 4.2 (1) and using Lemma 4.2 (2), we have

$$|\,u(j)\,|^2 \le \|\,\boldsymbol{u}\,\|_A^2 \|\,\boldsymbol{G}_*^n \boldsymbol{\delta}_j\,\|_A^2 = C_0(n)\,E(n,\boldsymbol{u})\,. \tag{5.1}$$

Taking the maximum with respect to j on both sides, we obtain discrete Sobolev inequality

$$\left(\max_{0\le j\le N-1} |\,u(j)\,|\right)^2 \le C_0(n)\,E(n,\boldsymbol{u})\,. \tag{5.2}$$

It should be noted that in performing Schwarz inequality in (5.1), equality holds if and only if $\boldsymbol{u} = k\boldsymbol{G}_*^n\boldsymbol{\delta}_j$ $(k \ne 0,\ 0 \le j \le N-1)$.

For any fixed number j_0 $(0 \le j_0 \le N-1)$, if we take $\boldsymbol{u} = \boldsymbol{G}_*^n\boldsymbol{\delta}_{j_0}$ in the above inequality, then we have

$$\left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{\delta}_{j_0}\,|\right)^2 \le C_0(n)E(n,\boldsymbol{G}_*^n\boldsymbol{\delta}_{j_0}) = C_0(n)^2\,.$$

Combining this and a trivial inequality $C_0(n)^2 \le \left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{\delta}_{j_0}\,|\right)^2$, we have

$$\left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}_*^n \boldsymbol{\delta}_{j_0}\,|\right)^2 = C_0(n)E(n,\boldsymbol{G}_*^n\boldsymbol{\delta}_{j_0})\,.$$

This shows that $C_0(n)$ is the best constant of (5.2) and the equality holds for each column vector of $\boldsymbol{G}_*^n$. This completes the proof of Theorem 1.1. ■

PROOF OF THEOREM 1.2. Applying Lemma 4.3 (1) to Schwarz inequality and using Lemma 4.3 (2), we see that

$$|\,u(j)\,|^2 \le \|\,\boldsymbol{u}\,\|_H^2 \|\,\boldsymbol{G}^n \boldsymbol{\delta}_j\,\|_H^2 = C_0(n,a)\,E(n,a,\boldsymbol{u})\,. \tag{5.3}$$

Taking the maximum with respect to j on both sides, we have the discrete Sobolev inequality

$$\left(\max_{0\le j\le N-1} |\,u(j)\,|\right)^2 \le C_0(n,a)\,E(n,a,\boldsymbol{u})\,. \tag{5.4}$$

It should be noted that in performing Schwarz inequality in (5.3), equality holds if and only if $\boldsymbol{u} = k\boldsymbol{G}^n\boldsymbol{\delta}_j$ $(k \ne 0,\ 0 \le j \le N-1)$.

For any fixed number j_0 $(0 \le j_0 \le N-1)$, if we take $\boldsymbol{u} = \boldsymbol{G}^n\boldsymbol{\delta}_{j_0}$ in the above inequality, then we have

$$\left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}^n \boldsymbol{\delta}_{j_0}\,|\right)^2 \le C_0(n,a)\,E(n,a,\boldsymbol{G}^n\boldsymbol{\delta}_{j_0}) = C_0(n,a)^2\,.$$

Combining this and a trivial inequality $C_0(n,a)^2 \le \left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}^n \boldsymbol{\delta}_{j_0}\,|\right)^2$, we have

$$\left(\max_{0\le j\le N-1} |\,{}^t\boldsymbol{\delta}_j \boldsymbol{G}^n \boldsymbol{\delta}_{j_0}\,|\right)^2 = C_0(n,a)E(n,a,\boldsymbol{G}^n\boldsymbol{\delta}_{j_0})\,.$$

This shows that $C_0(n,a)$ is the best constant of (5.4) and the equality holds for each column vector of $\boldsymbol{G}^n$. This completes the proof of Theorem 1.2. ■

PROOF OF THEOREM 1.3. Exchanging t and s in the first formula of (2.2), we have

$$\boldsymbol{u}(s)=\int_{-\infty}^{\infty}\boldsymbol{H}_*(s-t)\boldsymbol{f}(t)dt\,,$$

or equivalently

$$u(j,s)={}^t\boldsymbol{\delta}_j\boldsymbol{u}(s)=\int_{-\infty}^{\infty}{}^t\boldsymbol{\delta}_j\boldsymbol{H}_*(s-t)\boldsymbol{f}(t)dt=\int_{-\infty}^{\infty}{}^t\Big(\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j\Big)\boldsymbol{f}(t)dt\,. \tag{5.5}$$

Applying Schwarz inequality to (5.5), we have

$$\begin{aligned}|\,u(j,s)\,|^2&\le\int_{-\infty}^{\infty}\big\|\boldsymbol{H}_*(s-t)\boldsymbol{\delta}_j\big\|^2dt\int_{-\infty}^{\infty}\|\boldsymbol{f}(t)\|^2dt\\&=\int_{-\infty}^{\infty}\big\|\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\big\|^2dt\int_{-\infty}^{\infty}\left\|\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{u}(t)\right\|^2dt\\&=C_1(a)F(a,\boldsymbol{u}(t))\,,\end{aligned} \tag{5.6}$$

where we use (2.1) and (3.4). Taking the supremum with respect to j and s, we obtain Sobolev-type inequality,

$$\left(\sup_{0\le j\le N-1,\ -\infty<s<\infty}|\,u(j,s)\,|\right)^2\le C_1(a)\ F(a,\boldsymbol{u}(t)). \tag{5.7}$$

It should be noted that in performing Schwarz inequality in (5.6), equality holds if and only if the relation

$$\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{u}(t)=k\boldsymbol{H}_*(t)\boldsymbol{\delta}_j\quad(k\ne0)$$

holds.

For any fixed number j_0 $(0\le j_0\le N-1)$, we introduce a vector $\boldsymbol{U}(t)={}^t(U(0,t),U(1,t),\dots,U(N-1,t))\in\mathbf{C}^N$ defined by

$$\boldsymbol{U}(t)=\int_{-\infty}^{\infty}\boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}ds\,,\quad U(j,t)=\int_{-\infty}^{\infty}{}^t\boldsymbol{\delta}_j\boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0}ds\,. \tag{5.8}$$

Then we have

$$\begin{aligned}&\left(\sup_{0\le j\le N-1,\ -\infty<s<\infty}|\,U(j,s)\,|\right)^2\le C_1(a)F(a,\boldsymbol{U}(t))\\&\quad=C_1(a)\int_{-\infty}^{\infty}\left\|\left(\frac{d}{dt}+\boldsymbol{A}+a\boldsymbol{I}\right)\boldsymbol{U}(t)\right\|^2dt=C_1(a)\int_{-\infty}^{\infty}\|\boldsymbol{H}_*(-t)\|^2dt=C_1(a)^2\,.\end{aligned}$$

Combining a trivial inequality $C_1(a)^2 = |\,U(j_0, 0)\,|^2 \leq \left(\sup_{0 \leq j \leq N-1,\ -\infty < s < \infty} |\,U(j, s)\,| \right)^2$,

we have

$$\left(\sup_{0 \leq j \leq N-1,\ -\infty < s < \infty} |\,U(j, s)\,| \right)^2 = C_1(a)\, F(a, \boldsymbol{U}(t))\,.$$

This shows that $C_1(a)$ is the best constant of (5.7) and the equality holds for $\boldsymbol{u}(t) = \boldsymbol{U}(t)$. From (5.8), we have (1.7) as follows:

$$\begin{aligned}
\boldsymbol{U}(t) &= \int_{-\infty}^{\infty} \boldsymbol{H}_*(t-s)\boldsymbol{H}_*(-s)\boldsymbol{\delta}_{j_0} ds \\
&= \int_{-\infty}^{\infty} Y(t-s)e^{-a(t-s)}\boldsymbol{H}(t-s)Y(-s)e^{-a(-s)}\boldsymbol{H}(-s)\boldsymbol{\delta}_{j_0} ds \\
&= \int_{-\infty}^{0\wedge t} e^{-a(t-2s)}\boldsymbol{H}(t-2s)\boldsymbol{\delta}_{j_0} ds = \int_{|t|}^{\infty} \frac{1}{2} e^{-a\sigma}\boldsymbol{H}(\sigma)\boldsymbol{\delta}_{j_0} d\sigma\,,
\end{aligned}$$

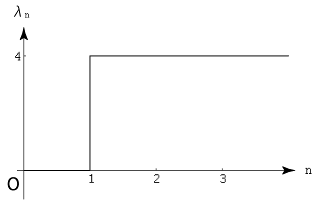
where we note that $x \wedge y = \min\{x, y\}$. This completes the proof of Theorem 1.3. ■

Appendix : Distribution of eigenvalues

We here illustrate distributions of eigenvalues of $\boldsymbol{A} = \boldsymbol{A}(M)$ ($M = 4, 8, 6, 20, 12$). We put $p = \sqrt{5}$.

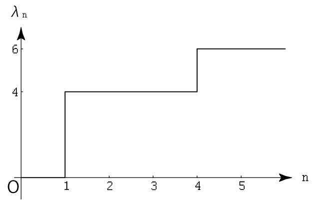
$\boldsymbol{A}(4)$

Eigenvalue	Multiplicity
0	1
4	3



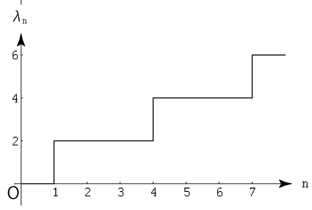
$\boldsymbol{A}(8)$

Eigenvalue	Multiplicity
0	1
4	3
6	2



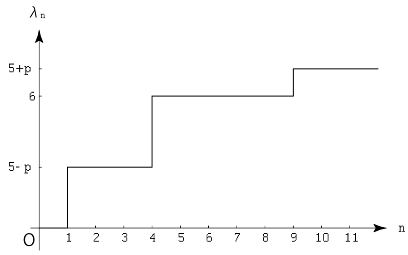
$\boldsymbol{A}(6)$

Eigenvalue	Multiplicity
0	1
2	3
4	3
6	1



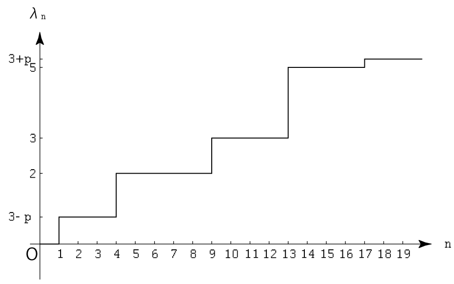
$\boldsymbol{A}(20)$

Eigenvalue	Multiplicity
0	1
$5-p$	3
6	5
$5+p$	3



$\boldsymbol{A}(12)$

Eigenvalue	Multiplicity
0	1
$3-p$	3
2	5
3	4
5	4
$3+p$	3



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References

[1] F. R. K. CHUNG and S.-T. YAU, Eigenvalues of graphs and Sobolev inequalities, Combin. Probab. Comput. **4** (1995), 11–25.

[2] Y. KAMETAKA, K. WATANABE, H. YAMAGISHI, A. NAGAI and K. TAKEMURA, The Best Constant of Discrete Sobolev Inequality on Regular Polyhedron, Transactions of the Japan Society for Industrial and Applied Mathematics **21** (2011), 289–308 [in Japanese].

[3] A. NAGAI, Y. KAMETAKA, H. YAMAGISHI, K. TAKEMURA and K. WATANABE, Discrete Bernoulli polynomials and the best constant of discrete Sobolev inequality, Funkcial. Ekvac. **51** (2008), 307–327.

[4] H. YAMAGISHI, A. NAGAI, K. WATANABE, K. TAKEMURA and Y. KAMETAKA, The best constant of discrete Sobolev inequality corresponding to a bending problem of a string, Kumamoto J. Math. **25** (2012), 1–15.

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