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RSK Type Correspondence of Pictures and Littlewood-Richardson Crystals

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Abstract. We present a Robinson-Schensted-Knuth type one-to-one correspondence between the set of pictures and the set of pairs of Littlewood-Richardson crystals.

1. Introduction

Combinatorics of pictures has been initiated in [1, 2, 6, 14]. Picture is a certain bijective order morphism between two skew Young diagrams with some partial/total orders. The remarkable result for pictures is that there exists a kind of RSK type one to one correspondence as follows. Let κ^i $(i = 1, 2)$ be skew Young diagrams with $|\kappa^1| = |\kappa^2| (= N)$. There exists a bijection:

$$\mathbf{P}(\kappa^1, \kappa^2) \quad \overset{1:1}{\longleftrightarrow} \quad \coprod_{\mu} \left(\mathbf{P}(\mu, \kappa^1) \times \mathbf{P}(\mu, \kappa^2) \right), \tag{1.1}$$

where μ runs over the set of Young diagrams with $|\mu| = N$ and $\mathbf{P}(\kappa^1, \kappa^2)$ is a set of pictures from κ^1 to κ^2. Since some set of pictures can be identified with a set of permutations, this correspondence can be seen as an analogue of the RSK correspondence. In [3, 13], certain generalizations have been done using various combinatorial methods.

In [11, 12], we introduced the one to one correspondence between "Littlewood-Richardson crystals" and pictures.

$$\mathbf{P}(\mu, \nu \setminus \lambda) \quad \overset{1:1}{\longleftrightarrow} \quad \mathbf{B}(\mu)^{\nu}_{\lambda}, \tag{1.2}$$

where λ, μ, ν are Young diagrams with $|\lambda| + |\mu| = |\nu|$. This seems to give a new interpretation of pictures from the view point of the theory of crystal bases.

In this article, we shall describe the following bijections

$$\mathbf{P}(\kappa^1, \kappa^2) \quad \overset{1:1}{\longleftrightarrow} \quad \mathbf{S}(\kappa^1, \kappa^2) \quad \overset{1:1}{\longleftrightarrow} \quad \mathbf{W}(\kappa^1, \kappa^2) \quad \overset{1:1}{\longleftrightarrow} \quad \coprod_{\mu} \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2} \right), \tag{1.3}$$

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where $\mathbf{P}(\kappa^1, \kappa^2)$ is the set of pictures from κ^1 to κ^2, $\mathbf{S}(\kappa^1, \kappa^2)$ is the set of Littlewood-Richardson skew tableaux associated with (κ^1, κ^2), $\mathbf{W}(\kappa^1, \kappa^2)$ is the set of lexicographic two-rowed array (of column type) associated with (κ^1, κ^2) and the last one is a set of pairs of Littlewood-Richardson crystals. Thus, applying (1.2) to the last one in (1.3) we obtain the original correspondence (1.1). The pictures treated in this article are defined by the order J (see Sect.2), which is a kind of admissible orders. More general setting, namely defined by general admissible orders will be discussed elsewhere.

As is well known that the crystal $\mathbf{B}(\mu)$ of type $\mathfrak{gl}_n$ (or $\mathfrak{sl}_n$) is realized as the set of Young tableaux [9] and the Littlewood-Richardson crystal $\mathbf{B}(\mu)^\nu_\lambda$ is a subset of $\mathbf{B}(\mu)$ with the certain special conditions 'highest conditions' [10, 11, 12]. Thus, the last term in (1.3) is a set of pairs of same shaped Young tableaux and then bijections in (1.3) turn out to be a generalization of the RSK correspondence.

As claimed in [11, 12], these methods would open the door to generalize the theory of pictures to wider classes. Indeed, in preparing this manuscript, we received the preprint 'Admissible pictures and $U_q(gl(m; n))$- Littlewood-Richardson tableaux' by J. H. Jung, S-J. Kang and Y-W. Lyoo, which gives the first bijection in (1.3) and generalizes it to the the super case $U_q(gl(m; n))$. This is a kind of the evidence of our claims, unfortunately, which was not done by us.

The organizations of the article is as follows: in Sect.2 and 3, the basics of pictures and crystals are reviewed. In Sect.4, we introduce several combinatorial procedures and notions required in this article; column bumping, RSK correspondence, Knuth equivalence, crystal equivalence and etc. The main theorem is given in Sect.5. and its proof is described separately in the subsequent sections.

2. Pictures

2.1. Young diagrams and Young tableaux. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ be a Young diagram or a partition, which satisfies $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$. For Young diagrams λ and μ with $\mu \subset \lambda$, a *skew diagram* $\lambda \setminus \mu$ is obtained by subtracting set-theoretically μ from λ.

In this article we frequently consider a (skew) Young diagram as a subset of $\mathbb{N} \times \mathbb{N}$ by identifying the box in i-th row and j-th column with $(i, j) \in \mathbb{N} \times \mathbb{N}$.

EXAMPLE 2.1. A Young diagram $\lambda = (2, 2, 1)$ is expressed by $\{(1, 1), (1, 2), (2, 1), (2, 2), (3, 1)\}$.

As in [4], in the sequel, a "(skew) Young tableau" means a semi-standard (skew) tableau. For a skew Young tableau S of shape $\lambda \setminus \mu$, we also consider a "coordinate" in $\mathbb{N} \times \mathbb{N}$ like as a skew diagram $\lambda \setminus \mu$. Then an entry of S in (i, j) is denoted by $S_{i,j}$ and called (i, j)-entry. For $k > 0$, define ([11])

$$S^{(k)} = \{(l, m) \in \lambda \setminus \mu | S_{l,m} = k\} . \tag{2.1}$$

There is no two elements in one column in $S^{(k)}$. For a skew Young tableau S with (i, j)-entry $S_{i,j} = k$, we define $p(S; i, j)$ ([11]) as the number of (i, j)-entry from the right in $S^{(k)}$.

2.2. Picture. First, we shall introduce the original notion of "picture" as in [14].

We define the following two kinds of orders on a subset $X \subset \mathbb{N}\times\mathbb{N}$: For $(a, b),\ (c, d) \in X$,

(i) $(a, b) \leqslant_P (c, d)$ iff $a \le c$ and $b \le d$.

(ii) $(a, b) \leqslant_J (c, d)$ iff $a < c$, or $a = c$ and $b \ge d$.

Note that the order $\leqslant_P$ is a partial order and $\leqslant_J$ is a total order.

DEFINITION 2.2 ([14]). Let $X, Y \subset \mathbb{N} \times \mathbb{N}$.

(i) A map $f : X \to Y$ is said to be *PJ-standard* if it satisfies

$$\text{For } (a, b), (c, d) \in X, \text{ if } (a, b) \leqslant_P (c, d),\ \text{ then } f(a, b) \leqslant_J f(c, d)\,.$$

(ii) A map $f : X \to Y$ is a *picture* if it is bijective and both f and f^{-1} are PJ-standard.

Taking two skew Young diagrams $\kappa^1, \kappa^2 \subset \mathbb{N} \times \mathbb{N}$, denote the set of pictures by:

$$\mathbf{P}(\kappa^1, \kappa^2) := \{f : \kappa^1 \to \kappa^2 \mid f \text{ is a picture.}\}$$

Next, we shall generalize the notion of pictures by using a total order on a subset $X \subset \mathbb{N} \times \mathbb{N}$, called an "*admissible order*", though we do not treat this generalization in this article:

DEFINITION 2.3. (i) A total order $\leqslant_A$ on $X \subset \mathbb{N} \times \mathbb{N}$ is called *admissible* if it satisfies:

$$\text{For any } (a, b),\ (c, d) \in X \text{ if } a \le c \text{ and } b \ge d \text{ then } (a, b) \leqslant_A (c, d).$$

(ii) For $X, Y \subset \mathbb{N} \times \mathbb{N}$ and a map $f : X \to Y$, if f satisfies that if $(a, b) \leqslant_P (c, d)$, then $f(a, b) \leqslant_A f(c, d)$ for any $(a, b),\ (c, d) \in X$, then f is called PA-standard.

(iii) Let $\leqslant_A$ (resp. $\leqslant_{A'}$) be an admissible order on X(resp. Y) $\subset \mathbb{N} \times \mathbb{N}$. A bijective map $f : X \to Y$ is called an (A, A')-*admissible picture* or simply, an *admissible picture* if f is PA-standard and f^{-1} is PA'-standard.

3. Crystals

The basic references for the theory of crystals are [7], [8].

3.1. Readings and Additions. Let $\mathbf{B} = \{\boxed{i} \mid 1 \le i \le n + 1\}$ be the crystal of the vector representation $V(\Lambda_1)$ of the quantum group $U_q(A_n)$ ([9]). As in [11], we shall identify a dominant weight of type A_n with a Young diagram in the standard way, *e.g.,* the fundamental weight Λ_1 is identified with a square box $\square$. For a Young diagram λ, let $B(\lambda)$ be the crystal of the finite-dimensional irreducible $U_q(A_n)$-module $V(\lambda)$. Set $N := |\lambda|$. Then there exists an embedding of crystals: $B(\lambda) \hookrightarrow \mathbf{B}^{\otimes N}$ and an element in $B(\lambda)$ is realized by a Young tableau of shape λ ([9]). Note that this embedding can be extended to skew tableaux, that is, there exists an embedding of crystals $S(\kappa) \hookrightarrow \mathbf{B}^{\otimes N}$, where $S(\kappa)$ is the set of skew tableaux of shape κ with $N = |\kappa|$ ([5]). Indeed, there are some dominant weights $\lambda_1, \dots, \lambda_k$ such that $S(\kappa) \cong B(\lambda_1) \oplus \cdots \oplus B(\lambda_k)$. Such an embedding is not unique, which is called a 'reading' and described by:

DEFINITION 3.1 ([5]). Let A be an admissible order on a (skew) Young diagram λ with $|\lambda| = N$. For $T \in B(\lambda)$ (resp. $S(\lambda)$), by reading the entries in T according to A, we obtain the map

$$R_A : B(\lambda)(\text{resp. } S(\lambda)) \longrightarrow \mathbf{B}^{\otimes N} \quad (T \mapsto \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}),$$

which is called an *admissible reading* associated with the order A. The map R_A is an embedding of crystals. In particular, in case that taking the order J as an admissible order, we denote the embedding R_J by ME and call it a *middle-eastern reading*.

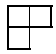
DEFINITION 3.2. For $i \in \{1, 2, \dots, n+1\}$ and a Young diagram $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we define

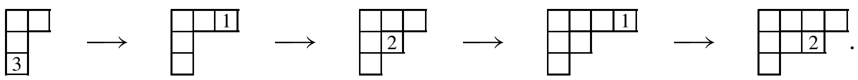
$$\lambda[i] := (\lambda_1, \lambda_2, \dots, \lambda_i + 1, \dots, \lambda_n)$$

which is said to be the *addition* of i to λ. In general, for $i_1, i_2, \dots, i_N \in \{1, 2, \dots, n+1\}$ and a Young diagram λ, we define

$$\lambda[i_1, i_2, \dots, i_N] := (\cdots((\lambda[i_1])[i_2])\cdots)[i_N],$$

which is called the *addition* of $i_1, \dots, i_N$ to λ.

EXAMPLE 3.3. For a sequence $\mathbf{i} = 31212$, the addition of $\mathbf{i}$ to $\lambda =$  is:



REMARK. For a Young diagram λ, the addition $\lambda[i_1, \dots, i_N]$ is not necessarily a Young diagram. For instance, a sequence $\mathbf{i}' = 22133$ and $\lambda = (2, 2)$, the addition $\lambda[\mathbf{i}'] = (3, 3, 2)$ is a Young diagram. But, in the second step of the addition, it becomes the diagram $\lambda[2, 2] = (2, 3)$, which is not a Young diagram.

3.2. Littlewood-Richardson Crystal. As an application of the description of crystal bases of type A_n, we see the so-called "Littlewood-Richardson rule" of type A_n.

For a sequence $\mathbf{i} = i_1 i_2 \cdots i_N$ ($i_j \in \{1, 2, \dots, n+1\}$) and a Young diagram λ, let $\tilde{\lambda} := \lambda[i_1, i_2, \dots, i_N]$ be an addition of $i_1, i_2, \dots, i_N$ to λ. Then set

$$\mathbf{B}(\lambda : \mathbf{i}) = \begin{cases} \mathbf{B}(\tilde{\lambda}) & \text{if } \lambda[i_1, \dots, i_k] \text{ is a Young diagram for any } k = 1, 2, \dots, N, \\ \emptyset & \text{otherwise.} \end{cases}$$

THEOREM 3.4 ([5, 10]). *Let λ and μ be Young diagrams with at most n rows. Then we have*

$$\mathbf{B}(\lambda) \otimes \mathbf{B}(\mu) \cong \bigoplus_{\substack{T \in \mathbf{B}(\mu),\\ \mathrm{ME}(T) = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}}} \mathbf{B}(\lambda : i_1, i_2, \ldots, i_N)\,. \tag{3.1}$$

Define

$$\mathbf{B}(\mu)^{\nu}_{\lambda} := \left\{ T \in \mathbf{B}(\mu) \;\middle|\; \begin{array}{l} \mathrm{ME}(T) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N}. \\ \text{For any } k = 1, \ldots, N, \\ \lambda[i_1, \ldots, i_k] \text{ is a Young diagram and} \\ \lambda[i_1, \ldots, i_N] = \nu. \end{array} \right\},$$

which is called the *Littlewood-Richardson crystal* associated with a triplet (λ, μ, ν).

4. Robinson-Schensted-Knuth(RSK) correspondence

In this section we review the Robinson-Schensted-Knuth(RSK) correspondence with respect to the column bumping procedure. For the contents of this section see [4] (in particular, Appendix A.).

4.1. Column Bumping and RSK Correspondence. For an integer x and a Young tableau T, we define the column bumping procedure:

DEFINITION 4.1. (i) (a) If all entries in the 1-st column of T are greater than x, put x just beneath the 1-st column and the procedure is over.

(b) Otherwise, let y be the top entry in the 1-st column that is equal to or smaller than x and put x in the box and bump the entry y out.

(c) Do the same one for y and the second column. If it does not stop at the last column, make a new box next to the last column and put the entry in the new box.

We denote the resulting tableau by $x \to T$.

(ii) The shape of $x \to T$ is a diagram added one box to the original shape of T. We shall denote the added new box by New(x) and call the new box by x.

The following lemma is known as the 'column bumping lemma'.

LEMMA 4.2. *Let T be a tableau and x, x' positive integers. In the column bumping $x' \to (x \to T)$, we have:*

(i) *If $x < x'$, then New(x') is weakly left of and strictly below New(x).*

(ii) *If $x \geq x'$, then New(x) is strictly left of and weakly below New(x').*

It is shown similarly to the row bumping lemma ([4]).

As is well-known that there is the reverse operation of this procedure, which is called an reverse (column) bumping.

DEFINITION 4.3. A two-rowed array $w = \begin{pmatrix} u_1 u_2 \cdots u_m \\ v_1 v_2 \cdots v_m \end{pmatrix}$ is in lexicographic order (of column type) if it satisfies: (i) $u_1 \le u_2 \le \cdots \le u_m$. (ii) If $u_k = u_{k+1}$, then $v_k \ge v_{k+1}$.

Let w be a two-rowed array in the lexicographic order with length m as above. We call the following procedure the RSK procedure:

(i) Set $P_1 = v_1$ and $Q_1 = u_1$.

(ii) We obtain (P_{k+1}, Q_{k+1}) from (P_k, Q_k) by $P_{k+1} = v_{k+1} \to P_k$ and put u_{k+1} to the same place in Q_k as the new box by v_{k+1} in P_{k+1}.

(iii) Set $R(w) := (P, Q) = (P_m, Q_m)$.

Note that P and Q are Young tableaux with entries $1, \dots, m$ and the same shape. We call the tableau Q the recording tableau of P. This procedure is reversible by using the reverse column bumping: For a pair of Young tableaux (P, Q), we apply the reverse bumping to P starting from the box in P which is in the same position as the box with the right-most maximum entry in Q and remove the entry from Q. Repeat this procedure until the tableaux become empty. We obtain the two-rowed array from (P, Q), which gives the reverse of the RSK procedure.

THEOREM 4.4 (RSK correspondence). *Let* $\mathbf{W}[n; m]$ *be the set of two-rowed array in the lexicographic order* (*of column type*) *with length m and entries* $1, \dots, n$ *and* $\mathbf{P}[n; m]$ *be the set of pairs of same-shaped Young tableaux with m boxes and entries* $1, \dots, n$. *Then the map R as above gives a bijection between* $\mathbf{W}[n; m]$ *and* $\mathbf{P}[n; m]$.

4.2. Knuth equivalence and Crystal equivalence. In this article, a *word* means a finite sequence of non-negative integers.

DEFINITION 4.5 (Knuth equivalence).

(i) Each of the following transformations between 3-letter words is called an elementary Knuth transformation:

(a) $K : yxz \longleftrightarrow yzx$ if $x < y \le z$

(b) $K' : xzy \longleftrightarrow zxy$ if $x \le y < z$.

(ii) If two words with same length w and w' are Knuth equivalent if one can be transformed to the other by a sequence of the elementary Knuth transformations and we denote it by $w \overset{k}{\sim} w'$.

Here let us mention the relation between the crystal $\mathbf{B}$ and the Knuth equivalence. The following lemma is well-known:

LEMMA 4.6. *There exists the following non-trivial isomorphism of crystals*: $\mathbf{R} : \mathbf{B} \otimes \mathbf{B} \otimes \mathbf{B} \to \mathbf{B} \otimes \mathbf{B} \otimes \mathbf{B}$ *by* :

$$\mathbf{R}(\boxed{b} \otimes \boxed{a} \otimes \boxed{c}) = \boxed{b} \otimes \boxed{c} \otimes \boxed{a}, \quad \mathbf{R}(\boxed{b} \otimes \boxed{c} \otimes \boxed{a}) = \boxed{b} \otimes \boxed{a} \otimes \boxed{c} \quad \textit{if } a \le b < c,$$

$$\mathbf{R}(\boxed{c} \otimes \boxed{a} \otimes \boxed{b}) = \boxed{a} \otimes \boxed{c} \otimes \boxed{b}, \quad \mathbf{R}(\boxed{a} \otimes \boxed{c} \otimes \boxed{b}) = \boxed{c} \otimes \boxed{a} \otimes \boxed{b} \quad \textit{if } a < b \le c,$$

$$\mathbf{R} = \mathrm{id}, \quad \textit{otherwise.}$$

This is known as a combinatorial R matrix. Indeed,

$$\mathbf{B}^{\otimes 3} \cong B(\Box\Box\Box) \oplus B\left(\Box\Box_{\Box}\right)^{\oplus 2} \oplus B\left(\Box^{\Box}_{\Box}\right),$$

and the map $\mathbf{R}$ flips two components $B(\Box\Box_{\Box})$ each other. Using this, we induce certain equivalent relation between elements in $\mathbf{B}^{\otimes m}$.

DEFINITION 4.7 (Crystal equivalence). Two elements b, b' in $\mathbf{B}^{\otimes m}$ are *crystal equivalent*, denoted by $b \overset{c}{\sim} b'$ if one is obtained by the others by applying a sequence of $\mathbf{R}$'s.

The following is trivial by the theory of crystal bases:

PROPOSITION 4.8. *If $b \overset{c}{\sim} b'$ ($b, b' \in \mathbf{B}^{\otimes m}$), then $\tilde{e}_i b \overset{c}{\sim} \tilde{e}_i b'$ or $\tilde{e}_i b = \tilde{e}_i b' = 0$ (resp. $\tilde{f}_i b \overset{c}{\sim} \tilde{f}_i b'$ or $\tilde{f}_i b = \tilde{f}_i b' = 0$) for any i.*

By the definitions we can easily see:

LEMMA 4.9. *For words $w = a_1 a_2 \cdots a_m$ and $w' = b_1 b_2 \cdots b_m$, set $b := \boxed{a_m} \otimes \cdots \otimes \boxed{a_1}$ and $b' := \boxed{b_m} \otimes \cdots \otimes \boxed{b_1}$. Then we have $w \overset{k}{\sim} w'$ if and only if $b \overset{c}{\sim} b'$.*

DEFINITION 4.10. For a skew Young tableau S, a word $w(S)$ is defined by reading the entries in each row from left to right and from the bottom row to the top row, which is called a *skew tableau word* of S.

The following is given in [4].

PROPOSITION 4.11. *For a Young tableau T and a positive integer x, we have $w(x \to T) \overset{k}{\sim} x \cdot w(T)$, and furthermore, for positive integers $x_1, \ldots, x_m$ we have*

$$w(x_1 \to (x_2 \to (\cdots (x_{m-1} \to x_m)))) \overset{k}{\sim} x_1 x_2 \cdots x_{m-1} x_m\,.$$

5. Main Theorem

Let κ^i $(i = 1, 2)$ be skew diagrams with $|\kappa^1| = |\kappa^2| =: N$ and λ^i, ν^i $(i = 1, 2)$ be Young diagrams satisfying $\kappa^i = \nu^i \setminus \lambda^i$. Now, let us define the the map $\mathcal{S}$:

$$\mathcal{S} : \mathbf{P}(\kappa^1, \kappa^2) \to \coprod_{\mu} \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2} \right) \quad (f \ \mapsto \ (T^1, T^2))\,,$$

where μ runs over the set of Young diagrams with $|\kappa^1| = |\kappa^2| = |\mu| (= N)$.

Set

$$\mathbf{S}(\kappa^1, \kappa^2) := \left\{ S \;\middle|\; \begin{array}{l} S \text{ is a skew tableau of shape } \kappa^1 \text{ and the number of entry } i \text{ is } \kappa^2_i, \\ ME(S) = \boxed{i_1} \otimes \boxed{i_2} \otimes \cdots \otimes \boxed{i_k} \otimes \cdots \otimes \boxed{i_N} \text{ satisfies that } \lambda^2[i_1, \ldots, i_k] \text{ is} \\ \text{a Young diagram for } k = 1, \ldots, N \text{ and } \lambda^2[i_1, \ldots, i_N] = \nu^2. \end{array} \right\},$$

$$\mathbf{W}(\kappa^1,\kappa^2) := \left\{ w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \;\middle|\; \begin{array}{l} w \text{ is a lexicographic two-rowed array of length } N, \\ \sharp\{i \in w^j\} = \kappa^j_i \ \ (j=1,2), \\ \text{the column bumping of } w^2 \text{ is in } \mathbf{B}(\mu)^{\nu^2}_{\lambda^2} \text{ and} \\ \text{the recording tableau by } w^1 \text{ is in } \mathbf{B}(\mu)^{\nu^1}_{\lambda^1}. \end{array} \right\}$$

where an element in $\mathbf{S}(\kappa^1,\kappa^2)$ is called a Littlewood-Richardson skew tableau associated with (κ^1,κ^2). Let us define maps:

$$\mathcal{S}_1 : \mathbf{P}(\kappa^1,\kappa^2) \to \mathbf{S}(\kappa^1,\kappa^2)\,, \quad \mathcal{S}_2 : \mathbf{S}(\kappa^1,\kappa^2) \to \mathbf{W}(\kappa^1,\kappa^2)\,,$$
$$\mathcal{S}_3 : \mathbf{W}(\kappa^1,\kappa^2) \to \coprod_{\mu} \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right).$$

DEFINITION 5.1. (i) For a picture $f = (f_1, f_2) \in \mathbf{P}(\kappa^1,\kappa^2)$ (where f_1, f_2 mean a coordinate of a box in κ^2), let S be a skew tableau of shape κ^1 whose (i,j)-entry $S_{i,j} = f_1(i,j)$. Define $\mathcal{S}_1(f) := S$.

(ii) For $S \in \mathbf{S}(\kappa^1,\kappa^2)$, writing $ME(S) = \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_N}$, define a word $w^2 = a_1 a_2 \cdots a_N$. Let b_i $(i = 1,2,\ldots,N)$ be the row number of the place of a_i in S and set $w^1 = b_1 b_2 \cdots b_N$. Define

$$\mathcal{S}_2(S) := w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \ldots & b_N \\ a_1 & a_2 & \ldots & a_N \end{pmatrix}.$$

(iii) For a two-rowed array $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 & \ldots & b_N \\ a_1 & a_2 & \ldots & a_N \end{pmatrix} \in \mathbf{W}(\kappa^1,\kappa^2)$, applying the column bumping procedure to w^2, obtain the tableau $T^2 = a_N \to (\cdots(a_2 \to a_1))$. Let T^1 be the recording tableau of T^2 using w^1. Define $\mathcal{S}_3(w) = (T^1, T^2)$.

(iv) Finally, define $\mathcal{S} = \mathcal{S}_3 \circ \mathcal{S}_2 \circ \mathcal{S}_1$.

Next, let us define a map $\mathcal{C}$

$$\mathcal{C} : \coprod_{\mu} \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right) \to \mathbf{P}(\kappa^1,\kappa^2).$$

To carry out this task, we define the following maps:

$$\mathcal{C}_3 : \coprod_{\mu} \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right) \to \mathbf{W}(\kappa^1,\kappa^2)\,,$$
$$\mathcal{C}_2 : \mathbf{W}(\kappa^1,\kappa^2) \to \mathbf{S}(\kappa^1,\kappa^2)\,, \quad \mathcal{C}_1 : \mathbf{S}(\kappa^1,\kappa^2) \to \mathbf{P}(\kappa^1,\kappa^2)\,.$$

DEFINITION 5.2. (i) For a pair of tableaux $(T^1, T^2) \in \coprod_\mu \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right)$, apply the reverse column bumping to T^2 by using T^1 as a recording tableau and set $c_N c_{N-1} \cdots c_1$ the sequence obtained from T^2 (c_i is the $N+1-i$-th entry bumped out from

T^2.). Set $w^2 := c_1 \cdots c_N$ and let d_i be the entry in the same place in T^1 as the $(N-i+1)$-th removed box in T^2 and set $w^1 := d_1 \cdots d_N$. Define $\mathcal{C}_3(T^1, T^2) = w = \binom{w^1}{w^2}$.

(ii) For

$$w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} = \begin{pmatrix} d_1 d_2 \cdots d_N \\ c_1 c_2 \cdots c_N \end{pmatrix} \in \mathbf{W}(\kappa^1, \kappa^2),$$

put $c_1 c_2 \cdots c_N$ to κ^1 according to the middle-eastern ordering and set S the resulting skew tableau, whose shape is κ^1. Define $\mathcal{C}_2(w) = S$.

(iii) For $S \in \mathbf{S}(\kappa^1, \kappa^2)$, define $\mathcal{C}_1(S) = f$ by $f(i, j) := (S_{ij}, \lambda^2_{S_{ij}} + p(S; i, j))$ for $(i, j) \in \kappa^1$, where $p(S; i, j)$ is as above and S_{ij} is the (i, j)-entry of S.

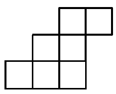
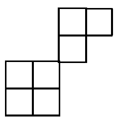
(iv) Finally, we define $\mathcal{C} = \mathcal{C}_1 \circ \mathcal{C}_2 \circ \mathcal{C}_3$.

Note that well-definedness of each map will be shown later.

THEOREM 5.3. *In the above setting, the maps $\mathcal{S}$ and $\mathcal{C}$ are both well-defined bijective maps between $\mathbf{P}(\kappa^1, \kappa^2)$ and $\coprod_\mu \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right)$, and inverse each other.*

Here note that the set $\coprod_\mu \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right)$ consists of pairs of same shaped Young tableaux, which means that this theorem is an analogue of the RSK correspondence.

EXAMPLE 5.4. We take the following skew diagrams:

$\kappa^1 =$  $\kappa^2 =$ 

Let $f_a \in \mathbf{P}(\kappa^1, \kappa^2)$ be

$$f_a = \begin{array}{c||c|c|c|c|c|c|c} \kappa^1 & (1,3) & (1,4) & (2,2) & (2,3) & (3,1) & (3,2) & (3,3) \\ \hline \kappa^2 & (1,3) & (3,1) & (1,4) & (3,2) & (2,3) & (4,2) & (4,1) \end{array}$$

Here we have

$$S_a = \mathcal{S}_1(f_a) = \begin{array}{cccc} & & 1 & 3 \\ & 1 & 3 & \\ 2 & 4 & 4 & \end{array} \quad \text{and then } \mathrm{ME}(S_a) = \boxed{3} \otimes \boxed{1} \otimes \boxed{3} \otimes \boxed{1} \otimes \boxed{4} \otimes \boxed{4} \otimes \boxed{2}.$$

Then we get $w_a = \mathcal{S}_2(S_a) = \begin{pmatrix} 1122333 \\ 3131442 \end{pmatrix}$ and then finally, we have

$$T^2 : \boxed{3} \dashrightarrow \begin{array}{cc} 1 & 3 \end{array} \dashrightarrow \begin{array}{cc} 1 & 3 \\ 3 & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 3 \\ 3 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 3 \\ 3 & & \\ 4 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 3 \\ 3 & 4 & \\ 4 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 4 & & \end{array} = T^2_a,$$

$$T^1 : \boxed{1} \dashrightarrow \begin{array}{cc} 1 & 1 \end{array} \dashrightarrow \begin{array}{cc} 1 & 1 \\ 2 & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & & \\ 3 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & \\ 3 & & \end{array} \dashrightarrow \begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & & \end{array} = T^1_a,$$

that is, $\mathcal{S}_3(w_a) = (T_a^1, T_a^2)$.

Conversely, for $(T^1, T^2) = \left(\begin{array}{ccc} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & & \end{array}, \begin{array}{ccc} 1 & 1 & 3 \\ 2 & 3 & 4 \\ 4 & & \end{array} \right)$, applying the reverse column bumping to T^2 using T^1, we get $c_7 = 2,\ c_6 = 4,\ c_5 = 4,\ c_4 = 1,\ c_3 = 3,\ c_2 = 1,\ c_1 = 3$ and $d_1 = d_2 = 1,\ d_3 = d_4 = 2, d_5 = d_6 = d_7 = 3$ and then

$$w = \mathcal{C}_3(T^1, T^2) = \begin{pmatrix} 1122333 \\ 3131442 \end{pmatrix}.$$

We obtain

$$S = \mathcal{C}_2(w) = \begin{array}{ccc} & c_2 & c_1 \\ c_4 & c_3 & \\ c_7 & c_6 & c_5 \end{array} = \begin{array}{cccc} & & 1 & 3 \\ & 1 & 3 & \\ 2 & 4 & 4 & \end{array} \quad \text{and then finally, we have}$$

$$\mathcal{C}_1(S) = \begin{array}{c||c|c|c|c|c|c|c} \kappa^1 & (1,3) & (1,4) & (2,2) & (2,3) & (3,1) & (3,2) & (3,3) \\ \hline \kappa^2 & (1,3) & (3,1) & (1,4) & (3,2) & (2,3) & (4,2) & (4,1) \end{array} = f_a.$$

To show the theorem, it suffices to prove:

(i) The well-definedness of $\mathcal{S}$.
(ii) The well-definedness of $\mathcal{C}$.
(iii) Bijectivity of $\mathcal{S}$ and $\mathcal{C}$.

We shall show these in the subsequent sections.

6. Well-definedness of $\mathcal{S}$

For the well-definedness of $\mathcal{S}$, we shall prove the following:

PROPOSITION 6.1. *The maps $\mathcal{S}_i$ $(i = 1, 2, 3)$ are well-defined.*

Indeed, the well-definedness of $\mathcal{S}_3$ is obvious by the definition.

6.1. Well-definedness of $\mathcal{S}_1$. For $f \in \mathbf{S}(\kappa^1, \kappa^2)$, by the similar argument in [11, 12], we can show that $S := \mathcal{S}_1(f)$ is a skew tableau. Thus, we may show:

LEMMA 6.2. *For any $k = 1, \ldots, n$ and the skew tableau $S = \mathcal{S}_1(f)$, we have*

$$\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(S)) = 0\,,$$

where Y_{λ^2} is a Young tableau of shape λ^2 satisfying that all the entries in k-th row are k $(k = 1, \ldots, n)$, which is called a highest tableau.

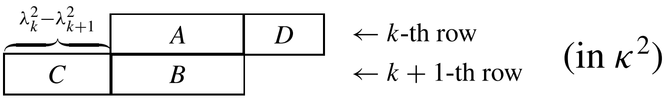
PROOF. Write

$$ME(Y_{\lambda^2}) \otimes ME(S) = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}\,.$$

By the rule of the action of $\tilde{e}_k$, we may show

$$\sharp\{j | i_j = k,\ \ j \le p\} \ge \sharp\{j | i_j = k+1,\ \ j \le p\} \tag{6.1}$$

for any $p = 1, \ldots, N$. In the skew diagram κ^2, we have



For boxes $(k, j), (k+1, j) \in \kappa^2$, by the fact $(k, j) \leqslant_P (k+1, j)$, we have

$$(x_1, y_1) := f^{-1}(k, j) \leqslant_J f^{-1}(k+1, j) =: (x_2, y_2)\,.$$

It is evident from the definition of the map $\mathcal{S}_1$ that

$$S_{x_1,y_1} = k\,, \quad S_{x_2,y_2} = k+1\,.$$

This implies that in the tensor product $ME(Y_{\lambda^2}) \otimes ME(S) = \boxed{i_1} \otimes \cdots \otimes \boxed{i_N}$, k's from A appear earlier than $k+1$'s from B and then they are cancelled each other with respect to the action of $\tilde{e}_k$. In $ME(Y_{\lambda^2})$, the number of k exceeds the one of $k+1$ by $\lambda^2_k - \lambda^2_{k+1}$. Thus, $k+1$'s from the part C in the figure also have been cancelled by k's in $ME(Y_{\lambda^2})$. Hence we obtain (6.1) and then $\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(S)) = 0$ for any k. □

Thus, we have the well-definedness of $\mathcal{S}_1$.

6.2. Well-definedness of $\mathcal{S}_2$. First, let us show that the two-rowed array $w := \mathcal{S}_2(S)$ ($S \in \mathbf{S}(\kappa^1, \kappa^2)$) is in the lexicographic order, that is, $b_1 \leq b_2 \leq \cdots \leq b_N$ and $a_j \geq a_{j+1}$ if $b_j = b_{j+1}$, where a_j, b_j are as in Definition 5.1. It follows immediately from the definition of b_i's that $b_1 \leq b_2 \leq \cdots \leq b_N$. Let k satisfy $b_1 \leq k \leq b_N$ and $\{b_i, b_{i+1}, \ldots, b_{i+r}\}$ the maximal subsequence of w^1 such that $b_i = \cdots = b_{i+r} = k$, which implies that $a_i, a_{i+1}, \ldots, a_{i+r}$ are the entries in the k-th row of S. Since S is a skew tableau, we obtain that $a_i \geq a_{i+1} \geq \cdots \geq a_{i+r}$, which means that w is in the lexicographic order. Let T^2 be the tableau from w^2 by the column bumping and show that $T^2 \in \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}$, *i.e.,*

$$\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(T^2)) = 0$$

for any $k = 1 \ldots, n$. For this purpose, we see the following lemma.

LEMMA 6.3. *$ME(S)$ is crystal equivalent to $ME(T^2)$.*

PROOF. For $w^2 = a_1 a_2 \cdots a_N$, since T^2 is obtained by the column bumping procedure of $a_N \cdots a_1$, we know that $w(S) = a_N a_{N-1} \ldots a_1 \overset{k}{\sim} w(T^2)$, which means $ME(S) \overset{c}{\sim} ME(T^2)$ by Lemma 4.9. □

By the Lemma 6.3, we have $ME(S) \overset{c}{\sim} ME(T^2)$ and then $ME(Y_{\lambda^2}) \otimes ME(S) \overset{c}{\sim} ME(Y_{\lambda^2}) \otimes ME(T^2)$. We also have

$$\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(S)) = 0\,,$$

for any k by Lemma 6.2. This and Proposition 4.8 show that

$$\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(T^2)) = 0\,,$$

for any k and then we have $T^2 \in \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}$.

For $w := \mathcal{S}_2(S)$, we set $(T^1, T^2) := \mathcal{S}_3(w)$. For our purpose, it suffices to show $T^1 \in \mathbf{B}(\mu)^{\nu^1}_{\lambda^1}$, that is, $\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(T^1)) = 0$ for any k.

LEMMA 6.4. *Let $1 \le c_1, \dots, c_k \le n$. For some $i \in \{1, \dots, k-1\}$ assume that*

$$b_1 := \cdots \otimes \boxed{c_{i-1}} \otimes \boxed{c_i} \otimes \boxed{c_{i+1}} \otimes \boxed{c_{i+2}} \otimes \cdots \overset{c}{\sim} \cdots \otimes \boxed{c_{i-1}} \otimes \boxed{c_{i+1}} \otimes \boxed{c_i} \otimes \boxed{c_{i+2}} \otimes \cdots =: b_2 .$$

Applying the column bumping procedure to both b_1 and b_2, the place of the new box New(c_i) (resp. New(c_{i+1})) from b_1 coincides with the one of the new box New(c_{i+1}) (resp. New(c_i)) from b_2.

PROOF. Set $x := c_i$, $y := c_{i-1}$ and $z := c_{i+1}$. First we consider the case $x \le y < z$. Let T_p (resp. T_q)be the tableau obtained from b_1 (resp. b_2) by the column bumping procedure. It follows immediately from the condition $x \le y < z$ that

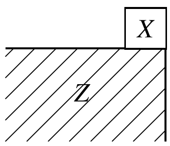
$$w(T_p) \overset{k}{\sim} c_k \cdots zxy \cdots c_1 \overset{k}{\sim} c_k \cdots xzy \cdots c_1 \overset{k}{\sim} w(T_q)\,,$$

which shows that $T_p = T_q$. Define the tableau T' by the column bumping

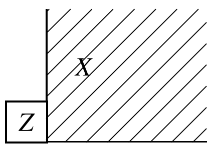
$$T' := z \to (x \to (y \to (\cdots (c_2 \to c_1)))) \tag{6.2}$$

$$= x \to (z \to (y \to (\cdots (c_2 \to c_1))))\,. \tag{6.3}$$

Let $X = \mathrm{New}(x)$ and $Z = \mathrm{New}(z)$ be the new boxes in each column bumping. Since $x < z$, applying the column bumping lemma to the bumping (6.2) we have:

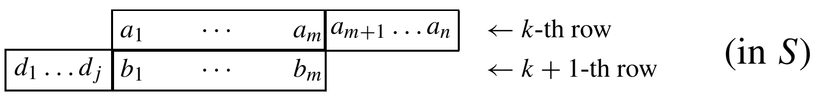


Similarly, in (6.3), we have



These mean that X (resp. Z) in (6.2) coincides with X (resp. Z) in (6.3). We can show the case $x < y \le z$ and the case $x = c_i$, $z = c_{i+1}$ and $y = c_{i+2}$ similarly. □

To show $\tilde{e}_k(ME(Y_{\lambda^1}) \otimes ME(T^1)) = 0$ for any k, we see the k-th and $k+1$-th rows of S.



By this figure, we know that for $i = 2, 3, \ldots, m$

$$a_1 < b_{i-1} \leq b_i \, .$$

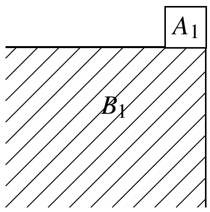
This induces the following transformations of $ME(S)$ by the map $\mathbf{R}$ in Lemma 4.6:

$$\begin{aligned} ME(S) &= \cdots \otimes \boxed{a_2} \otimes \boxed{a_1} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_1} \otimes \boxed{d_j} \otimes \cdots \\ &\overset{c}{\sim} \cdots \otimes \boxed{a_2} \otimes \boxed{b_m} \otimes \boxed{a_1} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_1} \otimes \boxed{d_j} \otimes \cdots \\ &\qquad \cdots\cdots\cdots\cdots \\ &\overset{c}{\sim} \cdots \otimes \boxed{a_2} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_2} \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{d_j} \otimes \cdots . \end{aligned}$$

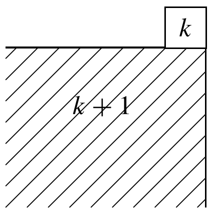
Furthermore, we have $a_j < b_{i-1} \leq b_i$ for $2 \leq j < i \leq m$. Thus, repeating the above transformations we get

$$ME(S) \overset{c}{\sim} \cdots \otimes \boxed{a_m} \otimes \boxed{b_m} \otimes \boxed{a_{m-1}} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{a_2} \otimes \boxed{b_2} \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{d_j} \otimes \cdots =: w' \, , \quad (6.4)$$

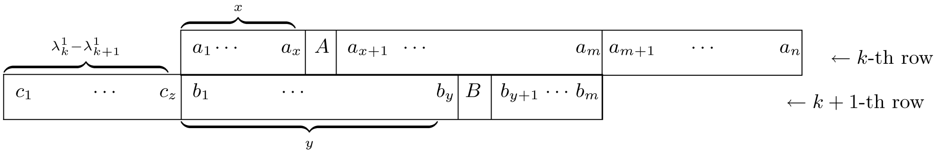
which means that the resulting tableaux by column bumping of $ME(S)$ and w' are same as T^2 by Lemma 6.4. Considering the column bumping of w', set $A_1 := \mathrm{New}(a_1)$ and $B_1 := \mathrm{New}(b_1)$ in T^2. We have



Since the entry a_1 (resp. b_1) has been placed at the k (resp. $k+1$)-th row in S, in T^1 we have



So, in $ME(T^1)$ the k as above appears earlier than the $k+1$. We know that the positions of $\mathrm{New}(a_i)$ and $\mathrm{New}(b_i)$ in T^1 are in the similar relation to the one of $\mathrm{New}(a_1)$ and $\mathrm{New}(b_1)$ and then in $ME(T^1)$ the k's from $a_1, \ldots, a_m$ cancel the $k+1$'s from $b_1, \ldots, b_m$. Moreover, in $ME(Y_{\lambda^1})$ we have $\sharp\{k\} - \sharp\{k+1\} = \lambda^1_k - \lambda^1_{k+1}$. Thus, $k+1$'s from $d_1, \ldots, d_j$ have been cancelled in $ME(T^1)$ and this implies $\tilde{e}_k(ME(Y_{\lambda^2}) \otimes ME(T^1)) = 0$ for any k. Now, we obtain $T^1 \in \mathbf{B}(\mu)^{\nu^1}_{\lambda^1}$ and the well-definedness of the map $\mathcal{S}_2$ and then $\mathcal{S}$, which completes the



proof of Proposition 6.1. □

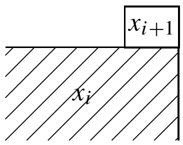
7. Well-definedness of $\mathcal{C}$

To show the well-definedness of the map $\mathcal{C}$, we should prove that $f := \mathcal{C}(T^1, T^2)$ is a PJ-picture from κ^1 to κ^2. In the course of the proof, we shall also show that the maps $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are well-defined. Indeed, the well-definedness of $\mathcal{C}_3$ is immediate from the definition.

PROPOSITION 7.1. *Let S be the filling of shape κ^1 appearing in the definition of $\mathcal{C}_2$. Then S is a skew tableau of shape κ^1.*

PROOF. For $w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \in \mathbf{W}(\kappa^1, \kappa^2)$, set $(T^1, T^2) := \mathcal{S}_3(w)$, which is in $\coprod_\mu \left(\mathbf{B}(\mu)^{\nu^1}_{\lambda^1} \times \mathbf{B}(\mu)^{\nu^2}_{\lambda^2}\right)$ as we have seen in the previous section. Since T^1 is in $\mathbf{B}(\mu)^{\nu^1}_{\lambda^1}$, the number of entry k's$(k = 1, \ldots, n)$ is $h := \nu^1_k - \lambda^1_k$. Let $X_1, \ldots, X_h$ be the positions of all k's in T^1 from right to left. Note that $(T^1)^{(k)} = \{X_1, \ldots, X_h\}$. And let x_j $(j = 1, \ldots, h)$ be the entry in T^2 at the same position as X_j. By the definition of $\mathcal{C}_2$, the entries in k-th row of S consist of the elements obtained by reverse column bumping, that is, the entry S_{k,λ^1+i} is the element by the inverse column bumping of x_i.

Now, assume that $S_{k,\lambda^1+i} > S_{k,\lambda^1+i+1}$. In the column bumping of $w^2 = ME(S)$ to T^2, the new box by S_{k,λ^1+i} (resp. S_{k,λ^1+i+1}) has x_i (resp. x_{i+1}) as an entry and it is placed at X_i (resp. X_{i+1}). Applying the column bumping lemma (Lemma 4.2) to these new boxes, we have



This contradicts to the fact that x_i is on the right side of x_{i+1} and shows that $S_{k,\lambda^1+i} \leq S_{k,\lambda^1+i+1}$.

Next, let us check the condition for vertical directions in S. Suppose that $S_{k,j} \geq S_{k+1,j}$. Then in S we obtain the following A, B: satisfying $A \geq B$, $a_i < b_j$ for $i \leq j$, $i = 1, \ldots, x$ and $j = 1, \ldots, m$. Indeed, we get these by the following way.

(i) Find the left-most pair (a_s, b_s) with $a_s \geq b_s$.

(ii) If $a_s \geq b_m$, then set $A := a_s$ and $B := b_m$.

(iii) Otherwise, compare a_s and b_{m-1} and if $a_s \geq b_{m-1}$, then set $A := a_s$ and $B := b_{m-1}$.

(iv) Otherwise, repeat the above procedure until getting $a_s \geq b_l$ for $l \geq s$. Then set $A := a_s$ and $B := b_l$.

Since we have $a_1 < b_{j-1} \leq b_j$ for $j = 2, \ldots, m$, and $a_1 < B \leq b_{y+1}$ we have

$$\begin{aligned}
ME(S) = {} & \cdots \otimes \boxed{a_n} \otimes \cdots\cdots \otimes \boxed{a_{x+1}} \otimes \boxed{A} \otimes \boxed{a_x} \otimes \cdots \otimes \boxed{a_1} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \\
& \cdots \otimes \boxed{b_{y+1}} \otimes \boxed{B} \otimes \boxed{b_y} \cdots \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots \otimes \boxed{c_1} \otimes \cdots \\
\overset{c}{\sim} {} & \cdots \otimes \boxed{a_n} \otimes \cdots\cdots \otimes \boxed{a_{x+1}} \otimes \boxed{A} \otimes \boxed{a_x} \otimes \cdots \otimes \boxed{b_m} \otimes \boxed{a_1} \otimes \boxed{b_{m-1}} \otimes \cdots \\
& \cdots \otimes \boxed{b_{y+1}} \otimes \boxed{B} \otimes \boxed{b_y} \cdots \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots \otimes \boxed{c_1} \otimes \cdots \\
\overset{c}{\sim} {} & \cdots \otimes \boxed{a_n} \otimes \cdots\cdots \otimes \boxed{a_{x+1}} \otimes \boxed{A} \otimes \boxed{a_x} \otimes \cdots \otimes \boxed{a_2} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \\
& \cdots \otimes \boxed{b_{y+1}} \otimes \boxed{B} \otimes \boxed{b_y} \cdots \otimes \boxed{b_2} \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots .
\end{aligned}$$

Due to the conditions $a_i < b_{k-1} \leq b_k$ and $a_i < B \leq b_{y+1}$ for $2 \leq k < i \leq x$, we can repeat the transformations above and get

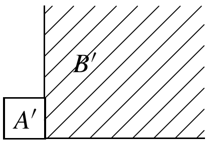
$$\begin{aligned}
ME(S) \overset{c}{\sim} {} & \cdots \otimes \boxed{a_n} \otimes \cdots \otimes \boxed{a_{x+1}} \otimes \boxed{A} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_{y+1}} \otimes \boxed{B} \otimes \boxed{b_y} \otimes \cdots \\
& \cdots \otimes \boxed{b_{x+1}} \otimes \boxed{a_x} \otimes \boxed{b_x} \otimes \cdots \otimes \boxed{a_2} \otimes \boxed{b_2} \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots .
\end{aligned}$$

It follows from the conditions $A < b_i \leq b_{i+1}$ for $i = y+1, \ldots, m$ and $B \leq A < b_{y+1}$ that

$$\begin{aligned}
ME(S) \overset{c}{\sim} {} & \cdots \otimes \boxed{a_n} \otimes \cdots \otimes \boxed{a_{x+1}} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_{y+2}} \otimes \boxed{A} \otimes \boxed{b_{y+1}} \otimes \boxed{B} \otimes \boxed{b_y} \otimes \cdots \\
& \cdots \otimes \boxed{b_{x+1}} \otimes \boxed{a_x} \otimes \boxed{b_x} \otimes \cdots \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots \\
\overset{c}{\sim} {} & \cdots \otimes \boxed{a_n} \otimes \cdots \otimes \boxed{a_{x+1}} \otimes \boxed{b_m} \otimes \boxed{b_{m-1}} \otimes \cdots \otimes \boxed{b_{y+2}} \otimes \boxed{A} \otimes \boxed{B} \otimes \boxed{b_{y+1}} \otimes \boxed{b_y} \otimes \cdots \\
& \cdots \otimes \boxed{b_{x+1}} \otimes \boxed{a_x} \otimes \boxed{b_x} \otimes \cdots \otimes \boxed{a_1} \otimes \boxed{b_1} \otimes \boxed{c_z} \otimes \cdots \otimes \boxed{c_1} \otimes \cdots . \quad (7.1)
\end{aligned}$$

Now, let us see the following Claim 1–3:

CLAIM 1. In (7.1) one can find that A and B ($A \geq B$) are neighboring each other. Thus, applying the column bumping of (7.1), by the column bumping lemma (Lemma 4.2) we obtain



in T^2,

where $A' := \mathrm{New}(A)$ and $B' := \mathrm{New}(B)$.

CLAIM 2. Next, in the column bumping of $ME(S)$, since $a_1 \le \cdots \le a_x \le A$, by the column bumping lemma (Lemma 4.2) the new boxes by $a_1, \dots, a_x$ are placed on the right-side of A'. Similarly, since $c_1 \le \cdots \le c_z \le b_1 \le \cdots \le b_y \le B$, the new boxes by $c_1, \dots, c_z, b_1, \dots, b_x$ are placed on the right-side of B'.

CLAIM 3. As the definition of the map $\mathcal{S}_3$, the tableau T^1 is the recording tableau of T^2. Then, it follows from Claim 2 that there are x entries k's on the right-side of A' and $z + y$ entries $k + 1$'s on the right-side of the same place as B' in T^1. We also know from Claim 1 that B' is on the right-side of A' and then there exist $z + y + 1$ entries $k + 1$'s on the right-side of A'.

In $ME(Y_{\lambda^1}) \otimes ME(T^1)$ let n_1 (resp. n_2) be the number of k (resp. $k+1$) on the left-side of A'. Claim 3 implies that

$$n_1 = \lambda^1 + x\,, \quad n_2 = \lambda^1 + z + y + 1\,. \tag{7.2}$$

Since $z = \lambda^1_k - \lambda^1_{k+1}$ and $x \le y$, one gets

$$n_2 - n_1 = (\lambda^1_{k+1} + z + y + 1) - (\lambda^1_k + x) \ge 1\,,$$

which contradicts that $T^1 \in \mathbf{B}(\mu)^{\nu^1}_{\lambda^1}$ and the case $S_{k,j} \ge S_{k+1,j}$ never occur. Thus, S is a skew tableau. It is immediate from the definition of $\mathcal{C}_2$ that $w(S) \overset{k}{\sim} w(T)$, which means S is a Littlewood-Richardson skew tableau and then $\mathcal{C}_2$ is well-defined. □

PROOF OF WELL-DEFINEDNESS OF $\mathcal{C}$. For the purpose we may show that f is bijective, f and f^{-1} are PJ-picture. The bijectivity of f is obtained by the similar way to that in [11, 12]. In order to show that f and f^{-1} are PJ-picture, we may see for any $(i, j), (i, j + 1), (i + 1, j) \in \kappa^1$ and any $(a, b), (a, b + 1), (a + 1, b) \in \kappa^2$,

$$f(i, j) \leqslant_J f(i, j + 1)\,, \quad f(i, j) \leqslant_J f(i + 1, j)\,,$$
$$f^{-1}(a, b) \leqslant_J f^{-1}(a, b + 1)\,, \quad f^{-1}(a, b) \leqslant_J f^{-1}(a + 1, b)\,.$$

These are also shown by the similar way to those in [11, 12]. □

8. Bijectivity of $\mathcal{S}$ and $\mathcal{C}$

It suffices to show that $\mathcal{C} \circ \mathcal{S} = \mathrm{id}$ and $\mathcal{S} \circ \mathcal{C} = \mathrm{id}$. To carry out these, we shall prove that $\mathcal{C}_i \circ \mathcal{S}_i = \mathrm{id}$ and $\mathcal{S}_i \circ \mathcal{C}_i = \mathrm{id}$ for $i = 1, 2, 3$.

8.1. $\mathcal{S}_1$ and $\mathcal{C}_1$. Take $S \in \mathbf{S}(\kappa^1, \kappa^2)$ and set $S' := \mathcal{S}_1 \circ \mathcal{C}_1(S)$. We have $\mathcal{C}_1(S)(i, j) = (S_{ij}, \lambda^2_{S_{ij}} + p(S; i, j))$. Hence, by the definition of $\mathcal{S}_1$ we have $S'_{ij} = S_{ij}$, which implies $S' = S$ and then $\mathcal{S}_1 \circ \mathcal{C}_1 = \mathrm{id}$.

For $f \in \mathbf{P}(\kappa^1, \kappa^2)$, set $g := \mathcal{C}_1 \circ \mathcal{S}_1(f)$. The following lemma can proved similarly to [11, Lemma 5.2], [12, Lemma 5.4].

LEMMA 8.1. *Set $S = \mathcal{S}_1(f)$. Considering $Y_{\lambda^2} \otimes ME(S)$, the entry $\boxed{S_{ij}}$ is added to the position $f(i, j) \in \kappa^2$.*

Since $S_{ij} = f_1(i, j)$ and $g(i, j) = (S_{ij}, \lambda^2_{S_{ij}} + p(S; i, j))$, we get $g_1(i, j) = f_1(i, j)$. We know that $S_{ij}(= k)$ is the $p(S; , i, j)$-th entry equal to k and $f_2(i, j) = \lambda^2_{S_{ij}} + p(S; , i, j) = g_2(i, j)$, which shows $f = g$ and then $\mathcal{C}_1 \circ \mathcal{S}_1 = \mathrm{id}$.

8.2. $\mathcal{S}_2$ and $\mathcal{C}_2$. Set $w' := \mathcal{S}_2 \circ \mathcal{C}_2(w)$ for $w \in \mathbf{W}(\kappa^1, \kappa^2)$ and write

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} d_1 d_2 \cdots d_N \\ c_1 c_2 \cdots c_N \end{pmatrix}, \quad w' = \begin{pmatrix} w'_1 \\ w'_2 \end{pmatrix} = \begin{pmatrix} b_1 b_2 \cdots b_N \\ a_1 a_2 \cdots a_N \end{pmatrix}.$$

Note that the number of i in w_1 is just equal to κ^1_i. For $S := \mathcal{C}_2(w)$, we have $ME(S) = \boxed{c_1} \otimes \boxed{c_2} \otimes \cdots \otimes \boxed{c_n}$ and then $w_2 = w'_2$ by the definition of $\mathcal{S}_2$. The number b_i is the row number of a_i in S. Thus, since the number of i in w'_1 is κ^1_i, $d_1 \leq \cdots \leq d_N$ and $b_1 \leq \cdots \leq b_N$, we have $w_1 = w' = 1$ and then $w = w'$, which means $\mathcal{S}_2 \circ \mathcal{C}_2 = \mathrm{id}$.

It is trivial from the definition of the maps $\mathcal{S}_2$ and $\mathcal{C}_2$ that $\mathcal{C}_2 \circ \mathcal{S}_2 = \mathrm{id}$.

8.3. $\mathcal{S}_3$ and $\mathcal{C}_3$. We have seen the well-definedness of the maps $\mathcal{S}_3$ and $\mathcal{C}_3$ and these maps are certain restriction of usual RSK correspondence in terms of column bumping. Thus, we obtain $\mathcal{S}_3 \circ \mathcal{C}_3 = \mathrm{id}$ and $\mathcal{C}_3 \circ \mathcal{S}_3 = \mathrm{id}$.

Now, we obtain $\mathcal{S}_i \circ \mathcal{C}_i = \mathrm{id}$ and $\mathcal{C}_i \circ \mathcal{S}_i = \mathrm{id}$ $(i = 1, 2, 3)$ and then $\mathcal{S} \circ \mathcal{C} = \mathrm{id}$ and $\mathcal{C} \circ \mathcal{S} = \mathrm{id}$. So, we have completed the proof of Theorem 5.3. □

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