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Metric Fibrations from Simply Connected Rank—One Projective Spaces

In memory of Irmgard Braunmueller Escobales, 1947–2005

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Abstract. In this paper we classify all non-trivial Riemannian submersions with connected fibers from any of the simply connected, rank-one projective spaces. The result follows from results of Gromoll, Grove, Wilking, Becker, Casson, Gottlieb, Schultz, Ucci, and Wolf, together with results of the author.

Throughout this paper all maps, functions and morphisms are assumed to be at least of class C^∞. All our manifolds are assumed to be without boundary. We will usually follow the terminology of [GW]. A surjective map $\pi : M^{n+p} \to B^n$, where M and B are manifolds of dimension $n + p$ and n respectively, is a *submersion* provided its derivative, π_{*x} , has maximal rank n for each $x \in M$ ([GW, page 1]). Now let $\pi : M^{n+p} \to B^n$ be a C^∞ map from a complete, connected, Riemannian manifold (M^{n+p}, g) onto a Riemannian manifold (B^n, g^*), where g and g^* are Riemannian metrics on M and B respectively. Let $\mathbf{V}$ denote the distribution tangent to the fibers $\pi^{-1}(x)$, for some $x \in B$, and $\mathbf{H}$ the distribution orthogonal to $\mathbf{V}$ in TM, the tangent bundle of M, determined by the metric g. If E is a vector field on M, $\mathcal{V}E$ and $\mathcal{H}E$ will denote the projections of E onto the distributions $\mathbf{V}$ and $\mathbf{H}$ respectively. Call the vector field E *vertical* if $\mathcal{V}E = E$. Call E *horizontal* if $\mathcal{H}E = E$. With this notation, the map π above is a *metric fibration*, that is a *Riemannian submersion*, provided for any horizontal vector fields X, Y at $x \in M$ on M^{n+p}, one has that $g_x(X, Y) = g^*_{\pi(x)}(\pi_*X, \pi_*Y)$. We say a horizontal vector field X on M is *basic* provided π_*X is a well defined vector field on B.

D will denote the Levi-Civita connection on M and, following [EP], we introduce the tensors T and A as follows. For vector fields E and F on M,

$$T_E F = \mathcal{V}D_{\mathcal{V}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{V}E}\mathcal{V}F\,, \quad \text{and} \tag{1}$$

$$A_E F = \mathcal{V}D_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}D_{\mathcal{H}E}\mathcal{V}F\,. \tag{2}$$

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Then T and A are tensors of type $(1, 2)$. These tensors satisfy the usual properties outlined in [EP]. We note that if X and Y are horizontal,

$$A_X Y = -A_Y X \,. \tag{3}$$

Following [GW, page 4], a surjective map $\pi : M \to B$ is a *fibration* provided it has the homotopy lifting property. As observed in [GW, page 4], a fibration is always a submersion. Moreover, a locally trivial fiber bundle is always a fibration [GW, page 5], but not conversely. By Theorem 1.3.1 of [GW], any Riemannian submersion, $\pi : M \to B$ with M complete is always a locally trivial fiber bundle and so is a fibration. We begin with the following elementary result that was just hinted at in [E2].

PROPOSITION 0.1. *Let (A, g), (B, g^*), and (C, g^{**}) be three Riemannian manifolds, with A connected and complete. If $\rho : A \to B$ and $\pi : B \to C$ are two non-trivial Riemannian submersions with connected fibers, then the composite Riemannian submersion $\pi \circ \rho : A \to C$ is also a non-trivial Riemannian submersion with connected fibers. In particular, B and C are also connected and complete.*

SKETCH OF PROOF. It follows easily from the assumptions on A and Theorem 1.3.1 of [GW] mentioned above that B and C are also connected and complete. As the composite of two submersions, one sees easily that $\pi \circ \rho : A \to C$ is surjective from A onto C and that $(\pi \circ \rho)_*$ has maximal rank. If X and Y are two tangent vectors orthogonal to the fibers of $\pi \circ \rho$ at some $x \in A$, then in particular, $\rho_* X$ and $\rho_* Y$ are orthogonal to the fibers of π at $\rho(x)$. Hence, $g_x(X, Y) = g^*_{\rho(x)}(\rho_* X, \rho_* Y) = g^{**}_{\pi(\rho(x))}(\pi_* \rho_* X, \pi_* \rho_* Y) = g^{**}_{(\pi \circ \rho)(x)}((\pi \circ \rho)_* X, (\pi \circ \rho)_* Y)$. Thus, the composite is a Riemannian submersion. Since the fibers of each submersion are manifolds, they are locally path connected and hence path connected. Using this observation it is easy to see that the fibers of $\pi \circ \rho$ are also connected and indeed path connected. The sketch of proof of the proposition is now complete.

We assume that the metric fibrations below are all *non-trivial*, that is, the dimension of the fibers of π is p, with $p \geq 1$. We will always assume that the fibers are connected. In the result below we assume that $\mathbf{S}^n$ is the n-sphere of radius 1, while $\mathbf{S}^m(r)$ is an m-sphere of radius r. $\mathbf{CP}(n)$ is complex projective n-space of real dimension $2n$, $\mathbf{QP}(n)$ is quaternionic n-space of real dimension $4n$, while $\mathbf{CaP}(2)$ is the Cayley projective two-plane of real dimension 16. Also, $\mathbf{CaP}(1) = S^8(1/2)$ is the Cayley one-plane. For $\mathbf{CP}(n)$, $\mathbf{QP}(n)$, and $\mathbf{CaP}(2)$, the sectional curvatures are understood to lie in the interval [1,4]. We assume that our rank-one symmetric spaces are simply connected to rule out those rank-one symmetric spaces with fundamental group $\mathbf{Z}_2$ as pointed out in [Chv], Corollary 2.2.

THEOREM 0.2. *The only non-trivial metric fibrations (i.e. Riemannian submersions), $\pi : M^{n+p} \to B^n$ from a simply connected rank-one projective space, M, having connected ,* *tangentially oriented fibers onto an oriented closed manifold B are those of the form*

$$\begin{array}{ccc} S^2(1) & \xrightarrow{\iota} & \mathbf{CP}(2n+1) \\ & & \downarrow{\scriptstyle\pi} \\ & & \mathbf{QP}(n) \end{array} \tag{4}$$

with $n \geq 1$. In fact, the fibers are totally geodesic. Moreover, any two such Riemannian submersions $\pi_i : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$ are equivalent in the sense that there is an isometry $f_- : \mathbf{CP}(2n+1) \to \mathbf{CP}(2n+1)$ that induces an isometry $f_= : \mathbf{QP}(n) \to \mathbf{QP}(n)$ so that the following diagram commutes:

$$\begin{array}{ccc} \mathbf{CP}(2n+1) & \xrightarrow{f_-} & \mathbf{CP}(2n+1) \\ {\scriptstyle\pi_1}\downarrow & & {\scriptstyle\pi_2}\downarrow \\ \mathbf{QP}(n) & \xrightarrow{f_=} & \mathbf{QP}(n)\,. \end{array} \tag{5}$$

REMARK. This result strengthens Theorem 5.2 (mislabelled Theorem 3.2) in [E2], where the fibers are already assumed to be complex and totally geodesic. These assumptions are not made in the above result, but will follow as a consequence of the classification results of Gromoll-Grove ([GG]), and Wilking ([W]), and equation (6) of [E2]. Together with the results of [GG] and [W] , this theorem classifies completely all metric fibrations from compact, simply connected, rank one symmetric spaces.

PROOF. Under the stated hypotheses, any possible Riemannian submersion from such a simply connected rank-one projective space is necessarily a fibration by Theorem 1.3.1 of [GW]. Since each of the the simply connected rank-one projective spaces M has positive Euler characteristic, it follows from Theorem 1 on page 481 of [Sp] that the fiber cannot be odd dimensional, since closed oriented odd-dimensional manifolds have an Euler characteristic of zero. Hence, the real fiber dimension cannot be odd, so the real dimensions of both fibers and bases are even.

According to Theorem 2 of [CG] (see also Theorem 2.1 of [S1]), there are no non-trivial fibrations from $\mathbf{CP}(2n)$, $\mathbf{QP}(2n)$, or $\mathbf{CaP}(2)$ onto a compact CW complex with compact fiber. Since the base space of a Riemannian submersion is a manifold and, in particular, a CW complex, it follows from this theorem and the remarks before the statement of the theorem that there are no non-trivial Riemannian submersions from these rank-one symmetric spaces onto a Riemannian manifold B. This means that the only possible non-trivial Riemannian submersions are from $\mathbf{CP}(2n+1)$ or from $\mathbf{QP}(2n+1)$, with $n \geq 0$.

From the work of Gromoll-Grove ([GG]) and Wilking ([W]), the only Riemannian submersions from spheres with connected fibers are the standard Hopf fibrations. Specifically, Theorem 4.4.3 of [GW] asserts that any non-trivial Riemannian submersion from $S^{n+k} \to B^n$ is congruent to a Hopf fibration. Since the congruence is achieved by a Euclidean motion, it follows that the fibers of *any* such Riemannian submersion are congruent to a Riemannian submersion with totally geodesic fibers, which exist in these cases. In particular, it follows that the fibers of *any* such Riemannian submersions are necessarily totally geodesic. This was assumed in [E1] and [R]. In particular, these submersions are:

$$\begin{array}{ccc} S^1 & \xrightarrow{\iota} & S^3 \\ & & {\scriptstyle\rho}\big\downarrow \\ & & S^2(1/2) \end{array} \tag{6}$$

$$\begin{array}{ccc} S^1 & \xrightarrow{\iota} & S^{2n+1} \\ & & {\scriptstyle\rho}\big\downarrow \\ & & \mathbf{CP}(n) \end{array} \tag{7}$$

$$\begin{array}{ccc} S^3 & \xrightarrow{\iota} & S^7 \\ & & {\scriptstyle\rho}\big\downarrow \\ & & S^4(1/2) \end{array} \tag{8}$$

$$\begin{array}{ccc} S^3 & \xrightarrow{\iota} & S^{4n+3} \\ & & {\scriptstyle\rho}\big\downarrow \\ & & \mathbf{QP}(n) \end{array} \tag{9}$$

$$\begin{array}{ccc} S^7 & \xrightarrow{\iota} & S^{15} \\ & & {\scriptstyle\rho}\big\downarrow \\ & & S^8(1/2) \end{array} \tag{10}$$

Moreover, any two submersions in the same class are congruent (see [GW, page 156] and [W, page 282]). Note, the first class, (6), is a special case of the second, (7), with $n = 1$, while the third class, (8), is a special case of the fourth, (9), again with $n = 1$. This occurs since when $n = 1$, $\mathbf{CP}(1) = S^2(1/2)$ and $\mathbf{QP}(1) = S^4(1/2)$. As noted, $\mathbf{CaP}(1) = S^8(1/2)$. It is easy to see using the list above that when $n = 0$, no such non-trivial Riemannian submersion exists from $\mathbf{CP}(1) = S^2(1/2)$ or from $\mathbf{QP}(1) = S^4(1/2)$. This follows easily from the above classification theorem for Riemannian submersions from spheres, using appropriate scaling of the metric on the total space. Also, there is no non-trivial Riemannian submersion from $\mathbf{CaP}(1) = S^8(1/2)$ onto some B, since if there were, there would also be a Riemannian submersion from S^8 onto some B^*, again obtained by appropriate scaling of the metric. Such a Riemannian submersion is not in the list above. If our projective space is $M = \mathbf{QP}(2n+1)$, our Riemannian submersions are $\pi : \mathbf{QP}(2n+1) \to B$. Thus, the only possible composite submersion is with the Hopf fibration $\rho : S^{4n+3} \to \mathbf{QP}(2n+1)$. The resulting submersion, $\pi \circ \rho : S^{4n+3} \to B$ is also a Riemannian submersion, and if the composite is to be non-trivial in the above sense, the only possible non-trivial composite is $\pi \circ \rho : S^{15} \to \mathbf{QP}(3) \to S^8$. The resulting possible Riemannian submersion from $\mathbf{QP}(3) \to B$ is thus,

$$\begin{array}{ccc} S^3(1) & \xrightarrow{\iota} & \mathbf{Q}(3) \\ & & \big\downarrow \pi \\ & & S^8 \end{array} \tag{11}$$

But this contradicts a result of Ucci ([U]) . Note, Ucci actually shows a more general result, namely, that there is no *Serre fibration* from $\mathbf{QP}(3) \to S^8$.

Let us now turn to Riemannian submersions $\pi : \mathbf{CP}(m) \to B$. As already noted, m must be odd, so our task is to classify Riemannian submersions $\pi : \mathbf{CP}(2n+1) \to B$. Then, from Proposition 1.1,

$$S^{4n+3} \xrightarrow{\rho} \mathbf{CP}(2n+1) \xrightarrow{\pi} B \tag{12}$$

is also Riemannian submersion from $S^{4n+3} \to B$. If $B = \mathbf{CP}(2n+1)$, the Riemannian submersion $\pi : \mathbf{CP}(2n+1) \to B$ is trivial. Thus, $B = \mathbf{QP}(n)$ or $B = S^8(1/2)$. The result of Ucci explicitly rules out a Riemannian submersion from $\mathbf{CP}(7) \to S^8(1/2)$. Note, once again, Ucci's result actually rules out any *Serre fibration* from $\mathbf{CP}(7) \to S^8(1/2)$. This is what is needed here. Thus, the only possible Riemannian submersion is from $\pi : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$. If such a Riemannian submersion exists, then the composite Riemannian submersion,

$$S^{4n+3} \xrightarrow{\rho} \mathbf{CP}(2n+1) \xrightarrow{\pi} \mathbf{QP}(n) \tag{13}$$

has totally geodesic fibers as follows from the result of Gromoll-Grove and Wilking. Then, $\pi : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$ has totally geodesic fibers. To see this, let $x \in \mathbf{QP}(n)$ and let $P = \pi^{-1}(x)$. We will show *explicitly* that P is totally geodesic in $\mathbf{CP}(2n+1)$. Now,

$$\rho^{-1}(P) = \rho^{-1}(\pi^{-1}(x)) = (\pi \circ \rho)^{-1}(x) \tag{14}$$

is totally geodesic in S^{4n+3}. We want to apply Proposition 2.1 (a) of [E2] with ρ as the Riemannian submersion of record rather than the π used there. Keeping in mind the label changes, Section 2 of [E2] identifies S_Y as the second fundamental form of $\rho^{-1}(P)$ (instead of $\pi^{-1}(P)$) in the horizontal direction of a vector Y orthogonal to $\rho^{-1}(P)$. If X is horizontal and tangent to $\rho^{-1}(P)$, then by Proposition 2.1 (a) of [E2], we have

$$S_Y X = C_Y X + A_Y X = 0\,, \tag{15}$$

since $\rho^{-1}(P)$ is totally geodesic in S^{4n+3} by the above remarks. Now $C_Y X$ is horizontal and tangent to $\rho^{-1}(P)$, while $A_Y X$ is vertical and tangent to $\rho^{-1}(P)$. Hence, (15) implies $C_Y X = 0$, since the second fundamental form $S_Y = 0$. Let Y_* be a vector orthogonal to P and X_* a vector tangent to P, with Y and X their respective ρ-horizontal lifts. Then,

$$C^*_{Y_*} X_* = \rho_*(C_Y X) = C^*_{\rho_* Y} \rho_* X = 0\,, \tag{16}$$

as was observed in the beginning of Section 2 of [E2] with the above mentioned submersion-map label changes. This means that the second fundamental form of $P = \pi^{-1}(x)$ in $\mathbf{CP}(2n+1)$ satisfies $C^*_{Y_*} X_* = 0$.

It was also observed in [E2] that if the Riemannian submersion

$$\pi : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$$

found there had complex fibers, then the connected complex totally geodesic fibers had to be isometric to $\mathbf{CP}(1) = S^2$. In fact, in section 5 of [E2], we actually constructed a Riemannian submersion, in our labeling $\pi_1 : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$, and showed that the fibers are complex and totally geodesic. We must show that *any other* such Riemannian submersion is equivalent to this standard construction in the sense of the statement of the theorem.

If $\pi_2 : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$ is any other Riemannian submersion, then by Proposition 1.1 above, the composite

$$S^{4n+3} \xrightarrow{\ \rho\ } \mathbf{CP}(2n+1) \xrightarrow{\ \pi_2\ } \mathbf{QP}(n) \tag{17}$$

is a Riemannian submersion. Appealing again to the results [GG] and [W] (see [GW, page 156]) and the above argument modified from [E2] (see also equation (6) in [E2]) , it follows that $\pi_2 : \mathbf{CP}(2n+1) \to \mathbf{QP}(n)$ has totally geodesic fibers. We want to apply Proposition 4.4 of [E2]. To do this we need to see that the hypotheses of that theorem are satisfied. Dimensional considerations force the fibers of the submersion to be even dimensional, namely 2. The exact sequence for a homotopy groups of a fibration ([St, page 91]) with total space $\mathbf{CP}(2n+1)$, fiber F, and base space $\mathbf{QP}(n)$ yields

$$\pi_2(\mathbf{QP}(n)) \xrightarrow{\ \triangle\ } \pi_1(F) \xrightarrow{\ p_*\ } \pi_1(\mathbf{CP}(n))\,. \tag{18}$$

Since both $\pi_2(\mathbf{QP}(n)) = 0$ and $\pi_1(\mathbf{CP}(n) = 0$, exactness forces $\pi_1(F) = 0$, so the fiber F is simply connected. Now by Theorem 1 of [Wo], the only 2- dimensional totally geodesic submanifolds of $\mathbf{CP}(2n+1)$ are real or complex projective spaces, $\mathbf{RP}(2)$ or $S^2 = \mathbf{CP}(1)$, respectively. But real projective space, $\mathbf{RP}(2)$, is not simply connected. Hence, the only admissible two-dimensional totally geodesic fiber is $S^2 = \mathbf{CP}(1)$, as follows from Wolf's theorem.

Using Proposition 4.4 of [E2] which we can now apply because the fibers are both totally geodesic and complex, we see that there is an isometry $f_- : \mathbf{CP}(2n+1) \to \mathbf{CP}(2n+1)$ so that the diagram above (and below), [(5) of Theorem 1.2], commutes, where the horizontal maps are isometries and where the vertical maps are Riemannian submersions.

$$\begin{array}{ccc} \mathbf{CP}(2n+1) & \xrightarrow{f_-} & \mathbf{CP}(2n+1) \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ \mathbf{QP}(n) & \xrightarrow{f_=} & \mathbf{QP}(n) \end{array} \tag{19}$$

This completes the proof of the theorem.

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