Isometries and Maps Compatible with Inverted Jordan Triple Products on Groups

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Abstract. Motivated by the famous Mazur-Ulam theorem in this paper we study algebraic properties of isometries between metric groups. We present some general results on so-called d-preserving maps between subsets of groups and apply them in several directions. We consider d-preserving maps on certain groups of continuous functions defined on compact Hausdorff spaces and describe the structure of isometries between groups of functions mapping into the circle group $\mathbb{T}$. Finally, we show a generalization of the Mazur-Ulam theorem for commutative groups and present a metric characterization of normed real-linear spaces among commutative metric groups.

1. Introduction

The study of morphisms of mathematical structures is of basic importance in all parts of mathematics. As for metric spaces, the natural morphisms are the isometries. In case the metric space under consideration carries an algebraic structure too, it is an interesting problem to investigate if the isometries somehow reflect also the algebraic character of the space. One of the most exciting questions relating to that problem is that whether the isometries necessarily preserve any relevant algebraic operation defined on the underlying space.

The most classical result in this direction is the famous Mazur-Ulam theorem [6] (see also [12]). It states that every bijective isometry T between normed real-linear spaces is affine and hence it equals a real-linear transformation (a surjective real-linear isometry) followed by a translation. The main point of a proof of that beautiful and important result is to show that the isometry T under consideration preserves the so-called algebraic midpoints $(x + y)/2$ in

the sense that

$$T\left(\frac{x + y}{2}\right) = \frac{Tx + Ty}{2}$$

holds for every pair x, y of points in the domain of T. In fact, using continuity, this easily implies that T is necessarily affine. There are other well-known results of the same spirit relating to non-linear structures, for example, to some classical groups. Any self-isometry (with respect to the Euclidean metric) of the circle group $\mathbb{T}$ is a rotation or reflection followed by a rotation. That means that any such isometry is an isometric group isomorphism multiplied by a fixed element. Similar result holds for the isometries of the integer group $\mathbb{Z}$. Its self-isometries are all group isomorphisms followed by translations. These facts have motivated us to study algebraic properties of isometries of general metric groups.

    With Mazur-Ulam theorem in mind, when studying isometries on groups, one may take a chance and consider algebraic midpoints defined in the general setting and examine if the isometries necessarily preserve them. The most natural definition of algebraic midpoints in groups seems to be the following. For any two given elements x, y in a group G, an algebraic midpoint of the pair x, y is any $c \in G$ for which $y = cx^{-1}c$ holds. Clearly, if $\mathcal{L}$ is a linear space, then in its additive structure $c = (x + y)/2$ is the unique algebraic midpoint of any pair x, y in $\mathcal{L}$. However, in the case of general groups serious difficulty may emerge due to the fact that in some cases there are not only one but many algebraic midpoints of a given pair of elements, while in other cases it might happen that there is no algebraic midpoint at all. For example, the pair 1, -1 has two algebraic midpoints in the circle group $\mathbb{T}$ while it has no algebraic midpoint in the multiplicative group of nonzero real numbers. We mention without proof that if the group G is uniquely 2-divisible (which means that for each $b \in G$ there exists a unique $a = b^{1/2} \in G$ such that $b = a^2$), then the element $x \# y = y^{1/2}(y^{-1/2}xy^{-1/2})^{1/2}y^{1/2}$ is the unique algebraic midpoint of any pair x, y in G. (For curiosity we recall that for positive invertible operators A, B on a complex Hilbert space, $A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ is just the geometric mean of A and B in the sense of Ando.) But there are important groups (for example, the circle group) which are not uniquely 2-divisible. Therefore, in order to keep the necessary level of generality we must look for another approach to attack our basic problem. The solution what we offer in the present paper is the following. Instead of considering the "operation" sending any pair x, y to the set of all algebraic midpoints of x, y, we consider the true operation sending any pair x, y to the well-defined element $yx^{-1}y$. We call this latter operation inverted Jordan triple product. Recall that the Jordan triple product yxy is a deeply studied operation in ring theory with extensive applications in other areas of mathematics. Clearly, the geometric meaning of $yx^{-1}y$ in the additive group of the Euclidean space is the reflection of x with respect to the center y. As for algebraic properties of isometries on general groups, instead of examining if they necessarily preserve algebraic midpoints, in what follows we present results showing that under different conditions surjective isometries $T : G_1 \to G_2$ between groups are compatible with the operation of inverted Jordan triple product (or, in other words, preserve inverted Jordan triple products) in the sense that they

satisfy

(1.1) $$T(yx^{-1}y) = T(y)(T(x))^{-1}T(y)$$

for all (or for some) $x, y \in G_1$. This equality clearly captures important information about the algebraic character of the isometries under consideration. In fact, further inspection of that equality played essential role in obtaining the complete description of the isometries of the unitary group on a Hilbert space [5]. Of course, without any conditions on the metric on the given groups the conclusion (1.1) fails to be true as the following simple example shows. Let S_3 be the symmetric group of permutations of $\{1, 2, 3\}$ with the discrete metric. Then every bijection of S_3 is a surjective isometry among which one can easily find such a map which does not satisfy equality (1.1).

The paper is organized as follows. In Section 2 we consider so-called d-preserving maps that are far reaching generalizations of isometries. Using the idea of the miraculous proof given by Väisälä [12] for Mazur-Ulam theorem we prove the general result Theorem 2.4 concerning a local property of d-preserving maps of abstract spaces on which all remaining results rest.

In Section 3 we consider d-preserving maps on groups or subsets of groups which are closed under the operation of inverted Jordan triple product. We introduce several conditions under which any d-preserving bijection T necessarily satisfies (1.1) for a given pair x, y of points in its domain. We present various examples for groups, subsets of groups and elements for which those conditions are fulfilled.

In Section 4 we first study the question how close bijective transformations on commutative groups satisfying the equation (1.1) for all x, y in their domains are to group isomorphisms. Then we consider certain groups of continuous functions defined on compact Hausdorff spaces. More exactly, we consider groups of functions mapping either into the group of non-zero complex numbers or into the circle group $\mathbb{T}$. We give a representation theorem for the isometries in the latter case and then describe the isometries of the group $\mathbb{T}^n$.

In section 5 we give a generalization of Mazur-Ulam theorem for commutative groups. We present a result on the extendibility of surjective isometries between additive subgroups of normed linear spaces to the generated linear spaces and apply the statement to describe the isometries of the group $\mathbb{Z}^n$. The last result of the section gives a metric characterization of normed real-linear spaces among commutative metric groups.

2. d-**preserving maps on abstract spaces**

In this section we present a very general result on the local behaviour of so-called d-preserving maps. This turns to be our key result on which the other statements we obtain in various directions all rest.

Again, we recall that in [12] Väisälä gave a surprisingly short proof for Mazur-Ulam theorem. In order to show that surjective isometries of normed real-linear spaces preserve algebraic midpoints he used (following former observations by Vogt [11]) transformations

which are reflections in algebraic midpoints of arbitrary pairs of points in the space. The proof of the main result Theorem 2.4 of the section is based on a similar idea. However, in contrast with the approach in [12] focusing on the transformation of algebraic midpoints, our goal with this result will be to see how the inverted Jordan triple products $yx^{-1}y$ are transformed.

We begin with some necessary definitions.

DEFINITION 2.1.   Let X be a non-empty set and d a real-valued function defined on $X \times X$. We say that the pair (X, d) satisfies the condition (p) if $d(x, y) \geq 0$ holds for every $(x, y) \in X \times X$.

Obviously, any metric space (X, d) satisfies the condition (p).

DEFINITION 2.2.   Suppose that the pairs (X, d_X) and (Y, d_Y) satisfy the condition (p). We say that a map $f : X \rightarrow Y$ is d-preserving if the equality $d_X(x, y) = d_Y(f(x), f(y))$ holds for every $(x, y) \in X \times X$.

Trivially, if both (X, d_X) and (Y, d_Y) are metric spaces, then a map $T : X \rightarrow Y$ as a transformation between metric spaces is an isometry if and only if T is d-preserving.

The following lemma is a simple generalization of a result of Vogt [11, Theorem 1.2].

LEMMA 2.3.   *Let (X, d) be a pair which satisfies the condition* (p). *Suppose that there exists a point $c \in X$, a bijective d-preserving map φ from X onto itself, and a constant $K > 1$ such that $d(\varphi(x), x) \geq K d(x, c)$ holds for every $x \in X$. If $\sup\{d(x, c) : x \in X\} < \infty$, then $d(f(c), c) = 0$ holds for every bijective d-preserving map f from X onto itself.*

PROOF.   Define

$$\lambda = \sup\{d(f(c), c) : f \text{ is a bijective } d\text{-preserving map from } X \text{ onto itself}\} .$$

Then $0 \leq \lambda < \infty$. Pick an arbitrary bijective d-preserving map f from X onto itself. Put $\tilde{f} = f^{-1} \circ \varphi \circ f$. Then $\tilde{f}$ is also a bijective d-preserving map. Thus

$$\lambda \geq d(\tilde{f}(c), c) = d(\varphi(f(c)), f(c)) \geq K d(f(c), c) .$$

Since f is arbitrary we have $\lambda \geq K \lambda$ implying $\lambda = 0$ which completes the proof.   □

The main result of the section which follows is our key result that will later be applied to verify algebraic properties of isometries between groups.

THEOREM 2.4.   *Let (X, d_X) and (Y, d_Y) be pairs which satisfy the condition* (p). *Pick two points $a, c \in X$. Suppose that $\varphi : X \rightarrow X$ is a d-preserving map such that $\varphi(c) = c$ and $\varphi \circ \varphi$ is the identity map on X. Let*

$$L = \{x \in X : d_X(a, x) = d_X(\varphi(a), x) = d_X(a, c)\} .$$

Suppose that

$$\sup\{d_X(x, c) : x \in L\} < \infty$$

and that there exists a constant $K > 1$ such that $d_X(\varphi(x), x) \geq K d(x, c)$ holds for every $x \in L$. If T is a bijective d-preserving map from X onto Y, and ψ is a bijective d-preserving map from Y onto itself such that $\psi(T(a)) = T(\varphi(a))$ and $\psi(T(\varphi(a))) = T(a)$, then we have

$$d_Y(\psi(T(c)), T(c)) = 0 \,.$$

Note that $\sup\{d_X(x, c) : x \in L\} < \infty$ holds automatically if d_X is a metric (just apply the triangle inequality). We remark that φ is modelled after the transformations $x \mapsto yx^{-1}y$.

PROOF. Since $\varphi(c) = c$ and φ is d-preserving, $d_X(a, c) = d_X(\varphi(a), c)$ holds, hence $c \in L$. Let

$$L_Y = \{y \in Y : d_Y(T(a), y) = d_Y(T(\varphi(a)), y) = d_X(a, c)\} \,.$$

We assert that $T(L) = L_Y$. Indeed, let $x \in L$. Then $d_Y(T(a), T(x)) = d_X(a, x) = d_X(a, c)$ and $d_Y(T(\varphi(a)), T(x)) = d_Y(\varphi(a), x) = d_X(a, c)$. This shows that $T(L) \subset L_Y$. Now, let $y \in L_Y$. Since $T(X) = Y$, there is an $x \in X$ with $T(x) = y$. Then $d_X(a, x) = d_Y(T(a), T(x)) = d_X(a, c)$ and $d_X(\varphi(a), x) = d_Y(T(\varphi(a)), T(x)) = d_X(a, c)$. Thus $x \in L$ and we obtain $L_Y \subset T(L)$.

We next show that $\varphi(L) = L$. Firstly, let $x \in L$. Then $d_X(\varphi(a), \varphi(x)) = d_X(a, x) = d_X(a, c)$ and $d_X(a, \varphi(x)) = d_X(\varphi(\varphi(a)), \varphi(x)) = d_X(\varphi(a), x) = d_X(a, c)$, so that $\varphi(x) \in L$. This gives us $\varphi(L) \subset L$. Since $\varphi \circ \varphi$ is the identity map, $L \subset \varphi(L)$ also holds.

We assert that $\psi(L_Y) = L_Y$. To see this, let $y \in L_Y$. We compute

$$d_Y(T(\varphi(a)), \psi(y)) = d_Y(\psi(T(a)), \psi(y)) = d_Y(T(a), y) = d_X(a, c) \,,$$

$$d_Y(T(a), \psi(y)) = d_Y(\psi(T(\varphi(a))), \psi(y)) = d_Y(T(\varphi(a)), y) = d_X(a, c) \,.$$

Thus $\psi(y) \in L_Y$ and we obtain $\psi(L_Y) \subset L_Y$. On the other hand, if $y \in L_Y$, then there is $y' \in Y$ with $\psi(y') = y$ since $\psi(Y) = Y$. We compute

$$d_Y(T(a), y') = d_Y(\psi(T(a)), \psi(y')) = d_Y(T(\varphi(a)), y) = d_X(a, c) \,,$$

$$d_Y(T(\varphi(a)), y') = d_Y(\psi(T(\varphi(a))), \psi(y')) = d_Y(T(a), y) = d_X(a, c) \,.$$

Thus $y' \in L_Y$, so $y = \psi(y') \in \psi(L_Y)$ and we obtain $L_Y \subset \psi(L_Y)$.

Let $\widetilde{T} = T^{-1} \circ \psi \circ T$. Then $\widetilde{T}|_L : L \to L$ is a bijective d-preserving map from L onto itself. Since $\sup\{d_X(x, c) : x \in L\} < \infty$ and $\varphi|_L : L \to L$ is a bijective d-preserving map, applying Lemma 2.3 we deduce

$$d_Y(\psi(T(c)), T(c)) = d_X(\widetilde{T}(c), c) = 0 \,.$$

$\square$

3. d-preserving maps on subsets of groups

In this section we consider d-preserving maps between subsets of groups and investigate the question what algebraic properties those transformations necessarily have. We are

primarily interested in seeing to what extent d-preserving maps respect the group operation. Obviously, without further conditions we can not expect any useful statement. Therefore, in what follows we introduce certain conditions under which applying the theorem in the previous section we obtain some results concerning our general problem.

The reason that below we consider transformations not only on subgroups but on more general subsets of groups is that we really need that generality among others in an application for the Thompson isometries of the space of all invertible positive elements in a C^*-algebra (see [5]). Clearly, the set of all such elements is not a subgroup however it is closed under the inverted Jordan triple product.

We fix the following: in Definitions 3.1 to 3.4, X denotes a nonempty subset of a group with the property that

$$yx^{-1}y \in X \ \text{ holds for every pair } \ x, y \in X$$

and X is equipped with a real-valued function d such that the pair (X, d) satisfies the condition (p).

In what follows we denote the unit element in a group by e.

DEFINITION 3.1. We say that X is 2-divisible (resp. uniquely 2-divisible) if for each $a \in X$ the equation $x^2 = a$ has a solution (resp. has a unique solution) $x \in X$. We say that X is 2-torsion free if the unit element e of the underlying group belongs to X and the equality $x^2 = e$ implies $x = e$.

Note that X automatically contains the unit element e if it is 2-divisible. Indeed, for any y in X there is $x \in X$ with $x^2 = y$ and hence we have $e = xy^{-1}x \in X$.

DEFINITION 3.2 (Condition B$(\cdot, \cdot)$). Let $a, b \in X$. We say that (X, d) satisfies the condition B(a, b) if the following (1) through (3) are fulfilled:

(1) the equality

$$d(bx^{-1}b, by^{-1}b) = d(x, y)$$

holds for every pair $x, y \in X$;

(2) $\sup\{d(x, b) : x \in L_{a,b}\} < \infty$, where

$$L_{a,b} = \{x \in X : d(a, x) = d(ba^{-1}b, x) = d(a, b)\};$$

(3) there exists a constant $K > 1$ such that the inequality

$$d(bx^{-1}b, x) \geq Kd(x, b)$$

holds for every $x \in L_{a,b}$.

One may say that $L_{a,b}$ is the set of "metrical midpoints" of the pair $a, ba^{-1}b$. Note that the algebraic midpoint b of the pair $a, ba^{-1}b$ is in $L_{a,b}$ by definition. Also observe that

if d is a metric, then by the triangle inequality we always have $\sup\{d(x, b) : x \in L_{a,b}\} \leq 2d(a, b) < \infty$.

DEFINITION 3.3 (Condition $C_1(\cdot, \cdot)$). Let $a, b \in X$. We say that (X, d) satisfies the condition $C_1(a, b)$ if the following (1) and (2) are fulfilled:

(1) $ax^{-1}b, bx^{-1}a \in X$ for every $x \in X$;
(2) the equality

$$d(ax^{-1}b, ay^{-1}b) = d(x, y)$$

holds for every pair $x, y \in X$.

We remark that the seemingly unreasonable condition that $bx^{-1}a \in X$ holds for every $x \in X$ will be applied in Corollary 3.9 to show the surjectivity of a certain transformation that we need in the proof.

DEFINITION 3.4 (Condition $C_2(\cdot, \cdot)$). Let $a, b \in X$. We say that (X, d) satisfies the condition $C_2(a, b)$ if the following is fulfilled: there exists a $c \in X$ with $ca^{-1}c = b$ such that the equality $d(cx^{-1}c, cy^{-1}c) = d(x, y)$ holds for every pair $x, y \in X$.

Concerning the previous condition observe the following. Assume that X is 2-divisible. Let $a, b \in X$. We assert that there exists $c \in X$ such that $ca^{-1}c = b$. To see this, take $d \in X$ with $d^2 = a$, and $f \in X$ with $f^2 = db^{-1}d$. Setting $c = df^{-1}d$ we have $c \in X$ and

$$ca^{-1}c = df^{-1}da^{-1}df^{-1}d = d(f^{-1})^2d = b.$$

Those $B(\cdot, \cdot), C_1(\cdot, \cdot), C_2(\cdot, \cdot)$ are the conditions under which we shall present results showing that d-preserving maps have a useful algebraic property. Namely, any such transformation T turns to be compatible with the operation of inverted Jordan triple product in the sense that $T(b)(T(a))^{-1}T(b) = T(ba^{-1}b)$ holds for certain a and b. We remark that the condition $B(\cdot, \cdot)$ mainly concerns the domains of the maps under consideration while the conditions $C_1(\cdot, \cdot)$ and $C_2(\cdot, \cdot)$ mainly concern their ranges.

Before presenting the results we give several examples for sets satisfying the conditions $B(\cdot, \cdot), C_1(\cdot, \cdot)$ or $C_2(\cdot, \cdot)$.

First observe the following triviality. If X is an additive subgroup of a normed space with norm $\| \cdot \|$ and $d(x, y) = \|x - y\|$ for all $x, y \in X$, then $B(\cdot, \cdot)$ and $C_1(\cdot, \cdot)$ are satisfied for every pair of points in X, moreover $C_2(\cdot, \cdot)$ is satisfied too in the case when X is 2-divisible.

This observation can be generalized introducing the following definition.

DEFINITION 3.5. Let G be a group and X a subgroup of G. Suppose that (X, d) is a pair which satisfies the condition (p). We say that $d(\cdot, \cdot)$ is inverse and translation invariant if

$$d(x^{-1}, y^{-1}) = d(ax, ay) = d(x, y)$$

holds for every triple $x, y, a \in X$.

Note that if X is a subgroup of a group, then $d(\cdot, \cdot)$ is inverse and translation invariant if and only if $C_1(a, b)$ holds for every pair $a, b \in X$. If X is a subgroup, 2-divisible, and $d(\cdot, \cdot)$ is inverse and translation invariant, then $C_2(a, b)$ holds for every pair $a, b \in X$.

In what follows we present some non-trivial examples.

EXAMPLE 3.6.   Let G be the group of all invertible elements in a unital C^*-algebra and X the subset of G which consists of the positive elements in G. Recall that the definition of the Thompson metric on X is

$$d(x, y) = \log \max\{\inf\{t > 0 : x \le ty\}, \inf\{t > 0 : y \le tx\}\}$$

for any $x, y \in X$. It is not difficult to see that we have

$$d(x, y) = \| \log x^{-\frac{1}{2}} y x^{-\frac{1}{2}} \|$$

(see, e.g., [8]). The condition $B(a, b)$ holds for every pair $a, b \in X$. Indeed, for any $x, y, b \in X$ the equality $d(bx^{-1}b, by^{-1}b) = d(x^{-1}, y^{-1}) = d(x, y)$ follows directly from the definition of the Thompson metric and, as we see in the proof of [8, Theorem 1], we also have $d(bx^{-1}b, x) = 2d(x, b)$. Since d is a metric, we conclude that $B(a, b)$ is satisfied for every pair $a, b \in X$. Moreover, since X is 2-divisible, we see that the condition $C_2(a, b)$ is also satisfied for every pair $a, b \in X$. It was presented in [5] that the surjective Thompson isometries globally satisfy the equation (1.1) (see also [8, 9]).

EXAMPLE 3.7.   Let $X = \mathbb{T}^n$ be the n-dimensional torus with the usual multiplication and metric defined by $d(\mathbf{x}, \mathbf{y}) = \max\{|x_j - y_j| : 1 \le j \le n\}$ for $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{T}^n$. We assert that for any $\mathbf{a}, \mathbf{b} \in \mathbb{T}^n$ with $d(\mathbf{a}, \mathbf{b}) < \sqrt{2}$ the condition $B(\mathbf{a}, \mathbf{b})$ is satisfied with $K = \sqrt{2}$. To see this, pick $\mathbf{a}, \mathbf{b} \in X$ with $d(\mathbf{a}, \mathbf{b}) < \sqrt{2}$. The equality in (1) of Definition 3.2 is trivial while (2) also follows immediately as $d(\cdot, \cdot)$ is a metric. We show that (3) holds, too. First let $a, b \in \mathbb{T}$, the unit circle in the complex plane. Suppose that $|a - b| < \sqrt{2}$. Then $|a - ba^{-1}b| \ge \sqrt{2}|a - b|$ holds. Indeed, since $\sqrt{2} > |a - b| = |1 - a^{-1}b|$ we have that $\mathrm{Re}(a^{-1}b) > 0$ where $\mathrm{Re}z$ denotes the real part of a given complex number z. It is easy to see that this implies $|1 + a^{-1}b| > \sqrt{2}$. It then follows that

$$
\begin{aligned}
|a - ba^{-1}b| &= |1 - (a^{-1}b)^2| \\
&= |1 + a^{-1}b||1 - a^{-1}b| \ge \sqrt{2}|1 - a^{-1}b| = \sqrt{2}|a - b|.
\end{aligned}
$$

Now pick $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{T}^n$ and suppose that $d(\mathbf{a}, \mathbf{b}) < \sqrt{2}$. We show that $d(\mathbf{bx}^{-1}\mathbf{b}, \mathbf{x}) \ge \sqrt{2}d(\mathbf{x}, \mathbf{b})$ holds for every $\mathbf{x} \in L_{\mathbf{a}, \mathbf{b}}$, where $L_{\mathbf{a}, \mathbf{b}} = \{\mathbf{x} \in \mathbb{T}^n : d(\mathbf{a}, \mathbf{x}) = d(\mathbf{ba}^{-1}\mathbf{b}, \mathbf{x}) = d(\mathbf{a}, \mathbf{b})\}$. The proof goes as follows. Put $\theta = 2\sin^{-1}\frac{d(\mathbf{a}, \mathbf{b})}{2}$. Since $d(\mathbf{a}, \mathbf{b}) < \sqrt{2}$, we have $0 \le \theta < \pi/2$. For each $1 \le j \le n$ we have $|a_j - b_j| \le d(\mathbf{a}, \mathbf{b})$, so for every j there exists θ_j with $-\theta \le \theta_j \le \theta$ such that $b_j = a_j e^{i\theta_j}$ holds. Since $|x_j - a_j| \le d(\mathbf{x}, \mathbf{a}) = d(\mathbf{a}, \mathbf{b})$ and $|x_j - b_j(a_j)^{-1}b_j| \le d(\mathbf{x}, \mathbf{ba}^{-1}\mathbf{b}) = d(\mathbf{a}, \mathbf{b})$ both hold, it can be verified that there exists η_j with $-\theta + |\theta_j| \le \eta_j \le \theta - |\theta_j|$ such that $x_j = b_j e^{i\eta_j}$ (recall that

$0 \leq \theta < \frac{\pi}{2}$). Since $0 \leq \theta - |\theta_j| \leq \theta < \pi/2$, we have $|x_j - b_j| < \sqrt{2}$ for every j. Thus, by the first part of the proof, $|x_j - b_j(x_j)^{-1}b_j| \geq \sqrt{2}|x_j - b_j|$ holds for every $1 \leq j \leq n$ and hence we have $d(\mathbf{b}\mathbf{x}^{-1}\mathbf{b}, \mathbf{x}) \geq \sqrt{2}d(\mathbf{x}, \mathbf{b})$.

Note that if $\mathbf{b} = -\mathbf{a}$, then $d(\mathbf{a}, \mathbf{b}) = 2$ and $d(\mathbf{ba}^{-1}\mathbf{b}, \mathbf{a}) = 0$. This shows that (3) of Definition 3.2 does not hold in general.

Finally, observe that since $\mathbb{T}^n$ is 2-divisible and $d(\cdot, \cdot)$ is inverse and translation invariant, $C_1(\mathbf{a}, \mathbf{b})$ and $C_2(\mathbf{a}, \mathbf{b})$ are satisfied for every $\mathbf{a}, \mathbf{b} \in \mathbb{T}^n$.

EXAMPLE 3.8. Let $\mathcal{X}$ be a compact Hausdorff space and E any real-linear subspace of the space of all complex-valued continuous functions on $\mathcal{X}$. Define $d(f, g) = \max\{\|\frac{f}{g} - 1\|_\infty, \|\frac{g}{f} - 1\|_\infty\}$ for $f, g \in \exp E$, where $\|\cdot\|_\infty$ denotes the supremum norm of continuous functions on $\mathcal{X}$. Then $\exp E$ is a (multiplicative) group and $(\exp E, d)$ satisfies the condition (p) (observe that d is not a metric). Moreover, $\exp E$ is 2-divisible, and $d(\cdot, \cdot)$ is inverse and translation invariant. Therefore, the conditions $C_1(a, b)$ and $C_2(a, b)$ are satisfied for every pair $a, b \in \exp E$.

Let us turn to the results announced after Definition 3.4. We fix the following. In Corollaries 3.9, 3.10 and 3.11 we suppose that: X_i is a non-empty subset of a group G_i and (X_i, d_i) is a pair which satisfies the condition (p) for $i = 1, 2$; we have $yx^{-1}y \in X_i$ for every pair $x, y \in X_i$ for $i = 1, 2$; $d_2(x, y) = 0$ holds only if $x = y$ (we do not require this for $d_1(\cdot, \cdot)$); $T : X_1 \to X_2$ is a bijective d-preserving map.

COROLLARY 3.9. *Let $a, b \in X_1$. Suppose that (X_1, d_1) satisfies $B(a, b)$ and (X_2, d_2) satisfies $C_1(T(a), T(ba^{-1}b))$. Then we have*

$$T(ba^{-1}b) = T(b)(T(a))^{-1}T(b).$$

PROOF. Let $\varphi(x) = bx^{-1}b$ for every $x \in X_1$. Then $\varphi : X_1 \to X_1$ is well-defined, $\varphi(b) = b$ and $\varphi \circ \varphi$ is the identity map on X, and as the condition $B(a, b)$ is satisfied, the map φ satisfies the relating hypotheses in Theorem 2.4. Let $\psi(x) = T(a)x^{-1}T(ba^{-1}b)$ for $x \in X_2$. Then $\psi : X_2 \to X_2$ is a well-defined bijective d-preserving map since (X_2, d_2) satisfies $C_1(T(a), T(ba^{-1}b))$. By definition, $\psi(T(a)) = T(\varphi(a))$ and $\psi(T(\varphi(a))) = T(a)$ hold. Applying Theorem 2.4 we have $d_2(\psi(T(b)), T(b)) = 0$. By the assumptions given before the corollary this implies that $\psi(T(b)) = T(b)$ which then gives us that $T(ba^{-1}b) = T(b)(T(a))^{-1}T(b)$. $\square$

As we shall see Corollary 3.9 is well-applicable in relation with isometries on groups. However, for example, to treat Thompson isometries of the set of all positive invertible elements of a unital C^*-algebra we need the following result.

COROLLARY 3.10. *Let $a, b \in X_1$. Suppose that (X_1, d_1) satisfies $B(a, b)$ and (X_2, d_2) satisfies $C_2(T(a), T(ba^{-1}b))$. If, in addition, X_2 is 2-divisible and 2-torsion free,*

then we have

$$T(ba^{-1}b) = T(b)(T(a))^{-1}T(b).$$

PROOF.   Let $\varphi(x) = bx^{-1}b$ for every $x \in X_1$. Just as in the proof of Corollary 3.9, φ satisfies the relating conditions in Theorem 2.4. Let $\psi(x) = cx^{-1}c$ for $x \in X_2$, where c is an element in X_2 such that $c(T(a))^{-1}c = T(ba^{-1}b)$ and $d_2(cx^{-1}c, cy^{-1}c) = d_2(x, y)$ holds for every pair $x, y \in X_2$. Then $\psi : X_2 \to X_2$ is a well-defined bijective d-preserving map. Simple calculation shows that $\psi(T(a)) = T(\varphi(a))$ and $\psi(T(\varphi(a))) = T(a)$. Applying Theorem 2.4 we see that $d_2(\psi(T(b)), T(b)) = 0$ implying $c(T(b))^{-1}c = T(b)$. Since X_2 is 2-divisible, there is $d \in X_2$ with $d^2 = c$. It follows that $d^2(T(b))^{-1}d^2 = T(b)$, and hence $d^2(T(b))^{-1}d^2(T(b))^{-1} = e$. Multiplying the last equality by d^{-1} from the left and by d from the right we have

$$(d(T(b))^{-1}d)^2 = d(T(b))^{-1}d^2(T(b))^{-1}d = e.$$

Since X_2 is 2-torsion free, we obtain $d(T(b))^{-1}d = e$. Hence we deduce $c = d^2 = T(b)$ and it follows by the definition of c that

$$T(ba^{-1}b) = T(b)(T(a))^{-1}T(b)$$

holds.   □

The conditions of 2-divisibility and 2-torsion freeness above can be replaced by a weak commutativity condition as seen in the next corollary.

COROLLARY 3.11.   *Let $a, b \in X_1$. Suppose that (X_1, d_1) satisfies $\mathrm{B}(a, b)$ and (X_2, d_2) satisfies $\mathrm{C}_2(T(a), T(ba^{-1}b))$. If, in addition, X_2 also satisfies the condition that $yx^{-1}y = y^2x^{-1}$ for every pair $x, y \in X_2$, then we have*

$$T(ba^{-1}b) = T(b)(T(a))^{-1}T(b).$$

PROOF.   Following the lines of the proof of Corollary 3.10 one can verify that $f(T(b))^{-1}f = T(b)$ holds for an $c \in X_2$ which satisfies $c(T(a))^{-1}c = T(ba^{-1}b)$ and $d_2(cx^{-1}c, cy^{-1}c) = d_2(x, y)$ for every $x, y \in X_2$. Since X_2 also satisfies $yx^{-1}y = y^2x^{-1}$ for every pair $x, y \in X_2$, it follows that $c^2 = (T(b))^2$ and next that $(T(b))^2(T(a))^{-1} = T(ba^{-1}b)$. Therefore, we have $T(ba^{-1}b) = T(b)(T(a))^{-1}T(b)$.   □

4.   *d-preserving maps on groups of continuous functions*

Let $\mathbb{C}$ be the set of all complex numbers. In this section we consider d-preserving maps between groups of complex-valued continuous functions, namely, groups of $\mathbb{C} \setminus \{0\}$-valued functions and groups of $\mathbb{T}$-valued functions. We note that surjective isometries between open subgroups of the groups of invertible elements in unital semisimple commutative Banach algebras were characterized in [2, 3].

First we study bijective transformations T on commutative groups which satisfy the equality $T(yx^{-1}y) = T(y)(T(x))^{-1}T(y)$ on their domains. In particular, we are interested in the question when such a map is necessarily a group isomorphism.

LEMMA 4.1. *Let G_1 and G_2 be commutative groups and $T : G_1 \to G_2$ a bijection such that*

(4.1) $$T(yx^{-1}y) = T(y)(T(x))^{-1}T(y) \quad \forall x, y \in G_1$$

holds. If at least one of G_1 and G_2 is 2-divisible or 2-torsion free, then for the bijective map $T_0(\cdot) = (T(e))^{-1}T(\cdot)$ we have

$$T_0(xy) = T_0(x)T_0(y) \quad \forall x, y \in G_1,$$

that is, T_0 is a group isomorphism.

PROOF. Clearly, $T_0 : G_1 \to G_2$ is bijective and satisfies

$$T_0(yx^{-1}y) = T_0(y)(T_0(x))^{-1}T_0(y)$$

for all $x, y \in G_1$. Inserting $y = e$ we see that $T_0(x^{-1}) = (T_0(x))^{-1}$. This implies that $T_0(yxy) = T_0(y)T_0(x)T_0(y)$ and next that $T_0(x^2) = (T_0(x))^2$ holds for all $x, y \in G_1$.

First we consider the case when G_2 is 2-divisible. Let $x, y \in G_1$ be arbitrary. Then there is $B \in G_2$ with $B^2 = T_0(y)$ and since T_0 is surjective there is $b \in G_1$ with $T_0(b) = B$. Thus $T_0(b^2) = (T_0(b))^2 = T_0(y)$ and hence $b^2 = y$ for T_0 is injective. It follows that

$$T_0(xy) = T_0(xb^2) = T_0(bxb)$$
$$= T_0(b)T_0(x)T_0(b) = T_0(x)(T_0(b))^2 = T_0(x)T_0(y)$$

since G_1 and G_2 are commutative.

Next we consider the case when G_2 is 2-torsion free. Let $x, y \in G_1$ be arbitrary. Since G_1 and G_2 are commutative we have

$$(T_0(xy))^2 = T_0((xy)^2) = T_0(yx^2y) = T_0(y)T_0(x^2)T_0(y)$$
$$= T_0(y)(T_0(x))^2T_0(y) = (T_0(x)T_0(y))^2$$

which gives us that

$$(T_0(xy)(T_0(x))^{-1}(T_0(y))^{-1})^2 = e.$$

As G_2 is 2-torsion free, it follows that

$$T_0(xy)(T_0(x))^{-1}(T_0(y))^{-1} = e$$

and hence

$$T_0(xy) = T_0(x)T_0(y)$$

holds.

The result for the case where G_1 is 2-divisible or 2-torsion free follows from the previous case simply applying the above argument for the inverse of T_0.   □

The following technical lemma which will be used in the rest of the paper can be found in [5]. For the sake of completeness we present it also here.

LEMMA 4.2.   *Let G_i be a group and X_i a non-empty subset of G_i such that $yx^{-1}y \in X_i$ for every pair $x, y \in X_i$, for $i = 1, 2$, and $T : X_1 \to X_2$ a map. Suppose that n is a positive integer. For every finite sequence $\{a_k\}_{k=0}^{2^n}$ with $2^n + 1$ terms in X_1, if*

$A(n)$ $a_{k+1}a_k^{-1}a_{k+1} = a_{k+2}$,

$$T(a_{k+1}a_k^{-1}a_{k+1}) = T(a_{k+1})(T(a_k))^{-1}T(a_{k+1}) \quad \text{for every } 0 \le k \le 2^n - 2$$

are satisfied, then

$C(n)$ $a_{2^n-1}a_0^{-1}a_{2^n-1} = a_{2^n}$,

$$T(a_{2^n-1}a_0^{-1}a_{2^n-1}) = T(a_{2^n-1})(T(a_0))^{-1}T(a_{2^n-1})$$

hold.

PROOF.   We prove the statement by induction on n. The condition $A(1)$ obviously implies the condition $C(1)$. Assume now that "if $A(n-1)$ is satisfied for a finite sequence with $2^{n-1} + 1$ terms, then $C(n-1)$ holds for that sequence". We will show that "if $A(n)$ is satisfied for a sequence with $2^n + 1$ terms, then $C(n)$ holds for that sequence". Suppose $A(n)$ is satisfied for $\{a_k\}_{k=0}^{2^n}$. We first show that for every $k = 0, \ldots, 2^n - 4$,

(4.2) $$a_{k+2}a_k^{-1}a_{k+2} = a_{k+4}$$

and

(4.3) $$T(a_{k+2}a_k^{-1}a_{k+2}) = T(a_{k+2})(T(a_k))^{-1}T(a_{k+2})$$

hold. Let $0 \le k \le 2^n - 4$. Then

$$a_{k+4} = a_{k+3}a_{k+2}^{-1}a_{k+3}$$
$$= (a_{k+2}a_{k+1}^{-1}a_{k+2})a_{k+2}^{-1}(a_{k+2}a_{k+1}^{-1}a_{k+2}) = a_{k+2}a_{k+1}^{-1}a_{k+2}a_{k+1}^{-1}a_{k+2}$$
$$= a_{k+2}a_{k+1}^{-1}(a_{k+1}a_k^{-1}a_{k+1})a_{k+1}^{-1}a_{k+2} = a_{k+2}a_k^{-1}a_{k+2},$$

so that (4.2) holds. We also see that

$$T(a_{k+2}a_k^{-1}a_{k+2}) = T(a_{k+3}a_{k+2}^{-1}a_{k+3})$$
$$= T(a_{k+3})(T(a_{k+2}))^{-1}T(a_{k+3})$$
$$= T(a_{k+2}a_{k+1}^{-1}a_{k+2})(T(a_{k+2}))^{-1}T(a_{k+2}a_{k+1}^{-1}a_{k+2})$$
$$= T(a_{k+2})(T(a_{k+1}))^{-1}T(a_{k+2})(T(a_{k+1}))^{-1}T(a_{k+2})$$

$$= T(a_{k+2})(T(a_{k+1}))^{-1}T(a_{k+1}a_k^{-1}a_{k+1})(T(a_{k+1}))^{-1}T(a_{k+2})$$

$$= T(a_{k+2})(T(a_k))^{-1}T(a_{k+2}),$$

so that (4.3) holds.

Next, letting $b_k = a_{2k}$ for $0 \leq k \leq 2^{n-1}$ the sequence $\{b_k\}_{k=0}^{2^{n-1}}$ satisfies $A(n-1)$ due to (4.2) and (4.3). By our assumption we infer that $C(n-1)$ holds for $\{b_k\}_{k=0}^{2^{n-1}}$, i.e.,

$$T(b_{2^{n-2}}b_0^{-1}b_{2^{n-2}}) = T(b_{2^{n-2}})(T(b_0))^{-1}T(b_{2^{n-2}}).$$

For $b_0 = a_0$ and $b_{2^{n-2}} = a_{2^{n-1}}$, we obtain that $C(n)$ holds for $\{a_k\}_{k=0}^{2^n}$. $\qquad \square$

The next theorem describes the algebraic structure of certain d-preserving maps on multiplicative groups of complex-valued continuous functions.

THEOREM 4.3. *Let Y_i be a compact Hausdorff space and $C(Y_i)$ the space of all complex-valued continuous functions on Y_i, for $i = 1, 2$. Suppose E_i is a real-linear subspace of $C(Y_i)$, for $i = 1, 2$. Define*

$$d_i(f, g) = \max \left\{ \left\| \frac{f}{g} - 1 \right\|_\infty, \left\| \frac{g}{f} - 1 \right\|_\infty \right\},$$

for $f, g \in \exp E_i$. If $T : \exp E_1 \to \exp E_2$ is a bijective d-preserving map with respect to d_1 and d_2, then for the transformation $T_0(\cdot) = (T(1))^{-1}T(\cdot)$ we have

$$T_0(fg) = T_0(f)T_0(g) \quad \forall f, g \in \exp E_1.$$

PROOF. Let $f, g \in \exp E_1$. Then $f = \exp u$ and $g = \exp v$ for some $u, v \in E_1$. For a sufficiently large integer n we have

$$\left| \exp \frac{2(u-v)}{2^n} - 1 \right| \leq \frac{1}{10}, \quad \left| \exp \frac{-2(u-v)}{2^n} - 1 \right| \leq \frac{1}{10}.$$

Then we obtain

$$\left| \exp \frac{2(u-v)}{2^n} + 1 \right| \geq \frac{19}{10}, \quad \left| \exp \frac{-2(u-v)}{2^n} + 1 \right| \geq \frac{19}{10}.$$

For each $0 \leq k \leq 2^n$, let

$$f_k = \exp \left(u - \frac{2k(u-v)}{2^n} \right).$$

Then

$$f_0 = f, \quad f_{2^n} = gf^{-1}g, \quad g = f_{2^{n-1}}$$

and

$$f_{k+1}f_k^{-1}f_{k+1} = f_{k+2}$$

for every $0 \le k \le 2^n - 2$.

Let $0 \le k \le 2^n - 2$ be arbitrary and put

$$L_{f_k, f_{k+1}} = \{h \in \exp E_1 : d_1(f_k, h) = d_1(f_{k+2}, h) = d_1(f_k, f_{k+1})\},$$

where

$$d_1(f_k, f_{k+1}) = \max \left\{ \left\| \exp \frac{2(u - v)}{2^n} - 1 \right\|_\infty, \left\| \exp \frac{-2(u - v)}{2^n} - 1 \right\|_\infty \right\} \le \frac{1}{10}.$$

We show that condition $B(f_k, f_{k+1})$ is satisfied. Let $h \in L_{f_k, f_{k+1}}$. Then we have

$$d_1(f_{k+1} h^{-1} f_{k+1}, h) = \max \left\{ \left\| \frac{f_{k+1}^2}{h^2} - 1 \right\|_\infty, \left\| \frac{h^2}{f_{k+1}^2} - 1 \right\|_\infty \right\}.$$

Since $f_{k+1} = f_k \exp \frac{-2(u-v)}{2^n}$, we compute

$$\left| \frac{f_{k+1}}{h} + 1 \right| = \left| \frac{f_k}{h} + \exp \frac{2(u - v)}{2^n} \right| \left| \exp \frac{-2(u - v)}{2^n} \right|$$

$$\ge \left(- \left| \frac{f_k}{h} - 1 \right| + \left| \exp \frac{2(u - v)}{2^n} + 1 \right| \right) \left| \exp \frac{-2(u - v)}{2^n} \right|$$

$$\ge \left(-d_1(f_k, h) + \frac{19}{10} \right) \frac{9}{10} \ge \frac{81}{50},$$

which gives us that

$$\left| \frac{f_{k+1}^2}{h^2} - 1 \right| = \left| \frac{f_{k+1}}{h} - 1 \right| \left| \frac{f_{k+1}}{h} + 1 \right| \ge \frac{81}{50} \left| \frac{f_{k+1}}{h} - 1 \right|,$$

and hence

$$\left\| \frac{f_{k+1}^2}{h^2} - 1 \right\|_\infty \ge \frac{81}{50} \left\| \frac{f_{k+1}}{h} - 1 \right\|_\infty.$$

In a similar way we obtain that

$$\left\| \frac{h^2}{f_{k+1}^2} - 1 \right\|_\infty \ge \frac{81}{50} \left\| \frac{h}{f_{k+1}} - 1 \right\|_\infty$$

holds. Thus we have

$$d_1(f_{k+1} h^{-1} f_{k+1}, h) \ge \frac{81}{50} d_1(h, f_{k+1})$$

for every $h \in L_{f_k, f_{k+1}}$, that is, (3) of Definition 3.2 holds with $K = \frac{81}{50}$. As d_1 is inverse and translation invariant, (1) of Definition 3.2 also holds. We show that (2) of Definition 3.2 is

satisfied, too. Indeed, let $h \in L_{f_k, f_{k+1}}$. Then

$$\left\| \frac{f_{k+1}}{h} - 1 \right\| = \left\| \frac{f_k \exp\left(-\dfrac{u-v}{2^{n-1}}\right)}{h} - 1 \right\|$$

$$\leq \left\| \frac{f_k}{h} - 1 \right\| \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) \right\| + \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) - 1 \right\|$$

$$\leq d_1(f_k, h) \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) \right\| + \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) - 1 \right\|$$

$$= d_1(f_k, f_{k+1}) \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) \right\| + \left\| \exp\left(-\frac{u-v}{2^{n-1}}\right) - 1 \right\|.$$

In the same way we have that

$$\left\| \frac{h}{f_{k+1}} - 1 \right\| \leq d_1(f_k, f_{k+1}) \left\| \exp\left(\frac{u-v}{2^{n-1}}\right) \right\| + \left\| \exp\left(\frac{u-v}{2^{n-1}}\right) - 1 \right\|.$$

Consequently we deduce

$$\sup \left\{ d_1(h, f_{k+1}) : h \in L_{f_k, f_{k+1}} \right\} < \infty.$$

Hence, we have obtained that $B(f_k, f_{k+1})$ is really satisfied for every $0 \leq k \leq 2^n - 2$. As d_2 is inverse and translation invariant, we infer that $C_1(T(f_k), T(f_{k+1} f_k^{-1} f_{k+1}))$ is satisfied for every $0 \leq k \leq 2^n - 2$. Applying Corollary 3.9 we obtain that

$$T(f_{k+1} f_k^{-1} f_{k+1}) = T(f_{k+1})(T(f_k))^{-1} T(f_{k+1})$$

holds for every $0 \leq k \leq 2^n - 2$. Lemma 4.2 gives us that

$$T(f_{2^{n-1}} f_0^{-1} f_{2^{n-1}}) = T(f_{2^{n-1}})(T(f_0))^{-1} T(f_{2^{n-1}}).$$

Since $f_0 = f$ and $f_{2^{n-1}} = g$, we have

$$T(g f^{-1} g) = T(g)(T(f))^{-1} T(g).$$

Finally, as $f, g \in \exp E_1$ are arbitrary and $\exp E_2$ is 2-divisible, by Lemma 4.1 we infer that

$$T_0(f g) = T_0(f) T_0(g) \quad \forall f, g \in \exp E_1.$$

This completes the proof. $\qquad\square$

We note that for uniform algebras A and B surjective maps $T : \exp A \to \exp B$ with the property that $\left\| \frac{f}{g} - 1 \right\|_\infty = \left\| \frac{T(f)}{T(g)} - 1 \right\|_\infty$ ($\forall f, g \in A$) were characterized in [7]. In [4] the case when E_j is the space of all real-valued continuous functions was considered.

We recall that by a theorem due to Sakai [10] any uniformly continuous group isomorphism between the unitary groups of AW^*-factors is implemented by a linear or conjugate

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linear $*$-isomorphism of the factors. In the following example we show that the assertion does not hold for general C^*-algebras. Let $C(Y)$ be the commutative C^*-algebra of all complex-valued continuous functions on a compact Hausdorff space Y. The unitary group $UC(Y)$ of $C(Y)$ consists precisely of the functions in $C(Y)$ which are of modulus one. In what follows we present a uniformly continuous group isomorphism between two such unitary groups which can not be extended to an (algebra) isomorphism between the underlying algebras of complex-valued continuous functions.

EXAMPLE 4.4. Let $Y_1 = [0, 1]$ be the unit closed interval and $Y_2 = \{(x, y) \in \mathbb{R}^2 : x \in [0, \frac{2}{3}],\ y = 0\} \cup \{(x, y) \in \mathbb{R}^2 : x = \frac{1}{3},\ y \in [0, \frac{1}{3}]\}$. Let $T : UC(Y_1) \to UC(Y_2)$ be defined as

$$T(f)(x, y) = \begin{cases} f(x), & y = 0 \\ \dfrac{f\left(\frac{1}{3}\right)}{f\left(\frac{2}{3}\right)} f\left(y + \frac{2}{3}\right), & 0 < y \le \frac{1}{3} \end{cases}$$

for every $f \in UC(Y_1)$. By a simple calculation we have $\|T(f) - T(g)\| \le 3\|f - g\|$, and hence T is a uniformly continuous (group) isomorphism. On the other hand, T can not be extended to an algebra isomorphism. The reason is that Y_i is the maximal ideal space of $C(Y_i)$ for $i = 1, 2$, and Y_1 and Y_2 are not homeomorphic to each other. Note that by the Arens-Royden theorem [1, 7.4. Corollary, p. 91] we have $\exp C(Y_i) = C(Y_i)^{-1}$ since the first Čech cohomology group on Y_i with integer coefficients vanishes. It follows that $\exp i C_{\mathbb{R}}(Y_i) = UC(Y_i)$ for $i = 1, 2$.

As an application of Theorem 4.3, in the next corollary we determine the general form of surjective isometries between groups of continuous functions that map into the unit circle. In particular, we obtain that if $\exp i C_{\mathbb{R}}(Y_1)$ and $\exp i C_{\mathbb{R}}(Y_2)$ are isometric (group operations need not be considered), then the corresponding C^*-algebras $C(Y_1)$ and $C(Y_2)$ are isometrically isomorphic.

COROLLARY 4.5. *Let Y_j be a compact Hausdorff space and define*

$$d_j(f, g) = \|f - g\|_\infty$$

for $f, g \in \exp i C_{\mathbb{R}}(Y_j)$ where $C_{\mathbb{R}}(Y_j)$ denotes the space of all real-valued continuous functions on Y_j, for $j = 1, 2$. Suppose that $T : \exp i C_{\mathbb{R}}(Y_1) \to \exp i C_{\mathbb{R}}(Y_2)$ is a surjective isometry with respect to d_1 and d_2. Then there exists a homeomorphisms Φ from Y_2 onto Y_1, and a decomposition $Y_2 = Y_{21} \cup Y_{22}$, where Y_{21} and Y_{22} are (possibly empty) clopen subsets of Y_2 with $Y_{21} \cap Y_{22} = \emptyset$ such that

$$T(f)(y) = T(1)(y) \times \begin{cases} f(\Phi(y)), & y \in Y_{21} \\ \overline{f(\Phi(y))}, & y \in Y_{22} \end{cases}$$

for all $f \in \exp i C_{\mathbb{R}}(Y_1)$ and $y \in Y_2$.

PROOF. Let $f, g \in \exp i C_{\mathbb{R}}(Y_j)$. Since $|f| = |g| = 1$ on Y_j we see that

$$\left\| \frac{f}{g} - 1 \right\|_\infty = \|f - g\|_\infty = \left\| \frac{g}{f} - 1 \right\|_\infty ,$$

so

$$d_j(f, g) = \max \left\{ \left\| \frac{f}{g} - 1 \right\|_\infty , \left\| \frac{g}{f} - 1 \right\|_\infty \right\}$$

for $j = 1, 2$. Then by Theorem 4.3, for the surjective isometry $T_0(\cdot) = (T(1))^{-1} T(\cdot)$ we have

$$T_0(fg) = T_0(f) T_0(g)$$

for every pair $f, g \in \exp i C_{\mathbb{R}}(Y_1)$. We will show that

$$T_0(\lambda)(Y_2) \subset \{\lambda, \overline{\lambda}\}$$

holds for every complex number λ with $|\lambda| = 1$. In $T_0(\lambda)$ the symbol λ denotes a constant function. We hope that it will cause no misunderstanding that λ means sometimes a complex number and sometimes a constant function. Let $\lambda_0 = \frac{-1 + \sqrt{3}i}{2}$. Then

$$(T_0(\lambda_0))^3 = T_0(\lambda_0^3) = T_0(1) = 1 ,$$

from which we deduce

$$T_0(\lambda_0)(Y_2) \subset \{1, \lambda_0, \overline{\lambda_0}\} .$$

Let $u \in \exp i C_{\mathbb{R}}(Y_1)$ with $T_0(u) = -1$. Such a u exists since T_0 is surjective. Then we have

$$T_0(u^2) = (T_0(u))^2 = 1$$

implying that $u^2 = 1$ for T_0 is injective. It follows that $u(Y_1) \subset \{-1, 1\}$. Hence

$$2 > \|u - \lambda_0\|_\infty = \|T_0(u) - T_0(\lambda_0)\|_\infty = \| - 1 - T_0(\lambda_0)\|_\infty$$

which implies that $T_0(\lambda_0)(Y_2) \subset \{\lambda_0, \overline{\lambda_0}\}$. Since $(T_0(-1))^2 = T_0(1) = 1$, we have $T_0(-1)(Y_2) \subset \{-1, 1\}$. On the other hand,

$$1 = \| - 1 - \lambda_0\|_\infty = \|T_0(-1) - T_0(\lambda_0)\|_\infty$$

and this implies $T_0(-1) = -1$ because $T_0(\lambda_0)(Y_2) \subset \{\lambda_0, \overline{\lambda_0}\}$. Since

$$\|T_0(\lambda) - 1\|_\infty = \|\lambda - 1\|_\infty$$

and

$$\|T_0(\lambda) + 1\|_\infty = \|T_0(\lambda) - T_0(-1)\|_\infty = \|\lambda + 1\|_\infty ,$$

we infer that

$$T_0(\lambda)(Y_2) \subset \{\lambda, \overline{\lambda}\}$$

for any complex number λ with $|\lambda| = 1$.

Let

$$Y_{21} = \{y \in Y_2 : T_0(i)(y) = i\}, \quad Y_{22} = \{y \in Y_2 : T_0(i)(y) = -i\}.$$

We have $Y_2 = Y_{21} \cup Y_{22}$ for $T(i)(Y_2) \subset \{-i, i\}$. Since $T_0(i)$ is continuous on Y_2, Y_{21} and Y_{22} are clopen subsets of Y_2. We will show that

$$T_0(\lambda) = \lambda \text{ on } Y_{21}, \quad T_0(\lambda) = \overline{\lambda} \text{ on } Y_{22}$$

holds for every complex number λ with $|\lambda| = 1$, $\lambda \neq \pm 1$. Suppose that the imaginary part of λ is greater than 0. Since

$$\|\lambda - i\|_\infty = \|T_0(\lambda) - T_0(i)\|_\infty$$

and $T_0(\lambda)(Y_2) \subset \{\lambda, \overline{\lambda}\}$, we see that $T_0(\lambda) = \lambda$ on Y_{21} and $T_0(\lambda) = \overline{\lambda}$ on Y_{22} by the definitions of Y_{21} and Y_{22} respectively. Next suppose that the imaginary part of λ is negative. Since the imaginary part of $\overline{\lambda}$ is positive, $T_0(\overline{\lambda}) = \overline{\lambda}$ on Y_{21} and $T_0(\overline{\lambda}) = \lambda$ on Y_{22}. From $T_0(\overline{\lambda})T_0(\lambda) = T_0(1) = 1$ we infer that $T_0(\lambda) = \lambda$ on Y_{21} and $T_0(\lambda) = \overline{\lambda}$ on Y_{22}.

Let $\widetilde{T}_0 : \exp i C_{\mathbb{R}}(Y_1) \to \exp i C_{\mathbb{R}}(Y_2)$ be defined by

$$\widetilde{T}_0(\exp iu) = \begin{cases} T_0(\exp iu) \text{ on } Y_{21}, \\ \overline{T_0(\exp iu)} \text{ on } Y_{22}. \end{cases}$$

Since

$$\|\widetilde{T}_0(\exp iu) - \widetilde{T}_0(\exp iv)\|_\infty = \|T_0(\exp iu) - T_0(\exp iv)\|_\infty,$$

it follows that $\widetilde{T}_0$ is an isometrical group isomorphism from $(\exp i C_{\mathbb{R}}(Y_1), d_1)$ onto $(\exp i C_{\mathbb{R}}(Y_2), d_2)$ and $\widetilde{T}_0(\lambda) = \lambda$ for every complex number λ with $|\lambda| = 1$. In addition, $\widetilde{T}_0(\exp iu)(Y_2) \subset (\exp iu)(Y_1)$ for every $u \in C_{\mathbb{R}}(Y_1)$. Indeed, suppose that $\lambda \in \widetilde{T}_0(\exp iu)(Y_2) \setminus (\exp iu)(Y_1)$. Then, using $\widetilde{T}_0(-\lambda) = -\lambda$, we have

$$2 > \|\exp iu - (-\lambda)\|_\infty = \|\widetilde{T}_0(\exp iu) - (-\lambda)\|_\infty = 2$$

which is a contradiction.

Let $x \in Y_1$ and define

$$A = \left\{u \in C_{\mathbb{R}}(Y_1) : u(x) = 0, 0 \leq u \leq \frac{\pi}{2} \text{ on } Y_1\right\}.$$

We assert that

$$\bigcap_{u \in A} \left(\widetilde{T}_0(\exp iu)\right)^{-1}(1) \neq \emptyset.$$

To see this, suppose on the contrary that $\bigcap_{u \in A} \left(\widetilde{T}_0(\exp iu) \right)^{-1}(1) = \emptyset$. By the finite intersection property there exists a finite number of functions $u_1, \ldots, u_n \in A$ such that

$$\bigcap_{k=1}^{n} \left(\widetilde{T}_0(\exp iu_k) \right)^{-1}(1) = \emptyset.$$

Let

$$f = \prod_{k=1}^{n} \exp i \frac{u_k}{N}$$

for a positive integer $N > n$. Then we have

$$f(Y_1) \subset \left\{ z \in \mathbb{C} : |z| = 1, 0 \le \arg z \le \frac{\pi}{2} \right\},$$

and $f(x) = 1$. Thus

$$2 = \|f - (-1)\|_\infty = \|\widetilde{T}_0(f) - \widetilde{T}_0(-1)\|_\infty = \|\widetilde{T}_0(f) - (-1)\|_\infty,$$

so there exists $y \in Y_2$ with $\widetilde{T}_0(f)(y) = 1$. Since $\widetilde{T}_0$ is multiplicative we have

$$\widetilde{T}_0(f) = \prod_{k=1}^{n} \widetilde{T}_0 \left(\exp i \frac{u_k}{N} \right).$$

Clearly,

$$\left(\exp i \frac{u_k}{N} \right)(Y_1) \subset \left\{ z \in \mathbb{C} : |z| = 1, 0 \le \arg z \le \frac{\pi}{2N} \right\}$$

implying

$$\left(\widetilde{T}_0 \left(\exp i \frac{u_k}{N} \right) \right)(Y_2) \subset \left\{ z \in \mathbb{C} : |z| = 1, 0 \le \arg z \le \frac{\pi}{2N} \right\}.$$

Hence we necessarily have

$$\left(\widetilde{T}_0 \left(\exp i \frac{u_k}{N} \right) \right)(y) = 1 \quad 1 \le \forall k \le n,$$

so that

$$\left(\widetilde{T}_0(\exp iu_k) \right)(y) = 1 \quad 1 \le \forall k \le n$$

which contradicts to

$$\bigcap_{k=1}^{n} \left(\widetilde{T}_0(\exp iu_k) \right)^{-1}(1) = \emptyset.$$

Consequently, we have proved that

$$\bigcap_{u \in A} \left(\widetilde{T}_0(\exp iu) \right)^{-1}(1) \neq \emptyset.$$

Let

$$y_0 \in \bigcap_{u \in A} \left(\widetilde{T}_0(\exp iu) \right)^{-1}(1)$$

be arbitrary. Suppose $v \in C_{\mathbb{R}}(Y_1)$ with $v(x) = 0$, and $|v| \leq \frac{\pi}{2}$ on Y_1. Let $v = v_+ - v_-$, where

$$v_+(t) = \max\{v(t), 0\}, \quad v_-(t) = \max\{-v(t), 0\}.$$

Then $v_+, v_- \in A$, so

$$\left(\widetilde{T}_0(\exp iv) \right)(y_0) = \left(\widetilde{T}_0(\exp iv_+) \right)(y_0) \left(\widetilde{T}_0(\exp(-iv_-)) \right)(y_0) = 1.$$

Suppose $w \in C_{\mathbb{R}}(Y_1)$ with $w(x) = 0$. Then for sufficiently large integer N with $\left| \frac{w}{N} \right| \leq \frac{\pi}{2}$ on Y_1 we have

$$\left(\widetilde{T}_0 \left(\exp i \frac{w}{N} \right) \right)(y_0) = 1$$

and this implies that

$$\left(\widetilde{T}_0(\exp iw) \right)(y_0) = \left(\left(\widetilde{T}_0(\exp i \frac{w}{N}) \right)(y_0) \right)^N = 1$$

since $\widetilde{T}_0$ is multiplicative. Therefore, we obtain that

$$y_0 \in \bigcap_{w \in C_{\mathbb{R}}(Y_1), w(x)=0} \left(\widetilde{T}_0(\exp iw) \right)^{-1}(1).$$

We next show that in fact we have

$$\{y_0\} = \bigcap_{w \in C_{\mathbb{R}}(Y_1), w(x)=0} \left(\widetilde{T}_0(\exp iw) \right)^{-1}(1).$$

Suppose that at least two points y_1 and y_2 are in the set on the right hand side. Then $g(y_1) \neq g(y_2)$ for some $g \in \exp iC_{\mathbb{R}}(Y_2)$ and by the surjectivity of $\widetilde{T}_0$ there is $w \in C_{\mathbb{R}}(Y_1)$ with $g = \widetilde{T}_0(\exp iw)$. Since

$$g(y_j) \exp(-iw(x)) = \widetilde{T}_0(\exp iw)(y_j) \widetilde{T}_0(\exp(-iw(x)))(y_j)$$
$$= \left(\widetilde{T}_0(\exp i(w - w(x))) \right)(y_j) = 1$$

holds for $j = 1, 2$, it follows that $g(y_1) = g(y_2)$ which is a contradiction. This proves that

$$\{y_0\} = \bigcap_{w \in C_{\mathbb{R}}(Y_1), w(x)=0} \left(\widetilde{T}_0(\exp i w)\right)^{-1}(1).$$

We define a map $\Psi : Y_1 \to Y_2$ by $\Psi(x) = y_0$ for arbitrary $x \in Y_1$. For any $x \in Y_1$ and $u \in C_{\mathbb{R}}(Y_1)$ we have

$$\left(\widetilde{T}_0(\exp iu)\right)(\Psi(x))\exp(-iu(x)) = \widetilde{T}_0(\exp i(u - u(x)))(\Psi(x)) = 1$$

which implies that

$$\left(\widetilde{T}_0(\exp iu)\right)(\Psi(x)) = (\exp iu)(x).$$

By a routine argument we see that Ψ is a continuous map from Y_1 into Y_2.

Applying a similar argument for $\widetilde{T}_0^{-1}$ instead of $\widetilde{T}_0$ we can prove that there exists a continuous map Φ from Y_2 into Y_1 such that

$$\left(\widetilde{T}_0^{-1}(\exp iv)\right)(\Phi(y)) = (\exp iv)(y)$$

holds for every $y \in Y_2$ and $v \in C_{\mathbb{R}}(Y_2)$. Thus for every $u \in C_{\mathbb{R}}(Y_1)$ we have that

$$(\exp iu)(\Phi(\Psi(x))) = \widetilde{T}_0^{-1}\left(\widetilde{T}_0(\exp iu)\right)(\Phi(\Psi(x)))$$
$$= \left(\widetilde{T}_0(\exp iu)\right)(\Psi(x)) = (\exp iu)(x)$$

holds for every $x \in Y_1$. Since $\exp i C_{\mathbb{R}}(Y_1)$ separates the points of Y_1 we see that

$$(\Phi \circ \Psi)(x) = x$$

for every $x \in Y_1$. In a similar way we see that $(\Psi \circ \Phi)(y) = y$ holds for every $y \in Y_2$. It follows that Φ is a homeomorphism from Y_2 onto Y_1. Considering the definitions of $\widetilde{T}_0$ and T_0 we conclude that

$$(T(\exp iu))(y) = T(1)(y) \times \begin{cases} (\exp iu)(\Phi(y)), & y \in Y_{21}, \\ \overline{(\exp iu)(\Phi(y))}, & y \in Y_{22} \end{cases}$$

holds for every $u \in C_{\mathbb{R}}(Y_1)$ and $y \in Y_2$ completing the proof. $\qquad\square$

The following corollary describing the isometries of the n-dimensional torus is an immediate consequence of the previous result.

COROLLARY 4.6. *Let $T : \mathbb{T}^n \to \mathbb{T}^n$ be a surjective isometry with respect to the metric d on $\mathbb{T}^n$ defined by*

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_j - y_j| : 1 \le j \le n\}.$$

Then there exists a bijection $\varepsilon : \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that

$$T(x_1, \ldots, x_n) = T(1, \ldots, 1)(\delta_1(x_{\varepsilon(1)}), \ldots, \delta_n(x_{\varepsilon(n)}))$$

for every $(x_1, \ldots, x_n) \in \mathbb{T}^n$, *where* δ_j *is either the identity or the conjugation on* $\mathbb{T}$ ($j = 1, \ldots, n$).

PROOF. Let $X = \{1, \ldots, n\}$ be with the discrete topology. Then $\mathbb{T}^n$ is identified by $\exp i C_\mathbb{R}(X)$ and the map T transforms to a surjective isometry of $\exp i C_\mathbb{R}(X)$ onto itself. Applying Corollary 4.5 we obtain the conclusion. □

5. Generalization of the Mazur-Ulam theorem and applications

In this section we study a generalization of the Mazur-Ulam theorem for certain metric groups. As is described in the introduction the main point of a proof of the classical Mazur-Ulam theorem is to show that the isometry under consideration preserves the algebraic midpoint $(x + y)/2$ for a given pair of points x and y in the underlying normed space. It can happen that there are several algebraic midpoints of a given pair of points while in other cases there might be no algebraic midpoint at all in the case of general groups. Therefore, we need an alternative approach to prove a generalization of the Mazur-Ulam theorem for isometries between groups. In this section applying results on the preservation of inverted Jordan triple products for isometries between groups obtained in the earlier sections we prove a generalization of the Mazur-Ulam theorem for certain commutative metric groups. Along with the several applications of the result we show in Corollary 5.4 a metric characterization of normed real-linear spaces among commutative metric groups with inverse and translation invariant metrics.

We begin with the following result concerning the algebraic character of isometries between commutative groups.

COROLLARY 5.1. *Let G_1 and G_2 be commutative groups. Suppose that d_j is a metric on G_j which is inverse and translation invariant. Assume that there exists a constant $K > 1$ such that*

$$(5.1) \qquad\qquad d_1(y^2, x^2) \geq K d_1(y, x)$$

holds for all $x, y \in G_1$. Suppose that $T : G_1 \to G_2$ is a surjective isometry. If at least one of G_1 and G_2 is 2-divisible or 2-torsion free, then for the transformation $T_0(\cdot) = (T(e))^{-1} T(\cdot)$ we have

$$T_0(xy) = T_0(x) T_0(y) \quad \forall x, y \in G_1 .$$

In particular, if the isometry T sends the unit to the unit, then it is a group isomorphism.

PROOF. We show that B(a, b) is satisfied for G_1 and C$_1(T(a), T(ba^{-1}b))$ is satisfied for G_2 for every pair $a, b \in G_1$. Indeed, as $d_1(\cdot, \cdot)$ is a metric which is inverse and translation invariant, we have (1) of Definition 3.2. The condition (2) of the same definition holds since $d_1(\cdot, \cdot)$ is a metric. Finally, the inequality (5.1) guarantees that (3) of Definition 3.2 also holds and hence we have that B(a, b) is satisfied for every pair $a, b \in G_1$.

As for the group G_2, having a look at Definition 3.3, it is apparent that $C_1(T(a), T(ba^{-1}b))$ is satisfied for G_2 simply because $d_2(\cdot, \cdot)$ is inverse and translation invariant. By Corollary 3.9 we deduce that

$$T(ba^{-1}b) = T(b)(T(a))^{-1}T(b)$$

holds for every pair $a, b \in G_1$. Applying Lemma 4.1 the conclusion follows. □

Observe that Corollary 5.1 generalizes the famous Mazur-Ulam theorem. Indeed, if G_1, G_2 are normed real-linear spaces and we consider their additive structures with the metrics induced by the norms, then the above corollary trivially applies and shows that for any surjective isometry $T : G_1 \to G_2$, the map $T_0(\cdot) = T(\cdot) - T(0)$ is additive and hence, by continuity, real-linear. Let us point out that the conditions in Corollary 5.1 are fulfilled by a really large class of commutative groups, so the result is a considerable generalization of the Mazur-Ulam theorem.

We also note that Corollary 5.1 removes the hypothesis of uniqueness of 2-divisibility of the groups in Corollary 1 in [13].

In the next corollary we present a result on the extendibility of surjective isometries between additive subgroups of normed linear spaces.

COROLLARY 5.2. *Let $\mathcal{E}_i$ be a normed real-linear space and X_i a subgroup of $\mathcal{E}_i$ (as an additive group) with the metric d_i induced by the norm for $i = 1, 2$. Suppose that $T : X_1 \to X_2$ is a surjective isometry. Then $T(\cdot) - T(0)$ is uniquely extendible to a real-linear isometry from the subspace $\widetilde{X}_1$ of $\mathcal{E}_1$ generated by X_1 onto the subspace $\widetilde{X}_2$ of $\mathcal{E}_2$ generated by X_2.*

PROOF. Applying Corollary 5.1 in the 2-torsion free case, we see that the bijective transformation $T_0(\cdot) = T(\cdot) - T(0)$ from X_1 onto X_2 is additive.

We first extend T_0 to a map $\widetilde{T}_0$ from the linear space over the rational number field generated by X_1 onto the linear space over the rational number field generated by X_2 as follows. Suppose that $\tilde{x} = \sum_{k=1}^{n} r_k x_k$, where r_k's are rational numbers and $x_1, \ldots, x_n \in X_1$. Set

$$\widetilde{T}_0(\tilde{x}) = \sum_{k=1}^{n} r_k T_0(x_k).$$

This transformation is well-defined. Indeed, suppose that $\tilde{x}$ also equals $\sum_{j=1}^{m} s_j y_j$ for some rational numbers $s_1, \ldots, s_m$ and $y_1, \ldots, y_m \in X_1$. Let N be a positive integer such that all $Nr_1, \ldots, Nr_n$ and $Ns_1, \ldots, Ns_m$ are all integers. Then

$$T_0\left(\sum_{k=1}^{n} Nr_k x_k \right) = \sum_{k=1}^{n} Nr_k T_0(x_k) = N \sum_{k=1}^{n} r_k T_0(x_k)$$

and

$$T_0\left(\sum_{j=1}^{m} N s_j y_j\right) = \sum_{j=1}^{m} N s_j T_0(y_j) = N \sum_{j=1}^{m} s_j T_0(y_j)$$

hold since T_0 is additive and

$$N\tilde{x} = \sum_{k=1}^{n} N r_k x_k = \sum_{j=1}^{m} N s_j y_j \in X_1\,.$$

This implies $\sum_{k=1}^{n} r_k T_0(x_k) = \sum_{j=1}^{m} s_j T_0(y_j)$. By definition $\widetilde{T_0}$ is linear (with respect to the rational number field) and surjective. We also see that $\widetilde{T_0}$ is an isometry with respect to the original norms of $\mathcal{E}_1$ and $\mathcal{E}_2$. To show this, suppose that $\tilde{x} = \sum_{k=1}^{n} r_k x_k$, where r_k's are rational numbers and $x_1, \ldots, x_n \in X_1$. Let N be a positive integer such that $N r_1, \ldots, N r_n$ are all integers. Then $N\widetilde{T_0}(\tilde{x}) = T_0(\sum_{k=1}^{n} N r_k x_k)$ and $\|T_0(\sum_{k=1}^{n} N r_k x_k)\|_2 = \|\sum_{k=1}^{n} N r_k x_k\|_1$ implies $\|\widetilde{T_0}(\tilde{x})\|_2 = \|\sum_{k=1}^{n} r_k x_k\|_1 = \|\tilde{x}\|_1$. Since $\widetilde{T_0}$ is linear, we see that it is an isometry.

Now it is a routine argument to extend $\widetilde{T_0}$ further to a surjective isometry from $\widetilde{X_1}$ onto $\widetilde{X_2}$. The uniqueness of the extension of T_0 is obvious.   □

As an application of the previous corollary, in the next example we describe the surjective isometries of the discrete group $\mathbb{Z}^n$.

EXAMPLE 5.3.   Let $T : \mathbb{Z}^n \to \mathbb{Z}^n$ be a surjective isometry with respect to the metric $d(\cdot, \cdot)$ defined by

$$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \max\{|x_j - y_j| : 1 \le j \le n\}$$

for $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \mathbb{Z}^n$. Then there are a bijection

$$\varepsilon : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

and a map

$$\eta : \{1, \ldots, n\} \to \{-1, 1\}$$

such that

$$T(x_1, \ldots, x_n) = (\eta(1)x_{\varepsilon(1)}, \ldots, \eta(n)x_{\varepsilon(n)}) + T(0, \ldots, 0)$$

for every $(x_1, \ldots, x_n) \in \mathbb{Z}^n$.

To see this, first observe that by Corollary 5.2 the transformation $T_0(\cdot) = T(\cdot) - T(0, \ldots, 0)$ can be uniquely extended to a linear isometry of $\mathbb{R}^n$ onto itself. Here $\mathbb{R}^n$ is equipped with the norm $\|(x_1, \ldots, x_n)\| = \max\{|x_j| : 1 \le j \le n\}$. The surjective linear isometries of that space are well-known. In fact, as this space can also be viewed as the space of all real-valued continuous functions on the set $\{1, \ldots, n\}$ with the supremum norm, we can apply Banach-Stone theorem to see that every surjective linear isometry is a composition

operator induced by some permutation of $\{1, \dots, n\}$ followed by multiplication by a fixed function on $\{1, \dots, n\}$ with values in $\{-1, 1\}$. The conclusion of the statement follows.

We conclude the paper with a result that gives a metric characterization of normed real-linear spaces among commutative metric groups with inverse and translation invariant metrics.

COROLLARY 5.4. *Let $\mathcal{E}$ be a normed real-linear space and G a commutative metric group with metric d which is inverse and translation invariant. Suppose that $T : \mathcal{E} \to G$ is a surjective isometry. Then G can be made to a normed real-linear space in a natural way, keeping the original operation of G as addition and the metric induced by the norm. In that setting T is a surjective linear isometry followed by a translation.*

PROOF. Hopefully it causes no confusion if we denote the operation on G by $+$, the same symbol as the one for the addition on $\mathcal{E}$. Applying Corollary 5.1 we obtain that $T_0(\cdot) = T(\cdot) - T(0)$ is a bijective additive map.

We next show that G can be made to a real-linear space in a natural way, keeping the original operation defined on G as the addition for a linear space. Let us define the scalar multiplication for $t \in \mathbb{R}$ and $g \in G$ by $tg = T_0(t(T_0)^{-1}(g))$. It is easy to check that in that way G becomes a real-linear space. Moreover, define $\|g\|_G = \|(T_0)^{-1}(g)\|_{\mathcal{E}}$ for $g \in G$. One can readily verify that this gives us a norm on G which induces the original metric on G. Therefore, T_0 is a surjective and additive isometry from the normed real-linear space $\mathcal{E}$ onto the normed real-linear space G and hence T_0 is necessarily real-linear. This completes the proof. $\square$

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