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Weyl's Theorems and Extensions of Bounded Linear Operators

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Abstract. A bounded operator $T \in L(X)$, X a Banach space, is said to satisfy Weyl's theorem if the set of all spectral points that do not belong to the Weyl spectrum coincides with the set of all isolated points of the spectrum which are eigenvalues and having finite multiplicity. In this article we give sufficient conditions for which Weyl's theorem for an extension $\overline{T}$ of T (respectively, for T) entails that Weyl's theorem holds for T (respectively, for $\overline{T}$).

1. Introduction

Let X be a complex infinite dimensional Banach space and denote by $L(X)$ the algebra of all bounded linear operators on X. Suppose that X is a subspace of another Banach space Y and assume that the embedding of X into Y is continuous, i.e. there is a constant $k > 0$ such that $\|x\|_Y \leq k\|x\|_X$ for all $x \in X$. Let $T \in L(X)$ and denote by $\overline{T} \in L(Y)$ is an extension of T to Y. In general, very few things can be said concerning the relationship between the spectral theory and Fredholm theory of T and $\overline{T}$, see Example 1 and Example 2 of [4]. In [4] it has been observed that the spectral theory and Fredholm theory of T and $\overline{T}$ are almost the same if we assume:

A) *X is dense in Y and $\overline{T}(Y) \subseteq X$.*

In [15] some aspects of Fredholm theory have been studied when we assume:B) *Y is a Hilbert space and T is symmetrizable* (see later for definitions).

In this paper we are mainly concerned with the transmission of Weyl's theorem (or some strong variants of Weyl's theorem) from $\overline{T}$ to T in both cases (A) and (B). We fix first our terminology. Let $T \in L(X)$ and denote by $\alpha(T)$ the dimension of the kernel ker T, and by $\beta(T)$ the codimension of the range $R(T)$. Recall that $T \in L(X)$ is said to be an *upper semi-Fredholm* operator if $\alpha(T) < \infty$ and $R(T)$ is closed, while $T \in L(X)$ is said to be a *lower semi-Fredholm* operators if $\beta(T) < \infty$. The class of all *semi-Fredholm operators* is

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defined by $\Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the class of all Fredholm operators is defined by $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$, where $\Phi_+(X)$ and $\Phi_-(X)$ denote the classes of upper semi-Fredholm operators and lower semi-Fredholm operators, respectively. If $T \in \Phi_\pm(X)$, the *index* of T is defined by $\text{ind}\,(T) := \alpha(T) - \beta(T)$. For a linear operator T the *ascent* $p := p(T)$ is defined as the smallest non-negative integer p such that $\ker T^p = \ker T^{p+1}$. If such integer does not exist then we put $p(T) = \infty$. Analogously, the *descent* $q := q(T)$ is defined as the smallest non-negative integer q such that $R(T^q) = R(T^{q+1})$, and if such integer does not exist then we put $q(T) = \infty$. It is well-known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [14, Proposition 38.3]. Moreover, $\lambda \in \sigma(T)$, $\sigma(T)$ the spectrum of T, is a pole of the resolvent precisely when $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$, see Proposition 50.2 of [14].

Two important classes of operators in Fredholm theory are the class of all *upper semi-Browder operators* defined by

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of all *lower semi-Browder operators* defined by

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of all *Browder operators* (known in the literature also as *Riesz-Schauder operators*) is defined by $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. The set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \text{ind}\, T = 0\}.$$

Define

$$W_+(X) := \{T \in \Phi_+(X) : \text{ind}\, T \leq 0\},$$

and

$$W_-(X) := \{T \in \Phi_-(X) : \text{ind}\, T \geq 0\}.$$

Clearly $W(X) = W_+(X) \cap W_-(X)$. The classes of operators defined above motivate the definition of the following spectra. The *upper semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{ub}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{lb}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

while the *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{b}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

Finally, the *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\text{w}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not Weyl}\}.$$

Note that $\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T)$ for every $T \in L(X)$. Let $\sigma_{\mathrm{a}}(T)$ denote the classical *approximate point spectrum* of T defined as

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not bounded below}\},$$

where $T \in L(X)$ is said to be bounded below if T is injective and has closed range. For a bounded operator $T \in L(X)$, if we write iso K for the set of all isolated points of $K \subseteq \mathbf{C}$, set

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Following Coburn [10], we say that *Weyl's theorem holds* for $T \in L(X)$ if

(1) $$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T).$$

Define

$$\pi_{00}^{a}(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Following Rakočević [19], we shall say that *a-Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}^{a}(T),$$

where $\sigma_{\mathrm{wa}}(T) = \cap_{K;\mathrm{compact}} \sigma_{\mathrm{a}}(T+K)$, the Weyl approximate point spectrum of T. Note that

$$a\text{- Weyl's theorem} \Rightarrow \text{Weyl's theorem},$$

see for instance [1, Chapter 3]. A weaker variant of Weyl's theorem is given by Browder's theorem: $T \in L(X)$ is said to satisfy *Browder's theorem* if $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$.

The concept of semi-Fredholm operator has been generalized by Berkani ([7], [9]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$ viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm* (resp. *B-Fredholm, upper semi B-Fredholm*, *lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator with $\operatorname{ind} T_{[n]} = \operatorname{ind} T_{[m]}$ for all $m \geq n$ ([9]). This enables one to define the index of a semi B-Fredholm as $\operatorname{ind} T = \operatorname{ind} T_{[n]}$.

An operator $T \in L(X)$ is said to be *B-Weyl* if, for some integer $n \geq 0$, $T^n(X)$ is closed and $T_{[n]}$ is Weyl. The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbf{C} : \lambda I - T \text{ is not } B\text{-Weyl}\}.$$

According Berkani and Koliha ([8]) an operator $T \in L(X)$ is said to satisfy *generalized Weyl's theorem*, if $\sigma(T) \setminus \sigma_{\mathrm{bw}}(T) = E(T)$, where

$$E(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T)\}.$$

In general we have

$$\text{generalized Weyl's theorem} \Rightarrow \text{Weyl's theorem}.$$

We now introduce a basic concept which has an important role in local spectral theory: $T \in L(X)$ is said to have *the single valued extension property* at $\lambda_0 \in \mathbf{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have the SVEP if T has the SVEP at every point $\lambda \in \mathbf{C}$.

Evidently, an operator $T \in L(X)$ has the SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$. The identity theorem for analytic function ensures that every $T \in L(X)$ has the SVEP at the points of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, every operator has the SVEP at every isolated point of the spectrum. It should be noted that if T is a normal operator on a Hilbert space, in particular if T is a selfadjoint operator, then both T and T^* have SVEP (in fact T is decomposable, see [17]).

The *quasi-nilpotent part* of T is defined as the set

$$H_0(T) := \left\{ x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0 \right\}.$$

Clearly, $\ker T^n \subseteq H_0(T)$ for every $n \in \mathbf{N}$.

THEOREM 1.1 [1, Theorem 3.74]. *Let $T \in L(X)$ and suppose that $\lambda_0 \in iso\,\sigma(T)$. If P_0 is the canonical spectral projection associated with λ_0 then $R(P_0) = H_0(\lambda_0 I - T)$. Furthermore, if λ_0 is a pole of order p then $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)^p$.*

Let $p_{00}(T) := \sigma(T) \setminus \sigma_b(T)$ denote the set of all spectral points λ for which $\lambda I - T \in \mathcal{B}(X)$. For operators having SVEP, Weyl's theorem may be characterized as follows:

THEOREM 1.2 ([2]). *Suppose that either T or T^* has* SVEP. *Then T satisfies Weyl's theorem if and only if $p_{00}(T) = \pi_{00}(T)$.*

If $T \in L(X)$, the *analytic core* $K(T)$ is the set of all $x \in X$ such that there exists a constant $c > 0$ and a sequence of elements $x_n \in X$ such that $x_0 = x, Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbf{N}$. Note that $T(K(T)) = K(T)$, see [1].

The condition $p_{00}(T) = \pi_{00}(T)$ may be characterized as follows:

LEMMA 1.3 . *$T \in L(X)$ satisfies $p_{00}(T) = \pi_{00}(T)$ if and only if for all $\lambda \in \pi_{00}(T)$ there exists $p := p(\lambda) \in \mathbf{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$.*

PROOF. Suppose that for $\lambda \in \pi_{00}(T)$ there exists $p := p(\lambda) \in \mathbf{N}$ such that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. Since every $\lambda \in \pi_{00}(T)$ is isolated in $\sigma(T)$ then, by [1, Theorem 3.74],

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

from which we obtain

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T),$$

so $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$ which implies, by [1, Theorem 3.6], that $p(\lambda I - T) = q(\lambda I - T) \leq p$. By definition of $\pi_{00}(T)$ we know that $\alpha(\lambda I - T) < \infty$ and this implies by Theorem 3.4 of [1] that $\beta(\lambda I - T)$ is also finite. Since $\lambda I - T$ has both ascent and descent finite then $\lambda I - T$ is Browder, see [1, Theorem 3.4]. Therefore $\lambda \in p_{00}(T)$ and hence $\pi_{00}(T) \subseteq p_{00}(T)$. Since the opposite inclusion holds for every operator we then conclude that $p_{00}(T) = \pi_{00}(T)$.

Conversely, if $p_{00}(T) = \pi_{00}(T)$ and $\lambda \in \pi_{00}(T)$ then $p := p(\lambda I - T) = q(\lambda I - T) < \infty$. By Theorem 3.16 of [1] it then follows that $H_0(\lambda I - T) = \ker(\lambda I - T)^p$. ∎

2. Extensions

As observed in the introduction the spectrum, and the Weyl spectrum, for $T \in L(X)$ and an extension $\overline{T} \in L(Y)$ may be completely different. An illuminating example is given by the following operator.

EXAMPLE 0.1 Let C denote the Cesàro matrix. C is a lower triangular matrix such that the nonzero entries of the n-th row are n^{-1} $(n \in \mathbf{N})$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1/2 & 1/2 & 0 & 0 & \dots \\ 1/3 & 1/3 & 1/3 & 0 & \dots \\ 1/4 & 1/4 & 1/4 & 1/4 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}.$$

Let $1 < p < \infty$ and consider the matrix C as an operator C_p acting on ℓ_p. Let q be such that $1/p + 1/q = 1$. In [20] it has been proved that $\sigma(C_p)$ is the closed disc Γ_q, where

$$\Gamma_q := \{\lambda \in \mathbf{C} : |\lambda - q/2| \le q/2\}.$$

Moreover, it has been proved in [13] that for each $\mu \in \operatorname{int} \Gamma_q$ the operator $\mu I - C_p$ is an injective Fredholm operator with $\beta(C_p) = 1$. Hence $\operatorname{int} \Gamma_q \subseteq \sigma_{\mathrm{w}}(C_p)$. Actually, we have $\Gamma_q = \sigma_{\mathrm{w}}(C_p)$. In fact, let λ be in the boundary of Γ_q and suppose that $\lambda I - C_p$ is Weyl. Since every operator, as well as its dual, has SVEP at the points of the boundary the spectrum then $p(\lambda I - C_p) = q(\lambda I - C_p) < \infty$, see [1, Theorem 3.16 and Theorem 3.17], hence λ is an isolated point of $\sigma(C_p)$ and this is impossible. Therefore, $\lambda \in \sigma_{\mathrm{w}}(C_p)$.

Now, choose $1 < p' < p < \infty$ and let q' be such that $1/p' + 1/q' = 1$. The $\ell_{p'}$ is continuously embedded and dense in ℓ_p. Let $C_{p'} : \ell_{p'} \to \ell_{p'}$ be the operator induced by the matrix C. The extension of $C_{p'}$ to ℓ_p is the operator C_p and, as above, $\sigma(C_{p'}) = \sigma_{\mathrm{w}}(C_{q'}) = \Gamma_{q'}$, with $q' \neq q$.

In the the sequel of this section we always assume that X and Y are Banach spaces with X a proper subspace of Y. Suppose that $T \in L(X)$ admits an extension $\overline{T} \in L(Y)$ and set

$$\mathcal{M}(X) := \{T \in L(X) : \overline{T}(Y) \subseteq X\}.$$

It is easily seen that $\mathcal{M}(X)$ is a left ideal of $L(X)$, i.e., if $T \in \mathcal{M}$ and $S \in L(X)$ then $ST \in \mathcal{M}(X)$. If $T \in \mathcal{M}(X)$, $\sigma(T)$ and $\sigma(\overline{T})$ may differ only in 0. Precisely, we have:

THEOREM 2.1. *If* $T \in \mathcal{M}(X)$ *then*

(i) $\ker(\lambda I - \overline{T}) = \ker(\lambda I - T)$ *for all* $\lambda \neq 0$.

(ii) $\sigma(T) \setminus \{0\} = \sigma(\overline{T}) \setminus \{0\}$.

(iii) $\sigma_{\mathrm{w}}(T) \setminus \{0\} = \sigma_{\mathrm{w}}(\overline{T}) \setminus \{0\}$.

(iv) $\sigma_{\mathrm{b}}(T) \setminus \{0\} = \sigma_{\mathrm{b}}(\overline{T}) \setminus \{0\}$.

PROOF. To show (i), note first that $\ker(\lambda I - T) = \ker(\lambda I - \overline{T}) \cap X$ for all $\lambda \in \mathbf{C}$. Suppose that $\lambda \neq 0$ and $y \in \ker(\lambda I - \overline{T})$. Then $y = \frac{1}{\lambda}\overline{T}y \in \overline{T}(Y) \subset X$, which proves the assertion (i). A direct proof of the assertions (ii) and (iii) can be found in [4], but it is possible to prove these by using an argument of [5]. Let $S \in L(X, Y)$ denote the canonical embedding of X into Y and define $R \in L(Y, X)$ by $Ry := \overline{T}y$ for all $y \in Y$. Then $T = RS$ and $\overline{T} = SR$, and hence the assertions (ii) and (iii) follows from [5, Theorem 6], while (iv) follows from [5, Theorem 6 and Proposition 10]. ∎

REMARK 2.2. Note that since X is dense in Y then $T \in \mathcal{M}(X)$ if and only if there exists $c > 0$ such that $\|Tx\|_X \leq c\|x\|_Y$ for all $x \in X$ ([4]).

REMARK 2.3. Since in the notation of the proof of Theorem 2.1 we have $T = RS$ and $\overline{T} = SR$, then $\overline{T}$ has SVEP if and only if T has SVEP, see [6, Proposition 2.1].

THEOREM 2.4. *Suppose that* X *is dense in* Y *and* $T \in \mathcal{M}(X)$. *Then* $0 \in \sigma_{\mathrm{w}}(T) \cap \sigma_{\mathrm{w}}(\overline{T}) \subseteq \sigma(T) \cap \sigma(\overline{T})$. *Consequently,* $\sigma(T) = \sigma(\overline{T})$ *and* $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(\overline{T})$.

PROOF. Suppose that $0 \notin \sigma_{\mathrm{w}}(\overline{T})$. Then $\overline{T} \in \Phi(Y)$, so $\overline{T}(Y)$ has finite codimension in Y and hence has finite codimension in X. Therefore there exists a finite-dimensional subspace Z such that $X = \overline{T}(Y) \oplus Z$. But $\overline{T}(Y)$ is closed in Y, hence X is a closed subspace of Y. Since X is assumed to be dense in Y, it then follows that $X = \overline{X} = Y$, contradicting our assumption that X is a proper subspace of Y.

Suppose now that $0 \notin \sigma_{\mathrm{w}}(T)$. Then $T \in W(X)$, hence there exists an invertible operator $U \in L(X)$ and a finite dimensional operator $K \in L(X)$ such that $T = U - K \in \mathcal{M}(X)$, see [1, Theorem 3.39]. From this we obtain

$$U^{-1}(U - K) = I - U^{-1}K = I - K_0 \in \mathcal{M}(X),$$

where $K_0 := U^{-1}K$ is a finite dimensional operator. By Remark 2.2 then $\ker K_0$ is closed in Y. Since $\ker K_0$ has finite codimension in X, then there is a finite dimensional subspace N such that $X = \ker K_0 \oplus N$. Therefore X is closed in Y and this implies $X = Y$, again contradicting the assumption that X is a proper subspace of Y.

The last assertion is clear by Theorem 2.1. ∎

COROLLARY 2.5. *Suppose that* X *is dense in* Y *and* $T \in \mathcal{M}(X)$. *Then* T *satisfies Browder's theorem if and only if* $\overline{T}$ *satisfies Browder's theorem.*

PROOF. By Theorem 2.4 we have $0 \in \sigma_{\mathrm{w}}(T) \cap \sigma_{\mathrm{w}}(\overline{T}) \subseteq \sigma_{\mathrm{b}}(T) \cap \sigma_{\mathrm{b}}(\overline{T})$. Therefore, by Theorem 2.1, $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{b}}(\overline{T})$, and hence $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$ if and only if $\sigma_{\mathrm{w}}(\overline{T}) = \sigma_{\mathrm{b}}(\overline{T})$. ∎

The equivalence of Weyl's theorem for T and $\overline{T}$ requires a very special condition on the range of T.

THEOREM 2.6. *Suppose that X is dense in Y, $T \in \mathcal{M}(X)$ and $T(X)$ is closed in X. Then $\overline{T}$ satisfies Weyl's theorem if and only if T satisfies Weyl's theorem. In particular, this equivalence holds if $\beta(T) < \infty$.*

PROOF. Suppose that Weyl's theorem holds for $\overline{T}$. By Theorem 2.4 then $0 \notin \sigma(\overline{T}) \setminus \sigma_{\mathrm{w}}(\overline{T}) = \pi_{00}(\overline{T})$. If $\lambda \in \pi_{00}(\overline{T})$ then $\lambda \neq 0$ so, by part (i) of Theorem 2.1, we have $\alpha(\lambda I - T) = \alpha(\lambda I - \overline{T})$. Since $\lambda \in \operatorname{iso} \sigma(\overline{T}) = \operatorname{iso} \sigma(T)$, it then follows that $\lambda \in \pi_{00}(T)$. Therefore, $\pi_{00}(\overline{T}) \subseteq \pi_{00}(T)$.

We now show that also the reverse inclusion holds. We claim that $\alpha(T) = \infty$. To see this, suppose $\alpha(T) < \infty$. Then $\ker T$ is complemented, since it is finite-dimensional, so there exists a closed subspace M of X such that $X = \ker T \oplus M$. The restriction $T|M : M \to T(X)$ admits an inverse $(T|M)^{-1}$. Define $V \in L(X)$ by

$$V : x \in X \to (T|M)^{-1} T x \in X .$$

Clearly, $V(\ker T) = \{0\}$ and $Vm = m$ for all $m \in M$. Consequently, $I - V$ is finite dimensional. We show that $V \in \mathcal{M}(X)$. Since $T \in \mathcal{M}(X)$ there exists $c > 0$ such that $\|Tx\|_X \leq c\|x\|_X$. Therefore,

$$\begin{aligned}\|Vx\|_X &= \|(T|M)^{-1}Tx\|_X \leq \|(T|M)^{-1}\|\|Tx\|_X \\ &\leq c\|(T|M)^{-1}\|\|x\|_Y ,\end{aligned}$$

from which we conclude that $V \in \mathcal{M}(X)$. Now, $\ker(I - V)$ is closed in Y. Indeed, let (x_n) be a sequence of elements of $\ker(I - V) \subset X$ such that $\|x_n - x_0\|_Y \to 0$ for some $x_0 \in Y$. Then

$$\|x_n - x_m\|_X = \|V(x_n - x_m)\|_X \leq c\|(T|M)^{-1}\|\|x - x - x_m\|_Y \to 0 ,$$

so (x_n) is a Cauchy sequence in X. Since X is a Banach space then there exists $z \in X$ such that $\|x_n - z\|_X \to 0$. Therefore, for some $c' > 0$ we have $\|x_n - z\|_Y \leq c'\|x_n - z\|_X \to 0$ and $z = x_0$. Consequently, $\|x_n - x_0\|_X \to 0$ which shows that $\ker(I - V)$ is closed in Y, as desired.

Since $I - V$ is finite dimensional, we have $X = \ker(I - V) \oplus N$, with N finite dimensional, and hence X is closed in Y. Hence $X = \overline{X} = Y$, a contradiction.

Therefore, $\alpha(T) = \infty$ and hence $0 \notin \pi_{00}(T)$. Consequently, by part (i) of Theorem 2.1, we have $\alpha(\lambda I - T) = \alpha(\lambda I - \overline{T})$ for all $\lambda \in \pi_{00}(T)$, so $\pi_{00}(T) \subseteq \pi_{00}(\overline{T})$. Therefore, $\pi_{00}(T) = \pi_{00}(\overline{T})$. Finally

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma(\overline{T}) \setminus \sigma_{\mathrm{w}}(\overline{T}) = \pi_{00}(\overline{T}) = \pi_{00}(T) ,$$

so T satisfies Weyl's theorem.

Suppose that T satisfies Weyl's theorem. Then $0 \notin \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$ and hence $\pi_{00}(T) \subseteq \pi_{00}(\overline{T})$. Suppose that $\alpha(\overline{T}) < \infty$. Then $\alpha(T) < \infty$ and as it has been proved before this is impossible. Therefore, $\alpha(\overline{T}) = \infty$ and hence $0 \notin \pi_{00}(\overline{T})$. As above, it then follows that $\pi_{00}(T) = \pi_{00}(\overline{T})$ and hence $\sigma(\overline{T}) \setminus \sigma_{\mathrm{w}}(\overline{T}) = \pi_{00}(\overline{T})$.

The last assertion is obvious: every finite codimensional subspace is closed. ∎

In the next corollary we consider the case that X is a dense subspace of a Hilbert space.

COROLLARY 2.7. *Suppose that X is dense in a Hilbert space H and let $T \in \mathcal{M}(X)$ be such that $T(X)$ is closed in X. If $\overline{T}$ is self-adjoint then T satisfies a-Weyl's theorem.*

PROOF. If $\overline{T}$ is self-adjoint then $\overline{T}$ is decomposable (see [17] for definition and basic results), hence the dual $\overline{T}^*$ (or equivalently, the Hilbert adjoint of $\overline{T}$) has SVEP. In the notation of the proof of Theorem 2.1 we have $\overline{T}^* = R^*S^*$ and $T^* = S^*R^*$, and this implies, again by [6, Proposition 2.1], that also T^* has SVEP. By Theorem 2.6 we know that T satisfies Weyl's theorem and the SVEP of T^* implies that also a-Weyl's theorem holds for T. ∎

Note that instead of assuming that $\overline{T}$ is self-adjoint we can assume that $\overline{T}$ is *generalized scalar*, see [17, p. 44 and §1.5] for definition and basic results. Indeed, every generalized scalar operator is decomposable and hence its dual has SVEP, so the argument of Corollary 2.7 still works.

A bounded operator $T \in L(X)$ is said to be *polaroid* if every isolated point λ of the $\sigma(T)$ is a pole of the resolvent, i.e., $p(\lambda I - T) = q(\lambda I - T) < \infty$.

THEOREM 2.8. *Suppose that X is dense in Y and $T \in \mathcal{M}(X)$.*

(i) *If T is polaroid and $\overline{T}$ satisfies Weyl's theorem then T satisfies generalized Weyl's theorem.*

(ii) *If $\overline{T}$ is polaroid and T satisfies Weyl's theorem then $\overline{T}$ satisfies generalized Weyl's theorem.*

PROOF. (i) Proceeding as in the first part of the proof of Theorem 2.6 we see that $\pi_{00}(\overline{T}) \subseteq \pi_{00}(T)$. The polaroid condition on T entails that $0 \notin \pi_{00}(T)$. Indeed if $0 \in \pi_{00}(T)$ then 0 is a pole of the resolvent and hence $p(T) = q(T) < \infty$. By definition of $\pi_{00}(T)$ we also have $\alpha(T) < \infty$, so $\beta(T) = \alpha(T)$, hence $0 \notin \sigma_{\mathrm{w}}(T)$, which is impossible by Theorem 2.4. By part (i) of Theorem 2.1 we then conclude that $\pi_{00}(T) \subseteq \pi_{00}(\overline{T})$. Therefore, $\pi_{00}(T) = \pi_{00}(\overline{T})$ and as in the proof of Theorem 2.6 this implies that T satisfies Weyl's theorem. Since T is polaroid, then T satisfies generalized Weyl's theorem, see [3, Theorem 3.7].

(ii) The proof is analogous to that of part (i). ∎

3. Symmetrizable operators

In this section we are concerned with proving that Weyl's theorem holds for symmetrizable operators. Indeed, in this very special situation the assumptions of Corollary 2.7 can be simplified. To see this, assume that the Banach space X is a subspace of a H a Hilbert space and assume that the embedding of X into H is continuous and X dense in H. Following Lax [15] $T \in L(X)$ is said to be *symmetrizable* if T is symmetric with respect to the inner product $\langle , \rangle$ induced by H on X, i.e.,

$$\langle Tx, y\rangle = \langle x, Ty\rangle \quad \text{for all } \ x, y \in X .$$

Note that every quasi-hermitian operator in the sense of Dieudonné [11] is symmetrizable. Applications of symmetrizable operators to partial differential equations may be found in Lax [15] and Gokheberg and Zambitsky [12].

The following important properties of symmetrizable operators $T \in L(X)$ may be found in [15]:

(a) T is bounded with respect to the Hilbert norm. Moreover, the natural extension $\overline{T}$ of T to H is a bounded self-adjoint operator.

(b) $\sigma(\overline{T}) \subseteq \sigma(T)$. Clearly, since $\overline{T}$ is a self-adjoint operator then $\sigma(\overline{T}) \subset \mathbf{R}$. This inclusion may be strict, since $\sigma(T)$ may contain non-real points, see in [18, Example 1, §4].

(c) If $\lambda I - T \in W(X)$ then $\lambda I - \overline{T} \in W(H)$. In this case $\ker(\lambda I - T) = \ker(\lambda I - \overline{T})$.

LEMMA 3.1 *If T is symmetrizable and λ_0 is an eigenvalue of T then $\lambda_0 \in \mathbf{R}$. Furthermore, if λ_0 is an isolated eigenvalue of T then λ_0 is an isolated eigenvalue of $\overline{T}$.*

PROOF. Clearly, every eigenvalue λ of T is an eigenvalue of $\overline{T}$. Since $\overline{T}$ is self-adjoint then $\lambda \in \mathbf{R}$. If λ_0 is an isolated eigenvalue of T then there exists a punctured open disc $\mathbf{D}_0$ centered at λ_0 such that $\lambda \notin \sigma(T)$ of all $\lambda \in \mathbf{D} \setminus \{\lambda_0\}$, and hence by (b) we have $\lambda \notin \sigma(\overline{T})$, from which we deduce that λ_0 is an isolated eigenvalue of $\overline{T}$. ∎

LEMMA 3.2. *Every symmetrizable operator T has* SVEP.

PROOF. Observe first that since $\overline{T}$ is self-adjoint then $\overline{T}$ has SVEP. This entails that also T has SVEP. In fact, let $\lambda \in \mathbf{C}$ be arbitrary given. Since every analytic function $f : U \to X$, defined on an open disc U centered at λ remains analytic when considered as a function from U to H, it is clear that T inherits the SVEP at λ.

THEOREM 3.3. *If $T \in L(X)$ is symmetrizable then Weyl's theorem holds for T.*

PROOF. Since T has SVEP, by Theorem 1.2 and Lemma 1.3 it suffices to prove that $H_0(\lambda I - T) = \ker(\lambda I - T)$ for all $\lambda \in \pi_{00}(T)$.

For this suppose that $\lambda_0 \in \pi_{00}(T)$. Then λ_0 is an isolated eigenvalue of finite multiplicity in $\sigma(T)$ and hence, by Lemma 3, it follows that λ_0 is also an isolated eigenvalue of $\overline{T}$. Since $\overline{T}$ is self-adjoint then λ_0 is a pole of first order of the resolvent of $\overline{T}$, see [14, Proposition

70.5]. Therefore, $p(\lambda_0 I - \overline{T}) = q(\lambda_0 I - \overline{T}) = 1$. By Theorem 1.1, if P_0 denotes the spectral projection of $\overline{T}$ associated with λ_0, then, by (c),

$$H_0(\lambda I - \overline{T}) = P_0(H) = \ker(\lambda_0 I - \overline{T}) = \ker(\lambda_0 I - T)\,.$$

Therefore, $(\lambda_0 I - T)P_0 x = 0$ for all $x \in H$.

On the other hand, the restriction of P_0 to X coincides with the spectral projection of T associated with the isolated point λ_0 of $\sigma(T)$. For all $x \in X$ then $(\lambda_0 I - T)P_0 x = 0$, which implies, again by Theorem 1.1 that

$$H_0(\lambda_0 I - T) = R(P_0|X) \subseteq \ker(\lambda_0 I - T),$$

where $R(P_0|X)$ is the range of $P_0|X$. This implies that $H_0(\lambda_0 I - T) = \ker(\lambda_0 I - T)$ for all $\lambda \in \pi_{00}(T)$, so Weyl's theorem holds for T. ∎

The result of Theorem 3.3 has been proved in [18] by using different arguments.

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