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On Wronskian Determinant Formulas of the General Hypergeometric Functions

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Abstract. The general hypergeometric functions of confluent type are studied. We establish a link between the general hypergeometric functions defined by 1-dimensional integrals and those defined by multi-dimensional integrals. The key point is to form an intermediate Wronskian determinant for the 1-dimensional ones and to rewrite it into a multi-dimensional integral using the generalized Veronese map.

1. Introduction

For a positive integer N and a partition λ of N, the general hypergeometric function (GHF, for short) was defined in [10] as a "Radon transform" of a character of the maximal abelian subgroup H_λ of $\mathrm{GL}_N(\mathbb{C})$ (see Sect. 2). In the case the partition is $\lambda = (1, \dots, 1)$, the GHF is also called the Aomoto-Gel'fand hypergeometric function, which is written as

$$F(z, \alpha, C) = \int_C \prod_{1 \le k < N} (z_{0k} + z_{1k}t_1 + \cdots + z_{rk}t_r)^{\alpha_k} dt_1 \wedge \cdots \wedge dt_r \,,$$

where C is an r-cycle of the homology group with coefficients in the local system $\mathcal{L}$ of rank 1 defined by the multivalued function $\prod_{1 \le k < N}(z_{0k} + z_{1k}t_1 + \cdots + z_{rk}t_r)^{\alpha_k}$. In the case $r = 1$, namely the Aomoto-Gelfand hypergeometric function is defined by 1-dimensional integral, it is essentially the same as the classically known Appell-Lauricella hypergeometric function $F = F_D(a, b_1, \dots, b_{N-3}, c; x)$:

$$\begin{aligned} F &= \sum \frac{(a)_{m_1+\cdots+m_{N-3}}(b_1)_{m_1}\cdots(b_{N-3})_{m_{N-3}}}{(c)_{m_1+\cdots+m_{N-3}} m_1! \cdots m_{N-3}!} x_1^{m_1} \cdots x_{N-3}^{m_{N-3}} \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_1^\infty \prod_{k=1}^{N-3} (t - x_k)^{-b_k} \cdot t^{b_1+\cdots+b_{N-3}-c}(t-1)^{c-a-1} dt \end{aligned} \tag{1.1}$$

which reduces further to the Gauss hypergeometric function in the case $N = 4$. It is known that the Aomoto-Gelfand hypergeometric functions are solutions of holonomic systems on the

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z-space with regular singularity. When $\lambda \neq (1, \dots, 1)$, GHFs give a natural generalization of the classical hypergeometric functions of confluent type, say, confluent hypergeometric function of Kummer, Bessel, Hermite and Airy functions (see [13]).

The purpose of this paper is to establish a link between the general hypergeometric functions defined by 1-dimensional integrals and those defined by multi-dimensional integrals. For the case $\lambda = (1, \dots, 1)$, the link mentioned above is studied by T. Terasoma. To make clear the theme of this paper, we review briefly the result of [15].

Let $\bar{S}$ be a complex affine line with the coordinate s and let $x_1 < x_2 < \cdots < x_{N-1}$ be distinct real points on $\bar{S}$. Given complex numbers $\alpha_1, \dots, \alpha_{N-1}$ satisfying $\operatorname{Re}\alpha_j > -1$. For each q $(1 \leq q \leq N-2)$, let ω_q be the 1-form:

$$\omega_q = \omega_q(s) := U(s)s^{q-1}ds, \quad U(s) = \prod_{1 \leq j < N} (s - x_j)^{\alpha_j} .$$

Note that the polynomials in ω_q are written as

$$s - x_j = \mathbf{s}\begin{pmatrix} -x_j \\ 1 \end{pmatrix}, \quad \mathbf{s} = (1, s)$$

using the column vector $\begin{pmatrix} -x_j \\ 1 \end{pmatrix}$, and hence they are specified by the matrix:

$$x = \begin{pmatrix} 1 & -x_1 & \dots & -x_{N-1} \\ 0 & 1 & \dots & 1 \end{pmatrix} \in \operatorname{Mat}_{2,N}(\mathbb{C}). \tag{1.2}$$

The form $\omega_q(s)$ is single-valued in the lower half plane and is continued analytically to the whole space $S = \bar{S} \setminus \{x_1, \dots, x_{N-1}\}$ along any path starting from a point of lower half plane. Let γ_p be the path in S joining the point x_p to x_{p+1} on the real line $\Im s = 0$:

$$\gamma_p := \{s \in S \,;\, x_p < s < x_{p+1}\}, \quad 1 \leq p \leq N-2 .$$

The branch of the multivalued function $U(s)$ in the lower half plane is specified as follows. We define the branch of $(s - x_j)^{\alpha_j}$ by assuming $\arg(s - x_j) = 0$ on γ_j when s approaches to a point on γ_j from the lower half plane. It amounts to fix the arguments of the functions $s - x_j$ on γ_p as

$$\arg(s - x_j) = \begin{cases} 0 & \text{if} \ \ 1 \leq j \leq p \\ -\pi & \text{if} \ \ p+1 \leq j < N . \end{cases}$$

Now we can define the integrals:

$$a_{pq}(x) := \int_{\gamma_p} \omega_q , \quad 1 \leq p, q \leq N-2 .$$

Note that $a_{p1}(x)$ is the Appell-Lauricella's hypergeometric function (1.1) if we restrict $a_{p1}(x)$ on the subset $x_{N-2} = 0, x_{N-1} = 1$. We consider the intermediate Wronskian determinants for these integrals, namely, for an index $P = (p_1, \dots, p_r),\ 1 \leq p_1 < \cdots < p_r \leq N-2$, we put

$$A_P(x) := \det\left(\int_{\gamma_{p_i}} \omega_j\right)_{1 \leq i,j \leq r} .$$

Let us relate this to Aomoto-Gel'fand hypergeometric function. To this end, let $X \subset \mathrm{Mat}_{2,N}(\mathbb{C})$ be the set of real matrices x of the form (1.2) such that $x_1, \dots, x_{N-1}$ are all distinct. Define the map $\Phi : X \to \mathrm{Mat}_{r+1,N}(\mathbb{C})$ by the correspondence

$$\begin{pmatrix} 1 & -x_1 & \dots & -x_{N-1} \\ 0 & 1 & \dots & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & (-x_1)^r & \dots & (-x_{N-1})^r \\ 0 & (-x_1)^{r-1} & \dots & (-x_{N-1})^{r-1} \\ \vdots & \vdots & & \vdots \\ 0 & 1 & \dots & 1 \end{pmatrix} \tag{1.3}$$

which is called Veronese map.

PROPOSITION 1.1 ([15]). *For the index $P = (1, 2, \dots, r)$, we have*

$$A_P(x) = \int_{D_P} \prod_{1 \leq j < N} L_j(t,x)^{\alpha_j} dt_1 \wedge \cdots \wedge dt_r , \tag{1.4}$$

where

$$L_j(t,x) := \mathbf{t}\Phi(x)_j = (-x_j)^r + t_1(-x_j)^{r-1} + \cdots + t_{r-1}(-x_j) + t_r ,$$

is the polynomial function of degree 1 *in T-space of variables of $t = (t_1, \dots, t_r)$ defined by the j-th column $\Phi(x)_j$ of $\Phi(x) \in \mathrm{Mat}_{r+1,N}(\mathbb{C})$ and D_P is the bounded connected component of $T_{\mathbb{R}} \setminus \bigcup_{1 \leq j < N} \{L_j(t,x) = 0\}$ in real t-space $T_{\mathbb{R}} := T \cap \mathbb{R}^r$:*

$$D_P = \{t \in T_{\mathbb{R}}\ ;\ (-1)^{P(j)} L_j(t,x) > 0\} ,$$

$P(j)$ being defined by

$$P(j) := \#\{i\ ;\ p_i < j\} .$$

The right hand side of (1.4) is just the Aomoto-Gel'fand hypergeometric function restricted to the image $\Phi(X)$ of X by the Veronese map.

Our aim is to extend this result to all the general hypergeometric functions of confluent type.

This paper is organized as follows. In Section 2, we recall the definition of the GHF. In Section 3, we fix notations for the twisted homology group and the twisted cohomology group which are needed in the formulation of the main theorem. Next in Section 4 we define the intermediate Wronskian for the GHF defined by 1-dimensional integral and state the main theorem (Theorem 4.2). In Section 5, we define the generalized Veronese map analogous to the map Φ in (1.3). Finally we prove the main theorem in the last section.

2. General hypergeometric functions

In this section we review briefly the definition of the general hypergeometric functions.

Let N be a positive integer and $\lambda = (n_1, \dots, n_\ell)$ be a partition of N. With the partition λ, we associate the maximal abelian subgroup H_λ of $\mathrm{GL}_N(\mathbb{C})$ of the form

$$H_\lambda = J(n_1) \times \cdots \times J(n_\ell),$$

where

$$J(n) := \left\{ h = \sum_{0 \le i < n} h_i \Lambda^i \; ; \; h_i \in \mathbb{C}, \;\; h_0 \neq 0 \right\} \subset \mathrm{GL}_n(\mathbb{C}),$$

$\Lambda = (\delta_{i+1,j})_{1 \le i,j \le n}$ being the shift matrix of size n. We sometimes denote an element $h \in J(n)$ as $[h_0, \dots, h_{n-1}]$.

REMARK 2.1. 1. $J(n)$ is the centralizer of an element

$$C(n,a) = \begin{pmatrix} a & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & a \end{pmatrix} \in \mathrm{GL}_n(\mathbb{C}).$$

2. An element $a \in \mathrm{GL}_N(\mathbb{C})$ is called a regular element if the orbit $O(a)$ of a by the adjoint action is of maximum dimension. a is a regular element if and only if there is a partition $\lambda = (n_1, \dots, n_\ell)$ such that Jordan normal form of a is

$$a \sim C(n_1, a_1) \oplus \cdots \oplus C(n_\ell, a_\ell)$$

with distinct eigenvalues $a_1, \dots, a_\ell$. The centralizer of this Jordan normal form is H_λ.

3. The Jordan group $J(n)$ is isomorphic to the group of units of the quotient ring $\mathbb{C}[X]/(X^n)$ by an obvious correspondence

$$h = \sum_{0 \le i < n} h_i \Lambda^i \mapsto h = \sum_{0 \le i < n} h_i X^i$$

Let $x = (x_0, x_1, x_2, \dots)$ be a sequence of variables and let $\theta_m(x)$ $(m \ge 0)$ be the function defined by

$$\sum_{0 \le m < \infty} \theta_m(x) T^m = \log(x_0 + x_1 T + x_2 T^2 + \cdots) \tag{2.1}$$

$$= \log x_0 + \log\left(1 + \frac{x_1}{x_0}T + \frac{x_2}{x_0}T^2 + \cdots\right). \tag{2.2}$$

Here $\theta_0(x) = \log x_0$, and $\theta_m(x)$ $(m \geq 1)$ is a quasihomogeneous polynomial of $x_1/x_0, \ldots, x_m/x_0$ of weight m if the weight of x_k/x_0 is defined to be k which is written explicitly as

$$\theta_m(x) = \sum (-1)^{k_1+\cdots+k_m-1} \frac{(k_1+\cdots+k_m-1)!}{k_1!\cdots k_m!} \left(\frac{x_1}{x_0}\right)^{k_1} \cdots \left(\frac{x_m}{x_0}\right)^{k_m},$$

where the sum is taken over the indices $(k_1, \ldots, k_m) \in \mathbb{Z}_{\geq 0}^m$ such that $k_1 + 2k_2 + \cdots + mk_m = m$.

LEMMA 2.2 ([6]). *We have the isomorphism $J(n) \simeq \mathbb{C}^\times \times \mathbb{C}^{n-1}$ by the correspondence*

$$h = \sum_{0 \leq i < n} h_i \Lambda^i \mapsto (h_0, \theta_1(h), \ldots, \theta_{n-1}(h)).$$

The following result is the consequence of the above lemma.

LEMMA 2.3. *The character $\chi_n : \tilde{J}(n) \to \mathbb{C}^\times$ is given by*

$$\chi_n(h; \alpha) = \exp\left(\sum_{0 \leq i < n} \alpha_i \theta_i(h)\right) = h_0^{\alpha_0} \exp\left(\sum_{1 \leq i < n} \alpha_i \theta_i(h)\right),$$

where $\alpha = (\alpha_0, \ldots, a_{n-1})$ are arbitrary complex constants.

Since H_λ is a product of $J(n_k)$, the character of $\tilde{H}_\lambda$ is the product of the characters χ_{n_k} of $\tilde{J}(n_k)$.

PROPOSITION 2.4. *A character $\chi_\lambda : \tilde{H}_\lambda \to \mathbb{C}^\times$ is given, for some $\alpha = (\alpha^{(1)}, \ldots, \alpha^{(\ell)}) \in \mathbb{C}^N$, $\alpha^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$, by*

$$\chi_\lambda(h; \alpha) = \prod_{1 \leq k \leq \ell} \chi_{n_k}(h^{(k)}; \alpha^{(k)}) = \prod_{1 \leq k \leq \ell} \exp\left(\sum_{0 \leq i < n_k} \alpha_i^{(k)} \theta_i(h^{(k)})\right), \tag{2.3}$$

where $h = (h^{(1)}, \ldots, h^{(\ell)}) \in \tilde{H}_\lambda$, $h^{(k)} \in \tilde{J}(n_k)$.

Next we consider the "Radon transform" of the character χ_λ. As in the case of Aomoto-Gelfand hypergeometric, we substitute polynomials of degree 1 in $t = (t_1, \ldots, t_r)$ into the character and integrate. We define the space of coefficients of these polynomials.

We sometimes identify a partition $\lambda = (n_1, \ldots, n_\ell)$ with the Yang diagram which is obtained by arraying boxes, n_1 boxes in the first row, n_2 boxes in the second row, and so on. See Figure. A sequence $\mu = (m_1, \ldots, m_\ell) \in \mathbb{Z}_{\geq 0}^\ell$ is called a subdiagram of λ if it satisfies

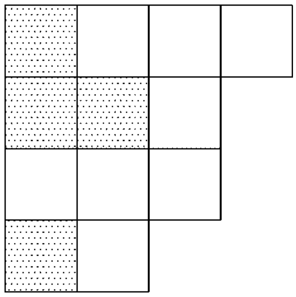

FIGURE 1. Subdiagram (1,2,0,1) in (4,3,3,2)

$0 \le m_k \le n_k$ $(1 \le k \le \ell)$ and is denoted as $\mu \subset \lambda$. The sum $|\mu| = m_1 + \cdots + m_\ell$ is called the weight of μ. See Figure 1. For a given $z = (z^{(1)}, \dots, z^{(\ell)}) \in \mathrm{Mat}_{r+1,N}(\mathbb{C})$ with $z^{(k)} = (z_0^{(k)}, \dots, z_{n_k-1}^{(k)})$ and for any subdiagram $\mu \subset \lambda$, $|\mu| = r + 1$, we put

$$z_\mu = (z_0^{(1)}, \dots, z_{m_1-1}^{(1)}, \dots, z_0^{(\ell)}, \dots, z_{m_\ell-1}^{(\ell)}) \in \mathrm{Mat}_{r+1}(\mathbb{C}).$$

DEFINITION 2.5. The generic stratum $Z_{r,\lambda} \subset \mathrm{Mat}_{r+1,N}(\mathbb{C})$ with respect to λ is defined by

$$Z_{r,\lambda} = \{z \in \mathrm{Mat}_{r+1,N}(\mathbb{C})\,;\ \det z_\mu \neq 0 \ \text{ for any } \ \mu \subset \lambda, |\mu| = r+1\}\,.$$

For the character $\chi_\lambda(\cdot\,; \alpha)$ given in (2.3), we assume

$$\sum_{1 \le k \le \ell} \alpha_0^{(k)} = -r - 1\,. \tag{2.4}$$

Moreover we define a biholomorphic map

$$\iota: \ H_\lambda \to \prod_{1 \le k \le \ell} \left(\mathbb{C}^\times \times \mathbb{C}^{n_k-1}\right) \subset \mathbb{C}^N$$

by

$$\iota(h) = (h_0^{(1)}, \dots, h_{n_1-1}^{(1)}, \dots, h_0^{(\ell)}, \dots, h_{n_\ell-1}^{(\ell)})$$

for $h = \bigoplus_k [h_0^{(k)}, \dots, h_{n_k-1}^{(k)}] \in H_\lambda$. The map ι can be lifted to that from $\tilde{H}_\lambda$ to $\prod_{1 \le k \le \ell} \left(\tilde{\mathbb{C}}^\times \times \mathbb{C}^{n_k-1}\right)$. This lift is also denoted by ι.

DEFINITION 2.6. The general hypergeometric function of type λ (GHF of type λ, for short) is the function of $z \in Z_{r,\lambda}$ defined by

$$F_\lambda(z; \alpha) = \int_{\Delta_z} \chi(\iota^{-1}(\mathbf{t}z), \alpha)\, dt_1 \wedge \cdots \wedge dt_r\,, \tag{2.5}$$

where we assumed (2.4) and Δ_z is an r-dimensional cycle in $\mathbb{C}^r \setminus \cup_{1\le k\le \ell}\{\mathbf{t}z_0^{(k)} = 0\}$ of the homology group defined by the integrand $\chi(\iota^{-1}(\mathbf{t}z), \alpha)$ in $\mathbb{C}^r$ (see also Sect.3).

3. Twisted homology and cohomology

In this section, we explain about the twisted algebraic de Rham cohomology group and twisted homology group associated with the GHF defined by 1-dimensional integral.

3.1. Algebraic de Rham cohomology. Let $\bar{S}$ be a complex affine line with the coordinates s. For a partition $\lambda = (n_1, \dots, n_\ell)$ of N and $z = (z^{(1)}, \dots, z^{(\ell)}) \in Z_{1,\lambda}$, we have the multivalued function $U(s) = \chi_\lambda(\iota^{-1}(\mathbf{s}z); \alpha)$ appeared as an integrand of GHF defined by 1-dimensional integral. $U(s)$ has the singular locus $D = \{p^{(1)}, \dots, p^{(\ell)}\}$, where $p^{(k)} = \{s \in \bar{S}\ ;\ \mathbf{s}z_0^{(k)} = 0\}$. Here, we regard $p^{(k)}$ as ∞ when $\mathbf{s}z_0^{(k)}$is a constant function of s. Note that $p^{(1)}, \dots, p^{(\ell)}$ are distinct by virtue of the condition for $z \in Z_{1,\lambda}$. We assume here $p^{(1)} = \infty$. Put $S = \bar{S} \setminus D$ and define the rational 1-form ω on S by

$$\omega := d \log U(s) = d \log\, \chi(\iota^{-1}(\mathbf{s}z); \alpha),$$

where d denotes the exterior differentiation with respect to s. Note that if $\alpha_{n_k-1} \neq 0$ for all k, the divisor of ω is $\sum_{1\le k\le \ell}(-n_k)p^{(k)}$. Let $\Omega^0(*D)$ and $\Omega^1(*D)$ be the modules of rational functions and rational 1-forms with poles at most on D. Let $\nabla : \Omega^0(*D) \to \Omega^1(*D)$ be the twisted differentiation defined by

$$\nabla f = df + \omega \wedge f\,, \quad f \in \Omega^0(*D)\,.$$

Since $\nabla = U(s)^{-1} \cdot d \cdot U(s)$, we have $\nabla^2 = U(s)^{-1} \cdot d^2 \cdot U(s) = 0$. Moreover we have $\nabla f \in \Omega^1(*D)$ since ω is a rational form with poles on D. Thus we have the twisted algebraic de Rham complex

$$(\Omega^\bullet(*D), \nabla):\quad 0 \to \Omega^0(*D) \xrightarrow{\nabla} \Omega^1(*D) \to 0.$$

We define twisted algebraic de Rham cohomology by

$$H^i(\Omega^\bullet(*D), \nabla) := \ker\{\nabla : \Omega^i(*D) \to \Omega^{i+1}(*D)\}/\nabla\Omega^{i-1}(*D)\,, \quad i = 0, 1\,.$$

PROPOSITION 3.1 ([8]). *Suppose that the parameter* $\alpha = (\alpha^{(1)}, \dots \alpha^{(\ell)}) \in \mathbb{C}^N$ *satisfies*

$$\alpha^{(k)}_{n_k-1} \neq 0 \ \textit{if}\ n_k \ge 2 \quad \textit{and} \quad \alpha^{(k)}_{n_k-1} \notin \mathbb{Z} \ \textit{if}\ n_k = 1\,. \tag{3.1}$$

Then we have

1. $H^p(\Omega^\bullet(*D), \nabla) = 0$ *for* $p \neq 1$,
2. $\dim_{\mathbb{C}} H^1(\Omega^\bullet(*D), \nabla) = N - 2$,
3. *the* 1*-forms* $s^{q-1}ds$, $1 \le q \le N-2$, *provide a basis of the vector space* $H^1(\Omega^\bullet(*D), \nabla)$.

3.2. Twisted homology. We consider the twisted homology groups associated with the 1-dimensional integral (2.5). We assume that the parameters α satisfy the condition (3.1).

Note that the integrand of the general hypergeometric integral for $z \in Z_1$ is written as

$$\chi(\iota^{-1}(\mathbf{s}z); \alpha) = \prod_{1 \le k \le \ell} \big(\mathbf{s}z_0^{(k)}\big)^{\alpha_0^{(k)}} \times e^{F(s)}$$

where $F(s)$ is the rational function of s:

$$F(s) = \sum_{1 \le k \le \ell} \sum_{1 \le i < n_k} \alpha_i^{(k)} \theta_i(\mathbf{s}z^{(k)}) .$$

Let $\mathcal{L}$ be the rank 1 local system of complex vector space on S whose local horizontal sections are constant multiples of the multivalued function $\chi(\iota^{-1}(\mathbf{s}z); \alpha)$. Equivalence class of the local system $\mathcal{L}$ is determined by the multivalued part $\prod_{1 \le k \le \ell} \big(\mathbf{s}z_0^{(k)}\big)^{\alpha_0^{(k)}}$. The rational function $F(s)$ carries an information of exponential growth and decay of the integrand. This information is formulated as follows.

Let Ψ be the family of closed sets such that $A \in \Psi$ if A satisfies the condition:

$$\text{For any } a \in \mathbb{R}, \quad A \cap \{s \in S; \Re F(s) \ge a\} \quad \text{is compact.}$$

Then we can define the homology group of locally finite chains with coefficients in the local system $\mathcal{L}$ and with the family of supports Ψ.

Consider an infinite sum $c = \sum_\sigma a_\sigma \otimes \sigma$, where σ is a singular p-simplex in S given by a smooth map $\sigma : \Delta_p \to S$ from the standard p-simplex Δ_p to S, and a_σ is a horizontal section of the local system on Δ_p obtained by pulling back the local system $\mathcal{L}$ on S by the map σ. We assume that the sum is locally finite and that $\operatorname{supp} c \in \Psi$. We call such infinite sum c a p-chain. With the obvious addition, the set of p-chains forms an abelian group which is denoted by $C_p^\Psi(S, \mathcal{L})$. We can define the boundary map $\partial : C_p^\Psi(S, \mathcal{L}) \to C_{p-1}^\Psi(S, \mathcal{L})$ as usual and hence we have the chain complex $(C_\bullet^\Psi(S, \mathcal{L}), \partial)$. The p-th homology groups of the chain complex is denoted by $H_p^\Psi(S; \mathcal{L})$.

PROPOSITION 3.2. *For $\chi(\iota^{-1}(\mathbf{s}z); \alpha)$ we assume the condition* (3.1) *on the parameter α. Then we have*

1. $H_p^\Psi(S; \mathcal{L}) = 0$ *if* $p \neq 1$
2. $\dim_{\mathbb{C}} H_1^\Psi(S; \mathcal{L}) = N - 2$.

For $[\gamma] \in H_1^\Psi(S; \mathcal{L})$ and $[\varphi] \in H^1(\Omega^\bullet(*D), \nabla)$ with $\gamma = \sum_\sigma a_\sigma \otimes \sigma$, we define

$$\langle [\gamma], [\varphi] \rangle = \sum_\sigma \int_{\Delta^1} a_\sigma \cdot \sigma^* \varphi .$$

We can see that the right hand side is independent of the choice of 1-cycle γ representing the class $[\gamma]$ and of 1-form φ representing the cohomology class $[\varphi]$. Hence the pairing $\langle \ , \ \rangle$

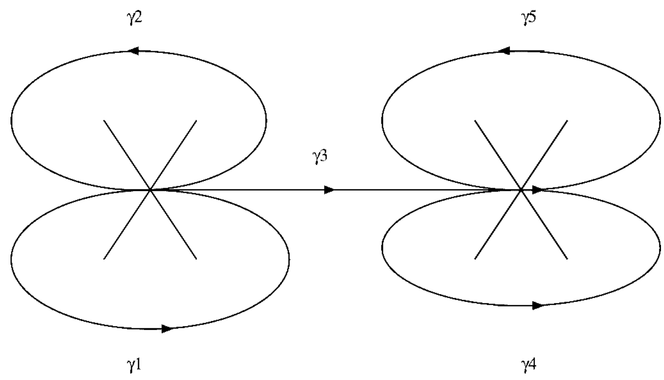

FIGURE 2. Cycles

defines a bilinear map $H_1^{\Psi}(S;\mathcal{L}) \times H^1(\Omega^{\bullet}(*D),\nabla) \to \mathbb{C}$.

By Proposition 3.2, we can choose cycles $\gamma_1,\dots,\gamma_{N-2}$ such that the homology classes $[\gamma_1],\dots,[\gamma_{N-2}]$ form a basis of $H_1^{\Psi}(S;\mathcal{L})$. Since we don't need the explicit form of a basis, we simply give an example of a choice of a basis.

EXAMPLE 3.3. $\lambda = (3,3,1)$. In this case $\dim_{\mathbb{C}} H_1^{\Psi}(S;\mathcal{L}) = 5$. we assume the stronger condition than (3.1):

$$\alpha_0^{(k)} \notin \mathbb{Z} \quad (k=0,1,2)\,, \qquad \alpha_2^{(k)} \neq 0 \quad (k=0,1)$$

Then we can take cycles γ_i $(i = 1,\dots,5)$ for a basis whose supports are drawn in the Figure 2.

One more convention. Let $[\gamma] \in H_1^{\Psi}(S;\mathcal{L})$ and assume that in the expression $\gamma = \sum_\sigma a_\sigma \otimes \sigma$, a_σ is a pull back of some branch of $\chi(\iota^{-1}(\mathbf{s}z);\alpha)$. In this case we write as

$$\langle[\gamma],[\varphi]\rangle = \int_\gamma \chi(\iota^{-1}(\mathbf{s}z);\alpha)\varphi$$

by abuse of notations.

4. Main theorem

In view of Proposition 3.1, we take $z \in Z_{1,\lambda}$ and put

$$\omega_q = \omega_q(s) := \chi(\iota^{-1}(\mathbf{s}z);\alpha)s^{q-1}ds\,, \quad 1 \le q \le N-2\,.$$

They are multivalued 1-forms on $S = \bar{S} \setminus D$. Let $\gamma_1,\dots,\gamma_{N-2}$ be cycles of the homology group $H_1^{\Psi}(S;\mathcal{L})$ as in Subsection 3.2.

DEFINITION 4.1. Let I, J be ordered subsets of $\{1, \ldots, N-2\}$ of the cardinality $|I| = |J| = r$, say, $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_r)$, The determinant

$$A_{IJ} = \det\left(\int_{\gamma_{i_p}} \omega_{j_q}\right)_{1\le p,q\le r}$$

is called the Wronskian determinant of level r for the pair (I, J).

For a technical reason, we assume that J satisfies $j_1 > \cdots > j_r$. But this assumption does not harm the generality of discussion.

Let S_i $(1 \le i \le r)$ be r copies of affine line $\mathbb{C}$ and let s_i be the coordinates of the i-th copy S_i. Let T be the r-dimensional complex affine space with coordinates $t = (t_1, \ldots, t_r)$. Define a map $\phi : S_1 \times \cdots \times S_r \to T$ by the correspondence $s = (s_1, \ldots, s_r) \mapsto t$ where the i-th coordinate t_i of the image of ϕ is the i-th elementary symmetric function of s:

$$t_1 = s_1 + s_2 + \cdots + s_r, \ldots, t_r = s_1 s_2 \ldots s_r .$$

Let $\mu = (\mu_1, \ldots, \mu_r)$ be a Young diagram and let $\sigma_\mu(s)$ be the Schur polynomial for μ:

$$\sigma_\mu(s) = \begin{vmatrix} s_1^{\mu_1+r-1} & \ldots & s_r^{\mu_1+r-1} \\ s_1^{\mu_2+r-2} & \ldots & s_r^{\mu_2+r-2} \\ \vdots & & \vdots \\ s_1^{\mu_r} & \ldots & s_r^{\mu_r} \end{vmatrix} \Bigg/ \begin{vmatrix} s_1^{r-1} & \ldots & s_r^{r-1} \\ s_1^{r-2} & \ldots & s_r^{r-2} \\ \vdots & & \vdots \\ 1 & \ldots & 1 \end{vmatrix}$$

Since $\sigma_\mu(s)$ is a symmetric polynomial in s, it can be expressed as a polynomial of elementary symmetric functions $t_1, \ldots, t_r$ of $s_1, \ldots, s_r$. We denote this polynomial in t as $S_\mu(t)$. Now the main theorem is stated as follows.

THEOREM 4.2. *Let $\lambda = (n_1, \ldots, n_\ell)$ be a partition of N and let $\Phi_\lambda : Z_{1,\lambda} \to Z_{r,\lambda}$ be the generalized Veronese map defined in Section* 5, *Definition* 5.5. *Then for any indices $I = (i_1, \ldots, i_r)$ and $J = (j_1 > \cdots > j_r)$, we have*

$$A_{IJ}(z) = \int_{\phi_*(\gamma_{i_1}\times\cdots\times\gamma_{i_r})} \chi(\mathbf{t}\Phi_\lambda(z); \alpha) S_Y(t) dt_1 \wedge \cdots \wedge dt_r ,$$

where Y is the Yang diagram corresponding to $(j_1 - r, j_2 - r + 1, \ldots, j_r - 1)$.

COROLLARY 4.3. *If we take $J = (r, r-1, \ldots, 1)$, then for any index $I \subset \{1, \ldots, N-2\}$ of $|I| = r$, $A_{IJ}(z)$ gives the general hypergeometric function of type λ on the image $\Phi_\lambda(Z_{1,\lambda}) \subset Z_{r,\lambda}$.*

5. Veronese map

We introduce the map $\Phi_\lambda : Z_{1,\lambda} \to Z_{r,\lambda}$ which is used in the statement of Theorem 4.2. First we treat the simple case $\lambda = (n)$. Let V be the complex vector space of $\dim V = 2$.

Let $R = \mathbb{C}[T]/(T^n)$, where (T^n) denote the ideal of $\mathbb{C}[T]$ generated by T^n. Put $W := V \otimes_{\mathbb{C}} R$. Then W is a free R-module of rank 2 as well as a left $\mathrm{GL}_V(\mathbb{C})$-module by the action $g \cdot (v \otimes h) = (gv) \otimes h$, where $g \in \mathrm{GL}_V(\mathbb{C})$, $v \in V$ and $h \in R$. The module W and the ring R above are also denoted as W_n and R_n, respectively, when it is necessary to emphasize their dependence on n.

Let $S^r(W)$ be the symmetric tensor product of W as R-module. Since $S^r(W) \simeq S^r(V) \otimes_{\mathbb{C}} R$, $S^r(W)$ is a free R-module of rank $r+1$. The symmetric tensor product $S^r(W)$ is endowed also with the structure of left $\mathrm{GL}_V(\mathbb{C})$-module induced from that for W.

DEFINITION 5.1. The $(\mathrm{GL}_V(\mathbb{C}), R)$-equivariant map $\Phi : W \to S^r(W)$ defined by $\Phi(w) = \overbrace{w \otimes \cdots \otimes w}^{r}$ is called the generalized Veronese map. Sometimes we write the map Φ as Φ_n in order to emphasize the dependence on n.

Let us write down the Veronese map Φ_n in terms of coordinates. Let e_0, e_1 be a basis of V by which we identify V with $\mathbb{C}^2$. Since $W = V \otimes R_n$, using its $\mathbb{C}$-basis $e_i \otimes T^j$ ($i = 0, 1;\ 0 \leq j < n$) we can identify W with $\mathrm{Mat}_{2,n}(\mathbb{C})$ as $\mathbb{C}$-vector spaces by the correspondence

$$W \ni w = \sum_{i=0,1} \sum_{0 \leq j < n} w_{ij} e_i \otimes T^j \mapsto (w_{ij}) \in \mathrm{Mat}_{2,n}(\mathbb{C})\,.$$

Similarly we can identify $S^r(W)$ with $\mathrm{Mat}_{r+1,n}(\mathbb{C})$ as $\mathbb{C}$-vector spaces. For this we take a basis $\mathbf{e}_0, \ldots, \mathbf{e}_r$ of $S^r(V)$ defined by

$$\mathbf{e}_k = \frac{1}{k!(r-k)!} \sum_{i_1 + \cdots + i_r = k} e_{i_1} \otimes \cdots \otimes e_{i_r}$$

and identify $S^r(V)$ with $\mathbb{C}^{r+1}$. Hence using the basis $\mathbf{e}_k \otimes T^j$ of $S^r(W) \simeq S^r(V) \otimes R_n$, we identify $S^r(W)$ with $\mathrm{Mat}_{r+1,n}(\mathbb{C})$ by the correspondence

$$S^r(W) \ni z = \sum_{0 \leq i \leq r} \sum_{0 \leq j < n} z_{ij} \mathbf{e}_i \otimes T^j \mapsto (z_{ij}) \in \mathrm{Mat}_{r+1,n}(\mathbb{C})\,.$$

For $w \in W$, we put

$$\Phi_n(w) = \overbrace{w \otimes \cdots \otimes w}^{r\text{times}} = \sum_{0 \leq i \leq r} \sum_{0 \leq j < n} \varphi_{ij}(w) \mathbf{e}_i \otimes T^j\,.$$

It is easily seen that the explicit form of polynomials $\varphi_{ij}(w)$ is given by

$$\varphi_{ij}(w) = \sum w_{0j_1} \ldots w_{0j_{r-i}} w_{1j_{r-i+1}} \ldots w_{1j_r}\,,$$

where the sum is taken over all the indices $(j_1, \ldots, j_r)$ satisfying $0 \leq j_p < n$ and $j_1 + \cdots + j_r = j$. Therefore $\varphi_{ij}(w)$ are homogeneous polynomials of $w \in \mathrm{Mat}_{2,n}(\mathbb{C})$ of degree r. Then the map

$$\mathrm{Mat}_{2,n}(\mathbb{C}) \ni w \mapsto (\varphi_{ij}(w)) \in \mathrm{Mat}_{r+1,n}(\mathbb{C})$$

gives the expression of the map Φ_n in terms of coordinates.

REMARK 5.2. When $n = 1$, the map Φ is given in terms of coordinates by

$$^t(w_0, w_1) \mapsto {}^t(w_0^r, w_0^{r-1}w_1, \dots, w_0 w_1^{r-1}, w_1^r)$$

and induces the map $\mathbb{P}^1 \to \mathbb{P}^r$ which coincides with the Veronese embedding in the usual sense.

REMARK 5.3. The Veronese map $\Phi_n : \mathrm{Mat}_{2,n}(\mathbb{C}) \to \mathrm{Mat}_{3,n}(\mathbb{C})$ is written as follows. Define a symmetric bilinear map $\varphi : \mathbb{C}^2 \times \mathbb{C}^2 \to \mathbb{C}^3$ by

$$\varphi(u, v) = \frac{1}{2}\begin{pmatrix} 2u_0v_0 \\ u_0v_1 + u_1v_0 \\ 2u_1v_1 \end{pmatrix}, \quad u = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \quad v = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}.$$

Then, for $z = (z_0, \dots, z_{n-1}) \in \mathrm{Mat}_{2,n}(\mathbb{C})$,

$$\begin{aligned} \Phi_n(z) = (\varphi(z_0, z_0), \varphi(z_0, z_1) + \varphi(z_1, z_0), \dots, \varphi(z_0, z_n) \\ + \varphi(z_1, z_{n-1}) + \cdots + \varphi(z_{n-1}, z_1) + \varphi(z_n, z_0)) . \end{aligned}$$

We give a proposition which will be used in the proof of Lemma 6.1. Take $z = (z_0, \dots, z_{n-1}) \in \mathrm{Mat}_{2,n}(\mathbb{C})$ and define the polynomials $l_k(s)$ in s by

$$l_k(s) := (1, s)z_k = z_{0k} + sz_{1k}, \quad 0 \le k < n .$$

Let $s_1, \dots, s_r$ be r copies of the variable s and let t_i be the i-th elementary symmetric function of $s_1, \dots, s_r$. The following proposition follows from the definition of the Veronese map $\Phi = \Phi_n$.

PROPOSITION 5.4. *Let $L_k(t), 0 \le k < n$, be the linear form of $\mathbf{t} = (1, t_1, \dots, t_r)$ defined by*

$$L_k(t) = \mathbf{t}\Phi(z)_k ,$$

where $\Phi(z)_k$ is the k-th column vector of the Veronese image $\Phi(z)$. Then we have the identity

$$\sum_{0 \le k < n} L_k(t)T^k \equiv \prod_{1 \le i \le r} \{l_0(s_i) + l_1(s_i)T + \cdots + l_{n-1}(s_i)T^{n-1}\} \mod (T^n) .$$

Let $\lambda = (n_1, \dots, n_\ell)$ be the partition of N as in the preceding sections. Put $W_\lambda = W_{n_1} \oplus \cdots \oplus W_{n_\ell}$ and $R_\lambda = R_{n_1} \times \cdots \times R_{n_\ell}$, where $W_n = V \otimes R_n$. Then W_λ is considered as a left $\mathrm{GL}_V(\mathbb{C})$-module as well as an R_λ-module.

DEFINITION 5.5. Define the map

$$\Phi_\lambda : W_\lambda \to \oplus_i S^r(W_{n_i})$$

by

$$\left(w^{(1)}, \dots, w^{(\ell)}\right) \mapsto \left(\Phi_{n_1}(w^{(1)}), \dots, \Phi_{n_\ell}(w^{(\ell)})\right).$$

This is called the *Veronese map of type* λ.

Expressing the map Φ_λ in terms of coordinates, we get the map $\mathrm{Mat}_{2,N}(\mathbb{C}) \to \mathrm{Mat}_{r+1,N}(\mathbb{C})$. We also denote it by Φ_λ. Let $R_n^\times$ denote the group of units in R_n and let $R_\lambda^\times := R_{n_1}^\times \times \cdots \times R_{n_\ell}^\times$ be the group of units in R_λ. $R_\lambda^\times$ acts on W_λ. In terms of the coordinates this action is interpreted as the action of $H_\lambda = J(n_1) \times \cdots \times J(n_\ell)$ on $\mathrm{Mat}_{2,N}(\mathbb{C})$, Cf. Remark 2.1.

PROPOSITION 5.6. *The map Φ_λ takes generic stratum $Z_{1,\lambda}$ into $Z_{r,\lambda}$: $\Phi_\lambda : Z_{1,\lambda} \to Z_{r,\lambda}$.*

Let $\rho : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{r+1}(\mathbb{C})$ denote the homomorphism which represents the action of $\mathrm{GL}_V(\mathbb{C})$ on $S^r(W)$ as the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathrm{Mat}_{r+1,N}(\mathbb{C})$.

To prove the proposition we first show the following lemma.

LEMMA 5.7. *The generic stratum $Z_{r,\lambda}$ is preserved by the action of* $\mathrm{GL}_{r+1}(\mathbb{C})$ *and H_λ. In particular it is preserved by the action of $\rho(\mathrm{GL}_2(\mathbb{C}))$.*

PROOF. It is clear that the generic stratum $Z_{r,\lambda}$ is preserved by the left multiplication of $g \in \mathrm{GL}_{r+1}(\mathbb{C})$ because we have $\det(gz_\mu) = \det g \cdot \det z_\mu$ for any subdiagram $\mu \subset \lambda$, $|\mu| = r + 1$ (see §2 for this notation). To see that $Z_{r,\lambda}$ is invariant by the action of H_λ, it will be sufficient to notice that for $h = (h^{(1)}, \dots, h^{(\ell)}) \in H_\lambda$ we have

$$\mathrm{span}\left\langle z_0^{(k)}, \dots, z_p^{(k)}\right\rangle = \mathrm{span}\left\langle (z^{(k)}h^{(k)})_0, \dots, (z^{(k)}h^{(k)})_p\right\rangle$$

for any $0 \le p < n_k$, which is easily seen by writing down $z^{(k)}h^{(k)}$ explicitly. $\square$

Let $\mathcal{I}_p$ be the set of multi-indices $I = (i_1, \dots i_r)$ satisfying $0 \le i_1 \le \cdots \le i_r$ and $|I| := i_1 + \cdots + i_r = p$. Put $\mathcal{I} = \bigcup_{0 \le p \le r} \mathcal{I}_p$. In $\mathcal{I}$ we consider the lexicographic order, namely, for $I = (i_1, \dots i_r)$ and $J = (j_1, \dots j_r)$ in $\mathcal{I}$ we define $I < J$ if there exists $1 \le s \le r$ such that $i_1 = j_1, \dots, i_{s-1} = j_{s-1}$ and $i_s < j_s$. In each $\mathcal{I}_p$ there is the maximum element $(\overbrace{0, \dots, 0}^{r-p}, \overbrace{1, \dots, 1}^{p})$ which we denote by I_p. For $I \in \mathcal{I}$ we put

$$z^{\otimes I} = z_{i_1} \otimes \cdots \otimes z_{i_r}$$

and denote its symmetrization by $\mathcal{S}(z^{\otimes I})$:

$$\mathcal{S}(z^{\otimes I}) = \frac{1}{r!} \sum_{\sigma \in \mathfrak{S}_r} z_{i_{\sigma(1)}} \otimes \cdots \otimes z_{i_{\sigma(r)}}.$$

LEMMA 5.8. *Let* $z = (z_0, \dots, z_{n-1}) \in \mathrm{Mat}_{2,n}(\mathbb{C})$ *such that* $\det(z_0, z_1) \neq 0$ *and let* $\Phi(z) = (\Phi(z)_0, \dots, \Phi(z)_{n-1}) \in \mathrm{Mat}_{r+1,n}(\mathbb{C})$. *Then for any* $0 \leq p < n$ *we have*

$$\mathrm{span}\langle \Phi(z)_0, \dots, \Phi(z)_p \rangle = \mathrm{span}\langle \mathcal{S}(z^{\otimes I_0}), \dots, \mathcal{S}(z^{\otimes I_p}) \rangle .$$

PROOF. We prove by induction on p. When $p = 0$, we have $\Phi(z)_0 = \mathcal{S}(z^{\otimes I_0})$ and the assertion is trivial. Suppose that the assertion holds for the indices up to $p - 1$. It is enough to show

$$\mathrm{span}\langle \mathcal{S}(z^{\otimes I_0}), \dots, \mathcal{S}(z^{\otimes I_{p-1}}), \Phi(z)_p \rangle = \mathrm{span}\langle \mathcal{S}(z^{\otimes I_0}), \dots, \mathcal{S}(z^{\otimes I_p}) \rangle . \tag{5.1}$$

By the definition of $\Phi(z)_p$ we have

$$\Phi(z)_p = \sum_{I \in \mathcal{I}_p} c_I \mathcal{S}(z^{\otimes I})$$

for some constants c_I. For $I \in \mathcal{I}_p$ such that $I < I_p$, there is an index $i_q \in I$ such that $2 \leq i_q$. Since z_0 and z_1 are linearly independent, z_{i_q} is a linear combination of them. Then $\mathcal{S}(z^{\otimes I})$ can be written as a linear combination of $\mathcal{S}(z^{\otimes I_0}), \dots, \mathcal{S}(z^{\otimes I_{p-1}})$. This proves the assertion (5.1) and the lemma. □

The following is the consequence of Lemma 5.8.

LEMMA 5.9. *For* $z = (z^{(1)}, \dots, z^{(\ell)}) \in Z_{1,\lambda}$ *with* $z^{(k)} = (z_0^{(k)}, \dots, z_{n_k-1}^{(k)})$, *put* $\zeta = (\zeta^{(1)}, \dots, \zeta^{(\ell)}) \in Z_{1,\lambda}$ *with* $\zeta^{(k)} = (z_0^{(k)}, z_1^{(k)}, 0, \dots, 0)$. *Then* $\Phi_\lambda(z) \in Z_{r,\lambda}$ *if and only if* $\Phi_\lambda(\zeta) \in Z_{r,\lambda}$.

LEMMA 5.10 ([8]). *Let* $z \in Z_{1,\lambda}$. *Then there exist* $g \in \mathrm{GL}_2(\mathbb{C})$ *and* $h \in H_\lambda$ *such that* $x = (x^{(1)}, \dots, x^{(\ell)}) := gzh \in Z_{1,\lambda}$ *has the form*

$$x^{(k)} = \begin{pmatrix} x_0^{(k)} & x_1^{(k)} & \dots & x_{n_k-1}^{(k)} \\ 1 & 0 & \dots & 0 \end{pmatrix} \tag{5.2}$$

Note that for x in Lemma 5.10, we have

$$x \in Z_{1,\lambda} \iff x_1^{(1)} \dots x_1^{(\ell)} \prod_{i<j} (x_0^{(i)} - x_0^{(j)}) \neq 0 .$$

In view of Lemmas 5.7, 5.9 and 5.10, to prove Proposition 5.6, it is sufficient to show that $\Phi_\lambda(x) \in Z_{r,\lambda}$ for $x \in Z_{1,\lambda}$ having the form (5.2) and satisfying

$$x_2^{(k)} = \cdots = x_{n_k-1}^{(k)} = 0 , \quad 1 \leq k \leq \ell . \tag{5.3}$$

Put

$$f(v) = {}^t(v^r, v^{r-1}, \dots, 1) . \tag{5.4}$$

Then $\Phi_\lambda(x) = \big(\Phi_{n_1}(x^{(1)}), \dots, \Phi_{n_\ell}(x^{(\ell)})\big)$ is written as

$$\Phi_{n_k}(x^{(k)}) = \big(f(x_0^{(k)}), x_1^{(k)}(Df)(x_0^{(k)}), \dots, (x_1^{(k)})^{n_k-1}(D^{n_k-1}f)(x_0^{(k)})\big),$$

where D is the differentiation with respect to v.

LEMMA 5.11. *Take a subdiagram $\mu = (m_1, \dots, m_\ell)$ of λ with $|\mu| = r+1$. Then*

$$\det(\Phi_\lambda(x))_\mu = C_\mu (x_1^{(1)})^{m_1(m_1-1)/2} \dots (x_1^{(\ell)})^{m_\ell(m_\ell-1)/2} \prod_{1\le i<j\le\ell} \big(x_0^{(i)} - x_0^{(j)}\big)^{m_i m_j},$$

where

$$C_\mu = \prod_{1\le k\le\ell} (-1)^{m_k(m_k-1)/2} \frac{\prod_{0\le q<m_k}\big(r+1-\sum_{1\le i\le k} m_i + q\big)!}{\big\{m_k(r+1-\sum_{1\le i\le k} m_i)\big\}!}. \tag{5.5}$$

PROOF. Define the polynomial $F(u)$ of $u = (u_1, \dots, u_\ell)$ by

$$F(u) := \det(F^{(1)}, \dots, F^{(\ell)}),$$

where $F^{(k)}$is defined by

$$F^{(k)} = \big(f(u_k), (Df)(u_k), \dots, (D^{m_k-1}f)(u_k)\big).$$

It is to be shown that

$$F(u) = C_\mu \prod_{1\le i<j\le\ell} (u_i - u_j)^{m_i m_j}. \tag{5.6}$$

For an index $I = (i_0, \dots, i_{m_k-1})$ such that $0 \le i_0 \le \dots \le i_{m_k-1}$, we put

$$D^I F^{(k)} := \big((D^{i_0}f)(u_k), (D^{i_1+1}f)(u_k), \dots, (D^{i_{m_k-1}+m_k-1}f)(u_k)\big).$$

It is easily seen from the explicit form of $F^{(k)}$ that $F(u)$ is of degree at most $m_k(r+1-m_k)$ as a polynomial of u_k. To show (5.6), first we consider F as a polynomial of u_1 regarding $u_2, \dots, u_\ell$ as constants. m times differentiation with respect to u_1 yields

$$\big(\partial/\partial u_1\big)^m F = \sum_{I=(i_0,\dots,i_{m_1-1}),|I|=m} \det(D^I F^{(1)}, F^{(2)}, \dots, F^{(\ell)}). \tag{5.7}$$

Evaluate $(\partial/\partial u_1)^m F$ at $u_1 = u_k$. The term $\det(D^I F^{(1)}, F^{(2)}, \dots, F^{(\ell)})$ in the right hand side of (5.7) vanishes at $u_1 = u_k$ if the index I satisfies $i_0 < m_k$. And this condition holds for all terms if $m < m_1 m_k$. This implies that F has the factor $\prod_{2\le k\le\ell}(u_1 - u_k)^{m_1 m_k}$ as a polynomial of u_1. We do the same thing for F by considering now u_2 as a variable and $u_1, u_3, \dots, u_\ell$ as constants and we can see that F has also a factor $\prod_{3\le k\le\ell}(u_2 - u_k)^{m_2 m_k}$. Proceeding in the same manner and noting that F is of degree less that $m_k(r+1-m_k)$ in u_k, we conclude that

F is written in the form (5.6), where the constant C_μ is yet to be determined. To determine the constant C_μ, consider the differential operator

$$P = \prod_{1\le k\le \ell} \big(\partial/\partial u_k\big)^{m_k(m_{k+1}+\cdots+m_\ell)}$$

and apply it to the both sides of (5.6). Then the left hand side of (5.6) becomes

$$\begin{aligned} P\cdot F(u) &= \det\big(D^{I^{(1)}}F^{(1)},\dots,D^{I^{(\ell)}}F^{(\ell)}\big) \\ &= \prod_{1\le k\le\ell}(-1)^{m_k(m_k+1)/2}\prod_{0\le q<m_k}\Big(r+1-\sum_{1\le i\le k} m_i+q\Big)!, \end{aligned} \tag{5.8}$$

where

$$I^{(k)} = \Big(r+1-\sum_{1\le i\le k} m_i,\dots,r+1-\sum_{1\le i\le k} m_i\Big).$$

This is seen by writing down explicitly the matrix $D^{I^{(k)}}F^{(k)}$. On the other hand, the right hand side is

$$P\cdot C_\mu \prod_{1\le i<j\le\ell}(u_i-u_j)^{m_im_j} = C_\mu\prod_{1\le k\le\ell}\{m_k(m_{k+1}+\cdots+m_\ell)\}!.$$

Equating the left and right hand sides, we get (5.5). □

6. Proof of the main theorem

We prove Theorem 4.2 only for $I=\{1,\dots,r\}$ and $J=\{j_1,\dots,j_r\}$ such that $j_1>\cdots>j_r$ in order to avoid the cumbersome complexity of notation. In the expression of A_{IJ} we denote the variable of integration on γ_i by s_i. We put $\mathbf{s}_i=(1,s_i)$. Then we have

$$\begin{aligned} A_{IJ} &= \det\Big(\int_{\gamma_i}\omega_{j_k}\Big) \\ &= \sum_{\sigma\in\mathfrak{S}_r}\operatorname{sgn}(\sigma)\int_{\gamma_1}\omega_{j_{\sigma(1)}}\int_{\gamma_2}\omega_{j_{\sigma(2)}}\cdots\int_{\gamma_r}\omega_{j_{\sigma(r)}} \\ &= \int_{\gamma_1\times\cdots\times\gamma_r}\prod_i\chi(\iota^{-1}(\mathbf{s}_iz);\alpha)\sum_{\sigma\in\mathfrak{S}_r}\operatorname{sgn}(\sigma)s_1^{j_{\sigma(1)}-1}\dots s_r^{j_{\sigma(r)}-1}ds_1\wedge\cdots\wedge ds_r \\ &= \int_{\gamma_1\times\cdots\times\gamma_r}\prod_i\chi(\iota^{-1}(\mathbf{s}_iz);\alpha)\begin{vmatrix} s_1^{j_1-1} & \dots & s_r^{j_1-1} \\ \vdots & & \vdots \\ s_1^{j_r-1} & \dots & s_r^{j_r-1}\end{vmatrix} ds_1\wedge\cdots\wedge ds_r\,. \end{aligned} \tag{6.1}$$

The first term in the integrand is calculated as

LEMMA 6.1. *We have*

$$\prod_i \chi(\iota^{-1}(\mathbf{s}_i z);\alpha) = \chi(\iota^{-1}(\mathbf{t}\Phi_\lambda(z));\alpha)\,, \tag{6.2}$$

where t_i is the i-th elementary symmetric function of $(s_1,\dots,s_r)$ and we put $\mathbf{t} = (1,t_1,\dots,t_r)$.

PROOF. Since χ is a character of $\tilde{H}_\lambda$, we have

$$\prod_i \chi(\iota^{-1}(\mathbf{s}_i z);\alpha) = \chi\left(\prod_i \iota^{-1}(\mathbf{s}_i z);\alpha\right).$$

On the other hand

$$\begin{aligned}\prod_i \iota^{-1}(\mathbf{s}_i z) &= \bigoplus_{1\le k\le \ell}\prod_i \left(\mathbf{s}_i z_0^{(k)} + \mathbf{s}_i z_1^{(k)}\Lambda_k + \cdots + \mathbf{s}_i z_{n_k-1}^{(k)}\Lambda_k^{n_k-1}\right) \\ &= \bigoplus_{1\le k\le \ell} \iota^{-1}(\mathbf{t}\Phi_{n_k}(z^{(k)})) \\ &= \iota^{-1}(\mathbf{t}\Phi_\lambda(z))\,,\end{aligned} \tag{6.3}$$

where $\Lambda_k = (\delta_{i+1,j})_{0\le i,j<n_k}$. Here we used Proposition 5.4. Thus we have (6.2). □

The following lemma can be shown by easy computation. See also [15].

LEMMA 6.2. *We have*

$$\frac{\partial(t_1,\dots,t_r)}{\partial(s_1,\dots,s_r)} = \begin{vmatrix} s_1^{r-1} & \dots & s_r^{r-1} \\ \vdots & & \vdots \\ s_1 & \dots & s_r \\ 1 & \dots & 1 \end{vmatrix}$$

It follows from Lemmas 6.1 and 6.2, the right hand side of (6.1) can be written as

$$\begin{aligned} A_{IJ} &= \int_{\phi_*(\gamma_1\times\cdots\times\gamma_r)} \prod_i \chi(\iota^{-1}(\mathbf{s}_i z);\alpha) \\ &\qquad \times \begin{vmatrix} s_1^{j_1-1} & \dots & s_r^{j_1-1} \\ \vdots & & \vdots \\ s_1^{j_r-1} & \dots & s_r^{j_r-1} \end{vmatrix} \Bigg/ \begin{vmatrix} s_1^{r-1} & \dots & s_r^{r-1} \\ \vdots & & \vdots \\ s_1 & \dots & s_r \\ 1 & \dots & 1 \end{vmatrix} dt_1\wedge\cdots\wedge dt_r \\ &= \int_{\phi_*(\gamma_1\times\cdots\times\gamma_r)} \chi(\iota^{-1}(\mathbf{t}\Phi_\lambda(z));\alpha) S_Y(t)\, dt_1\wedge\cdots\wedge dt_r\,, \end{aligned} \tag{6.4}$$

for the Young diagram $Y = (j_1 - r, j_2 - r + 1, \dots, j_r - 1)$. Thus we have completed the proof of the main theorem. □

References

[1] K. AOMOTO, Les équation aux différences linéaires et les intégrales des fonctions multiformes, J. Fac. Sci. Univ. Tokyo, Sec. IA **22**, (1975), 271–297.

[2] V. ARNOLD, A. VARCHENKO and S. GOUSSEIN-ZADÉ, Singularité des applications différentiables, Vol. 1, 2, Mir Moscou, 1986.

[3] ERDÉLYI et al., *Higher transcendental functions*, Vol. 1, MacGraw-Hill, 1953.

[4] I. M. GELFAND, General theory of hypergeometric functions, Dokl. Akad. Nauk. SSSR **288** (1986), 14–18, English translation: Soviet Math. Dokl. **33** (1986), 9–13.

[5] I. M. GELFAND and S. I. GELFAND, Generalized hypergeometric equations, Dokl. Akad. Nauk. SSSR **288** (1986), 279–283. English transl.: Soviet Math. Dokl. **33** (1986), 643–646.

[6] I. M. GELFAND, V. S. RETAHK and V. V. SERGANOVA, Generalized Airy funcitons, Schublrt cells, and Jordan groups. Dokl. Akad. Nauk. SSSR **298** (1988), 17–21. English transl.: Soviet Math. Dokl. **37** (1988), 8–12.

[7] K.IWASAKI, H. KIMURA, S. SHIMOMURA and M. YOSHIDA, From Gauss to Painlevé, a modern theory of special functions, Vieweg Verlag, Wiesbaden (1991).

[8] H. KIMURA, On rational de Rham cohomology associated with the generalized confluent hypergeometric functions I , $\mathbb{P}^1$ case. Royal Soc. Edinburgh **127A** (1997), 145–155.

[9] H. KIMURA, On rational de Rham cohomology associated with the generalized Airy functions, Annali della Scuola Norm. Sup. di Pisa **24** (1997), 351–366.

[10] H. KIMURA, Y. HARAOKA and K. TAKANO, The generalized confluent hypergeometric functions, Proc. Japan Acad. **68** (1992), 290–295.

[11] H. KIMURA, Y. HARAOKA and K. TAKANO, On confluences of the general hypergeometric systems, Proc. Japan Acad. **69** (1993), 99–104.

[12] H. KIMURA, Y. HARAOKA and K. TAKANO, On contiguity relations of the confluent hypergeometric systems, Proc. Japan Acad. **70** (1994), 47–49.

[13] H. KIMURA and T. KOITABASHI, Normalizer of maximal abelian subgroups of $GL(n)$ and the general hypergeometric functions, Kumamoto J. Math. **9** (1996), 13–43.

[14] F. PHAM, Introduction à l'étude topologique des singularités de Landau, Gauthier-Villars **9** (1967).

[15] T. TERASOMA, Exponential Kummer covering and determinant of hypergeometric function Tokyo J. Math. **16** (1993), 497–508.

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