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Geometric Limits and Length Bounds on Curves

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Abstract. In this paper, we present the new proof of the Length Upper Bounds Theorem on curves in surfaces, which is crucial in the proof of Ending Lamination Conjecture by Minsky et al. Our proof is based on arguments in Bowditch [Bow2] but we use geometric limit arguments fully.

Geometric limits of hyperbolic 3-manifolds describe extremal situations. By studying such limits, we often know the existence of uniform constants which are useful in hyperbolic geometry. Most results derived from geometric limit arguments do not give any computable bounds, but they are not needed for many applications, most notably, the Ending Lamination Conjecture and its consequences. Our plan in this paper and others is to reinterpret recently obtained important results on hyperbolic geometry by mainly using geometric limit arguments. Indeed, Soma [So] is one of papers written along the philosophy. Such reinterpretations will be useful to generalize theorems on hyperbolic 3-manifolds to those on 3-manifolds with pinched negatively curved metric.

The Ending Lamination Conjecture of Thurston [Th2] asserts that any open hyperbolic 3-manifold M with finitely generated fundamental group is determined up to isometry by its end invariants. In the case that $\pi_1(M)$ is isomorphic to the fundamental group of a surface S of finite type, the conjecture is proved by Minsky [Mi2] partially collaborating with Masur, Brock and Canary [MM1, MM2, BCM]. They also announced in [BCM] that the conjecture holds for all hyperbolic 3-manifolds N with $\pi_1(N)$ finitely generated. We refer to [Bow3, BBES, Re, So] for alternative approaches to this conjecture. In Minsky's proof of the conjecture, the a-Priori Bounds Theorem in [Mi2] plays an important role. This theorem shows that, for entries v of the tight geodesics in certain hierarchies on the curve graph $\mathcal{C}(S)$, the length of a closed geodesic in M representing v is uniformly bounded.

The Length Upper Bounds Theorem in Bowditch [Bow2] also presents a uniform bound for the length of closed geodetics in M representing entries of tight geodesics in $\mathcal{C}(F)$ for subsurfaces F of S. His result is essentially equivalent to Minsky's, for example see [Bow2, Section 8]. Bowditch proved his boundedness theorem by studying a nearly geometric limit

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situation of the relevant hyperbolic 3-manifold M_n. In general, the topological types of geometric limit manifolds are very complicated even if all M_n have simple topological types. So, he made a detour to avoid the difficulty. In fact, he stopped just before reaching at the geometric limit and studied the situation by using a certain stability for laminations in M_n and their lifts to the tangent line bundle over M_n.

In this paper, we will present the proof of the Length Upper Bounds Theorem based on that in [Bow2]. However, our proof relies fully on geometric limit arguments which enable us to skip rather harder discussions in [Bow2, Sections 6 and 7]. In our proof, the fact that the topological types of geometric limits are complicated does not matter, but just the existence of the limits does.

When F is either a one-holed torus or a four-holed sphere, the proof of the Length Upper Bounds Theorem in [Bow2] is quite different from that in other cases. In fact, he invoked then trace identities for representations $\pi_1(F) \to \mathrm{PSL}_2(\mathbf{C})$. In this paper, we will use geometric limit arguments even for the exceptional case, which may have an advantage in generalizing the Ending Lamination Conjecture on hyperbolic 3-manifolds to that on pinched negatively curved 3-manifolds. We note that this exceptional case is crucial in the proof of the conjecture in any case.

1. Preliminaries

We refer to Thurston [Th1], Benedetti and Petronio [BP], Matsuzaki and Taniguchi [MT], Marden [Ma] for details on hyperbolic geometry, and to Hempel [He] for those on 3-manifold topology.

Throughout this paper, all manifolds are assumed to be oriented and all homeomorphisms between manifolds orientation-preserving. Moreover, we always suppose that S is a connected surface with hyperbolic structure of finite area. An open subset F of S is an *open geodesic subsurface* (for short *o.g.-subsurface*) of S if each component of the topological boundary ∂F of F in S is a simple closed geodesic of S. In particular, this means that S itself is an o.g.-subsurface of S even if S is a closed surface. The complexity of an o.g.-subsurface F is defined by $\xi(F) = 3g + p - 3$, where g is the genus of F and p is the total number of components of ∂F and cusps of F. When $\xi(F) \geq 2$, we define the *curve graph* $\mathcal{C}(F)$ of F to be the simplicial graph whose vertices are homotopy classes of non-contractible and non-peripheral simple closed curves in F and whose edges are pairs of distinct vertices with disjoint representatives. We simply call a vertex of $\mathcal{C}(F)$ or any representative of the class a *curve* in F. For our convenience, we take a uniquely determined geodesic in F as a representative for any curve in F. The notion of curve graphs is introduced by Harvey [Har] and extended and modified versions are studied by [MM1, MM2, Mi1]. In the case that $\xi(F) = 1$, the curve graph $\mathcal{C}(F)$ is the simplicial graph such that its vertices are curves in F and two curves v, w form the end points of an edge if and only if they have the minimum geometric intersection number $i(v, w)$, that is, $i(v, w) = 1$ when F is a one-holed torus and $i(v, w) = 2$ when F is a four-holed sphere. In either case, $\mathcal{C}(F)$ is supposed to have a path metric such that each edge is isometric to the unit interval [0, 1]. The graph $\mathcal{C}(F)$ is not locally finite but is proved to be δ-hyperbolic by Masur and Minsky [MM2] (see also Bowditch [Bow1]) for some $\delta > 0$. Hence $\mathcal{C}(F)$ has the boundary $\partial\mathcal{C}(F)$ at infinity. The set of vertices in $\mathcal{C}(F)$ is denoted by $\mathcal{C}_0(F)$. We say that the union of $k+1$ elements of $\mathcal{C}_0(F)$ with mutually disjoint representatives is a *k-simplex* in $\mathcal{C}_0(F)$.

DEFINITION 1.1. A sequence $\{v_i\}_{i\in I}$ of simplices in $\mathcal{C}_0(F)$ is called a *tight geodesic* if it satisfies one of the following conditions, where I is a finite or infinite interval in $\mathbf{Z}$.

(i) When $\xi(F) \geq 2$, for any vertices w_i of v_i and w_j of v_j with $i \neq j$, $d(w_i, w_j) = |i-j|$. Moreover, if $\{i-1, i, i+1\} \subset I$, then v_i is represented by the topological boundary ∂F_{i-1}^{i+1} of F_{i-1}^{i+1} in F, where F_{i-1}^{i+1} is the minimum o.g.-subsurface of F containing the geodesic representatives of $v_{i-1} \cup v_{i+1}$.

(ii) When $\xi(F) = 1$, $\{v_i\}_{i\in I}$ is just a geodesic sequence in $\mathcal{C}_0(F)$.

This definition implies that, for a tight geodesic $\{v_i\}$, if a vertex w of $\mathcal{C}(F)$ meets v_i transversely, then w meets at least one of v_{i-1} and v_{i+1} transversely. In fact, this is just a property of tight geodesics which we use in this paper. According to Lemma 5.14 in [Mi1] (see also Theorem 1.2 in [Bow2]), any distinct points of $\mathcal{C}_0(F) \cup \partial\mathcal{C}(F)$ are connected by a tight geodesic in $\mathcal{C}_0(F)$.

A *geodesic pattern* $\mathcal{F}$ on F is a disjoint family of simple closed geodesics and connected o.g.-subsurfaces J in F with $\xi(J) \geq 1$. The notion of geodesic patterns is essentially same to that of efficient subsurfaces in F introduced by [Bow2]. However, components of an efficient subsurface are not necessarily supposed to be geodesic. By requiring any elements of $\mathcal{F}$ to be geodesic, one can determine them uniquely in their homotopy classes. For any geodesic pattern $\mathcal{F}$ on F, the union $\bigcup\mathcal{F} = F_1 \cup \cdots \cup F_n$ $(F_i \in \mathcal{F})$ is a subset of F. The distance of two geodesic patterns $\mathcal{F}, \mathcal{F}'$ on F is defined as $d(\mathcal{F}, \mathcal{F}') = \min\{d(v, v')\}$, where v (resp. v') ranges over simple geodesic loops contained in elements of $\mathcal{F}$ (resp. of $\mathcal{F}'$). Note that a curve in $\bigcup\mathcal{F}$ is not necessarily contained in an element of $\mathcal{F}$. Indeed, for any non-separating simple geodesic loop l in F, the union $\bigcup\mathcal{F}$ of the geodesic pattern $\mathcal{F} = \{l, F\setminus l\}$ is F itself and hence contains a simple geodesic loop not in either l or $F \setminus l$. Two geodesic patterns $\mathcal{F}, \mathcal{F}'$ on F are said to be *compatible* if the union $\mathcal{F} \cup \mathcal{F}'$ is also a geodesic pattern on F. A finite sequence $\{\mathcal{F}_i\}_{i=1}^p$ of non-empty geodesic patterns on F is *compatible* if $\mathcal{F}_i$ and $\mathcal{F}_{i+1}$ are compatible for any $i \in \{1, \ldots, p-1\}$. A compatible sequence $\{\mathcal{F}_i\}_{i=1}^p$ is *taut* if $\mathcal{F}_i \subset \mathcal{F}_{i-1} \cup \mathcal{F}_{i+1}$ for any $i \in \{1, \ldots, p\}$, where $\mathcal{F}_0 = \mathcal{F}_{p+1} = \emptyset$. The definition of a taut sequence implies that $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{F}_p \subset \mathcal{F}_{p-1}$, and hence in particular $p \geq 2$.

The following lemma given in [Bow2, Lemma 2.1] plays an important role in Bowditch's proof of the Length Upper Bounds Theorem and also in ours.

LEMMA 1.2 (2/3-Lemma). *For any taut sequence $\{\mathcal{F}_i\}_{i=1}^p$ of geodesic patterns on F, $d(\mathcal{F}_1, \mathcal{F}_p) \leq \left[\frac{2}{3}p\right] - 1$ holds.*

A non-empty compact subset λ of F is a *lamination* on F if λ is a union of mutually disjoint simple geodesics, called *leaves*, in F. We say that a lamination is *minimal* if it contains no proper sublaminations. Any lamination λ contains at most finitely many minimal laminations, which are mutually disjoint. The union of such minimal laminations is denoted by $\lambda_{\min}$.

For an $\varepsilon > 0$, the *ε-thin part* of a hyperbolic 3-manifold M is denoted by $M_{(0,\varepsilon]}$, that is, $M_{(0,\varepsilon]}$ is the set of points x in M admitting a non-contractible loop in M of length at most 2ε and passing through x. The *ε-thick part* $M_{[\varepsilon,\infty)}$ is the closure of $M \setminus M_{(0,\varepsilon]}$ in M. The ε-thin part $F_{(0,\varepsilon]}$ and thick part $F_{[\varepsilon,\infty)}$ of an o.g.-subsurface F of S are defined similarly. According to the Margulis Lemma, there exists a uniform constant $\varepsilon_0 > 0$ independent of M, called a *Margulis constant*, such that each component of $M_{(0,\varepsilon]}$ is either a solid torus with geodesic core, called a *Margulis tube*, or a parabolic cusp if $\varepsilon < \varepsilon_0$. This constant works also for S.

If $0 < \varepsilon < \varepsilon_0$ is taken sufficiently small, then $S_{(0,\varepsilon]}$ consists of parabolic cusps. Fix such an ε. We suppose that any hyperbolic 3-manifold M in this paper other than geometric limit manifolds admits a homeomorphism $h : M \to S \times \mathbf{R}$ such that $h^{-1}(S_{(0,\varepsilon]} \times \mathbf{R})$ is a union of parabolic cusps of M. The thin part $M_{(0,\varepsilon]}$ may contain parabolic cusps disjoint from $h^{-1}(S_{(0,\varepsilon]} \times \mathbf{R})$, which are called *accidental parabolic cusps* of M. The composition $\pi = \mathrm{pr} \circ h : M \to S$ is called a *marking* of M, where $\mathrm{pr} : S \times \mathbf{R} \to S$ is the direct projection to the first factor. Throughout the remainder of this paper, we assume that any hyperbolic 3-manifold M homeomorphic to $S \times \mathbf{R}$ is equipped with a marking. For an o.g.-subsurface F of S, a continuous map $f : F \to M$ is said to be *marking-preserving* if $\pi \circ f$ is homotopic to the inclusion $F \subset S$. Then, for any simple essential loop v in F, $v^{\natural}$ denotes the geodesic loop in M freely homotopic to $f(v)$ if any. Otherwise, $v^{\natural}$ represents the end of the parabolic cusp of M to which $f(v)$ is freely homotopic in M. We denote the M-length of $v^{\natural}$ by $l_M(v)$ if $v^{\natural}$ is a geodesic loop and set $l_M(v) = 0$ if $v^{\natural}$ is a parabolic cusp end. If w is a union of mutually disjoint and non-parallel simple essential loops $v_1, \dots, v_n$ in F, we set $l_M(w) = l_M(v_1) + \cdots + l_M(v_n)$.

Let λ be a geodesic lamination on an o.g.-subsurface F with $\xi(F) \geq 1$ and let μ be the union of loop components of λ corresponding to accidental parabolic cusps of M. A marking-preserving continuous map $\varphi : F \setminus \mu \to M$ is a *pleated map* realizing λ if it satisfies the following conditions.

- For each component H of $F \setminus \mu$, the restriction $\varphi|_H$ is a proper map sending each end of F to either a parabolic cusp of M or a geodesic loop in M.
- There exists a lamination ν_H on H containing $\lambda \cap H$ such that the restriction of φ on any leaf l of ν_H or any component of $H \setminus \nu_H$ is a totally geodesic immersion.

The union $\mu \cup \left(\bigcup_H \nu_H\right)$ is called a *pleating locus* of φ, where H ranges all components of $F \setminus \mu$. Then $F(\sigma) \setminus \mu$ means that $F \setminus \mu$ has the hyperbolic metric σ induced from that on M via φ. The length of any geodesic loop v in $F(\sigma) \setminus \mu$ is denoted by $l_{F(\sigma)\setminus\mu}(v)$ (or $l_\sigma(v)$ for short).

Let λ be a connected lamination in S. When λ is not a geodesic loop, we say that a connected o.g.-subsurface F in S *supports* λ if F contains λ and each component of $F \setminus \lambda$ is either an open disk or an open annulus. When λ is a geodesic loop, we suppose that λ is equal to its support. The support of λ is determined uniquely.

Though the following lemma is probably well known, the author does not know any suitable reference. So he presents the proof.

LEMMA 1.3. *Let F be the support of a non-loop connected lamination λ in S and $\varphi : F \to M$ a pleated surface realizing λ. If a geodesic segment α in M is not contained in the φ-image of any leaf of λ, then* $\mathrm{meas}_\alpha(\alpha \cap \varphi(\lambda)) = 0$. *In particular, if $\alpha \subset \varphi(\lambda)$, then α is contained in the φ-image of some leaf of λ.*

Here $\mathrm{meas}_a(\cdot)$ denotes the one-dimensional Lebesgue measure on a segment a with Riemannian metric.

PROOF. Let α be a geodesic segment in M which is not contained in the φ-image of any leaf of λ. We suppose that $\mathrm{meas}_\alpha(\alpha \cap \varphi(\lambda)) > 0$ and will derive a contradiction. Consider the natural lift $\boldsymbol{p} : \lambda \to \boldsymbol{P}(M)$ of $\varphi|_\lambda$ to the tangent line bundle $\boldsymbol{P}(M)$ over M. It is well known that $\boldsymbol{p}$ is a homeomorphism onto $\boldsymbol{p}(\lambda)$, for example see [Th3, Theorem 5.6] or [CEG, Subsection I.5.3]. For any sufficiently small $\varepsilon > 0$, one can take a subset τ of $\alpha \cap \varphi(\lambda)$ satisfying the following conditions.

(i) There exists a subsegment α_0 of α containing τ with $\mathrm{meas}_{\alpha_0}(\tau) > 0$ and $\mathrm{length}_M(\alpha_0) < \varepsilon$.

(ii) For each $x \in \tau$, there exists a vector $\boldsymbol{l}_x$ of $\boldsymbol{p}(\lambda)$ tangent to M at x with $\mathrm{diam}_{\boldsymbol{P}(M)}\{\boldsymbol{l}_x\,;\, x \in \tau\} < \varepsilon$ for a fixed Riemannian metric on $\boldsymbol{P}(M)$.

Since $\boldsymbol{p}$ is a homeomorphism to its image, one can choose $\varepsilon > 0$ so that $Y = \{\boldsymbol{p}^{-1}(\boldsymbol{l}_x)\,;\, x \in \tau\}$ is contained in an embedded open disk U in F with arbitrarily small radius. There exists a rectangle R in F which contains the closure $\overline{Y}$ of Y in F and has four sides a_1, a_2, b_1, b_2 such that a_i $(i = 1, 2)$ is a segment contained in a leaf of λ with $a_i \cap \overline{Y} \neq \emptyset$, b_j $(j = 1, 2)$ is a geodesic segment meeting $a_1 \cup a_2$ almost orthogonally and $\max\{\mathrm{length}_F(b_j)\}/\min\{\mathrm{length}_F(a_i)\}$ is sufficiently small, see Fig. 1.1. For any point y of $\overline{Y}$, the φ-image of the leaf l of $\lambda \cap R$ containing y meets α_0 transversely at $\varphi(y)$. If necessary replacing ε and R by smaller ones, one can suppose that all such leaves meet α_0 in single points. Consider another geodesic segment c in R meeting both a_1, a_2 almost orthogonally such that $\mathrm{dist}_R(c, b_1)/\mathrm{dist}_R(c, b_2)$ is sufficiently close to one. From our construction, any leaf l of λ with $l \cap R \neq \emptyset$ meets c almost orthogonally.

Now we define a Lipschitz map $f : c \to \alpha_0$ as follows. Let $\pi : \lambda \cap R \to c$ be the projection along the leaves of $\lambda \cap R$ and $d = \pi(\overline{Y})$. For any $z \in d$, $f(z)$ is a unique intersection point of $\varphi(\pi^{-1}(z))$ and α_0. The complement $c \setminus d$ consists of countably many open segments ι_n. We define $f|_{\bar{\iota}_n}$ to be an affine map onto the subsegment of α_0 bounded by $f(\partial\bar{\iota}_n)$. For any two points $w, w' \in d$, let u_i, u_i' $(i = 1, 2)$ be the points in b_i with $\pi(u_i) = w$, $\pi(u_i') = w'$. Since any leaves of $\lambda \cap R$ meet $b_1 \cup b_2$ almost orthogonally, by applying elementary hyperbolic geometry we have a constant $K > 0$ independent of the

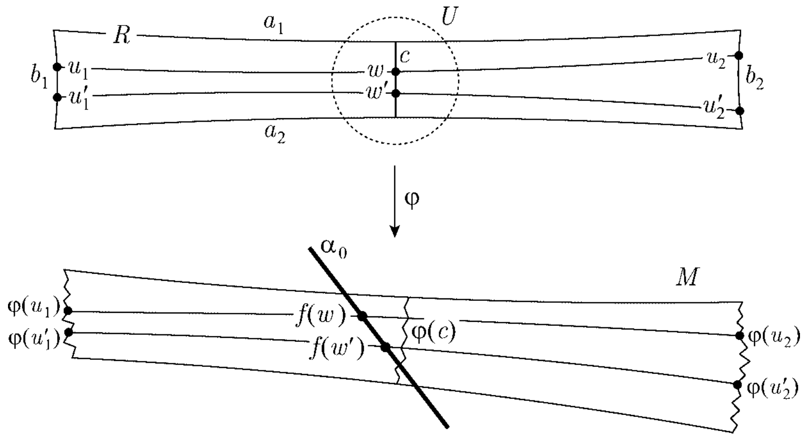

FIGURE 1.1

choice of w, w' with

$$K\,\mathrm{dist}_c(w, w') \geq \mathrm{dist}_F(u_1, u'_1) + \mathrm{dist}_F(u_2, u'_2)\,.$$

Since φ is a pleated map, $\mathrm{dist}_M(\varphi(u_i), \varphi(u'_i)) \leq \mathrm{dist}_F(u_i, u'_i)$. Since we took $\varepsilon > 0$ sufficiently small, by the condition (ii) the angle formed by $\varphi(\pi^{-1}(z))$ and α_0 at $f(z)$ for any $z \in d$ is uniformly bounded away from zero (or π). It follows that there exists a constant $K' > 0$ independent of the choice of w, w' with

$$\mathrm{dist}_{\alpha_0}(f(w), f(w')) \leq K'\big(\mathrm{dist}_M(\varphi(u_1), \varphi(u'_1)) + \mathrm{dist}_M(\varphi(u_2), \varphi(u'_2))\big)\,.$$

This implies that f is a KK'-Lipschitz map. Note that λ has zero two-dimensional Lebesgue measure in F, for example see [Th1, Subsection 8.5]. It follows from Fubini's Theorem that $\mathrm{meas}_c(d) = 0$. Since f is Lipschitz and $\tau \subset f(d)$, we also have $\mathrm{meas}_{\alpha_0}(\tau) = 0$. This contradicts the condition (i) and hence completes the proof. □

LEMMA 1.4. *For $i = 1, 2$, let F_i be the support of a connected lamination λ_i in S and $\varphi_i : F_i \to M$ a pleated surface realizing λ_i. If $\varphi_1(\lambda_1) = \varphi_2(\lambda_2)$ in M, then (F_1, λ_1) is isotopic to (F_2, λ_2) in S.*

PROOF. When λ_i are loops, the proof is obvious. So, we may assume that λ_i are not loops. Let $\boldsymbol{p}_i : \lambda_i \to \boldsymbol{P}(M)$ $(i = 1, 2)$ be the homeomorphism defined as above. The assumption $\varphi_1(\lambda_1) = \varphi_2(\lambda_2)$ together with the particular case of Lemma 1.3 implies $\boldsymbol{p}_1(\lambda_1) = \boldsymbol{p}_2(\lambda_2)$. Thus, $h_0 = \boldsymbol{p}_2^{-1} \circ \boldsymbol{p}_1 : \lambda_1 \to \lambda_2$ is a well-defined homeomorphism with $\varphi_2 \circ h_0 = \varphi_1|_{\lambda_1}$.

The total space $\widetilde{F}_i$ of the universal covering $q_i : \widetilde{F}_i \to F_i$ is realized as a convex subset of $\mathbf{H}^2$. Let $\mathcal{D}_i$ be the set of all simply connected components of $F_i \setminus \lambda_i$. Each element Δ_1 of $\mathcal{D}_1$ is lifted to a polygon $\widetilde{\Delta}_1$ in $\widetilde{F}_1$ with ideal vertices whose boundary $\widetilde{\Lambda}_1$ is a union of finitely many geodesic lines. Set $\widetilde{\lambda}_i = q_i^{-1}(\lambda_i)$ $(i = 1, 2)$ and $\Lambda_1 = q_1(\widetilde{\Lambda}_1)$. Since $\varphi_1|_{\Lambda_1} = \varphi_2 \circ h_0|_{\Lambda_1}$ extends to a map from Δ_1 to M, it follows from the π_1-injectivity of φ_2 that $\Lambda_2 = h_0(\Lambda_1)$ is contractible in F_2. Thus Λ_2 is lifted to a union $\widetilde{\Lambda}_2$ of geodesic lines in $\widetilde{F}_2$ bounding a polygon $\widetilde{\Delta}_2$. If $\widetilde{\Delta}_2 \cap \widetilde{\lambda}_2$ were not empty, then $\widetilde{\lambda}_2$ would contain a geodesic line which divides $\widetilde{\Delta}_2$ into two polygons. For the side $\widetilde{L}_2$ of one of them, $L_1 = h_0^{-1} \circ q_2(\widetilde{L}_2)$ has a lift $\widetilde{L}_1$ in $\widetilde{F}_1$ which bounds a proper subpolygon of $\widetilde{\Delta}_1$. This contradicts that $\widetilde{\Delta}_1 \cap \widetilde{\lambda}_1 = \emptyset$. Thus we have $\widetilde{\Delta}_2 \cap \widetilde{\lambda}_2 = \emptyset$ and hence $q_2|_{\widetilde{\Delta}_2} : \widetilde{\Delta}_2 \to F_2$ is an embedding the image of which is a component Δ_2 of $\mathcal{D}_2$. Using this fact repeatedly, one can extend h_0 to a homeomorphism $h_1 : \lambda_1 \cup \left(\bigcup \mathcal{D}_1\right) \to \lambda_2 \cup \left(\bigcup \mathcal{D}_2\right)$ such that $\varphi_2 \circ h_1$ is homotopic to $\varphi_1|_{\lambda_1 \cup (\bigcup \mathcal{D}_1)}$ rel. λ_1. Moreover, h_1 is extended to a homeomorphism h_2 between small regular neighborhoods C_i $(i = 1, 2)$ of $\lambda_i \cup \left(\bigcup \mathcal{D}_i\right)$ in F_i. Since each component of $F_i \setminus C_i$ is an open annulus, h_2 is also extended to a homeomorphism $h_3 : F_1 \to F_2$ such that $\varphi_2 \circ h_3$ is homotopic to φ_1 rel. λ_1. Since the pleated maps φ_i preserve the markings of F_i and M, h_3 also preserves the markings of F_1 and F_2. This shows that (F_1, λ_1) is isotopic to (F_2, λ_2) in S. □

2. Quasi-convexity Theorem

For $L > 0$, let $\mathcal{C}_{0M}(F, L)$ be the subset of $\mathcal{C}_0(F)$ consisting of elements v with $l_M(v) \leq L$ and $\mathcal{C}_M(F, L)$ the maximal subgraph of $\mathcal{C}(F)$ with vertex set $\mathcal{C}_{0M}(F, L)$. A non-empty subset Y of a geodesic metric space X is *r-quasi-convex* for an $r > 0$ if any geodesic in X connecting two points of Y is contained in the r-neighborhood of Y in X. Minsky proved the Quasi-convexity Theorem (Theorem 3.1 in [Mi1]) by using standard arguments of hyperbolic geometry and geometric group theory, which says that the subgraph $\mathcal{C}_M(F, L_1)$ for some $L_1 > 0$ is uniformly quasi-convex in $\mathcal{C}(F)$.

Let F be a connected o.g.-subsurface of S with $\xi(F) \geq 1$ and $l_F(\partial F) \leq L$ for a given $L > 0$. Any *uniform constant* in the remainder of this paper means a number depending only on L, $\xi(F)$ (and previously determined uniform constants). According to Bers [Be], there exists a uniform constant $L_1 \geq L$ (and hence independent of the hyperbolic metric on F) such that there exists a disjoint union v of simple geodesic loops in F with $l_F(v) \leq L_1$ and such that each component of $F \setminus v$ is a three-holed sphere. From now on, we fix $L_1 = L_1(L, \xi(F))$ satisfying this condition.

THEOREM 2.1 (Quasi-convexity Theorem [Mi1]). *For any $L > 0$ with $l_M(\partial F) \leq L$, there exists a uniform constant $r > 0$ such that $\mathcal{C}_M(F, L_1)$ is r-quasi-convex in $\mathcal{C}(F)$.*

We say that a pleated map $\varphi : F(\sigma) \setminus \mu \to M$ realizing λ is *ε-stable* if it satisfies the following conditions.

- Each component of $\lambda \setminus \mu$ is either a geodesic core of some annulus component of $(F(\sigma) \setminus \mu)_{(0,\varepsilon]}$ or disjoint from $(F(\sigma) \setminus \mu)_{(0,\varepsilon]}$.
- The geodesic core c of any component of $(F(\sigma) \setminus \mu)_{(0,\varepsilon]}$ is contained in the pleating locus of φ, and hence in particular $l_\sigma(c) = l_M(c)$.

Then the union $\mu^{(\varepsilon)}$ of μ and all such geodesic cores in the ε-thin part of $F(\sigma) \setminus \mu$ is called the *ε-thin locus* of the ε-stable pleated map φ. A lamination in F is said to be *ε-stable* in M if it is realized by an ε-stable pleated map from F to M. Let $\mathcal{C}_M(F, L)$ be the maximal subgraph of $\mathcal{C}(F)$ whose vertex set consists of curves v with $l_M(v) \leq L$.

As an application of Theorem 2.1, Tube Penetration Lemma for F with $\xi(F) \geq 2$ is given in [Mi2, Lemma 7.7] and also in [Bow2, Lemmas 5.1, 5.2].

LEMMA 2.2 (Tube Penetration Lemma). *Suppose that $\xi(F) \geq 2$. For any $0 < \varepsilon < \varepsilon_0$, $r \geq 0$, $L > 0$, there exists a constant $\varepsilon' = \varepsilon'(\varepsilon, r, L, \xi(F))$ with $0 < \varepsilon' < \varepsilon$ and satisfying the following condition. If $\{v^i\}_{i=0}^p$ is a tight geodesic in $\mathcal{C}_0(F)$ such that v^k ($k = 0, p$) is ε-stable in M and satisfies $d(v^k, \mathcal{C}_M(F, L)) \leq r$, then v^j is ε'-stable in M for any $j \in \{1, \dots, p-1\}$.*

We note that the constant ε' is independent of the length p of the tight geodesic $\{v^i\}_{i=0}^p$.

3. Geometric limits

Let M_n ($n \in \mathbf{N}$) be hyperbolic 3-manifolds with markings $\pi_n : M_n \to S$ and base points $x_n \in M_n$. We say that the sequence $\{(M_n, x_n)\}$ converges *geometrically* to a hyperbolic 3-manifold (M_∞, x_∞) with base point if there exist monotone decreasing and increasing sequences $\{K_n\}$, $\{R_n\}$ with $\lim_{n\to\infty} K_n = 1$, $\lim_{n\to\infty} R_n = \infty$ and K_n-bi-Lipschitz maps

$$g_n : \mathcal{N}_{R_n}(x_n, M_n) \to \mathcal{N}_{R_n}(x_\infty, M_\infty)\,, \tag{3.1}$$

where $\mathcal{N}_R(x, M)$ denotes the closed R-neighborhood of x in M. It is well known that, if $\inf\{\mathrm{inj}_{M_n}(x_n)\} > 0$, then $\{(M_n, x_n)\}$ has a geometrically convergent subsequence, for example see [JM, BP]. In this case, we say that $\{(M_n, x_n)\}$ *subconverges geometrically* to (M_∞, x_∞) and denote the subsequence again by $\{(M_n, x_n)\}$ for simplicity. The limit manifold M_∞ is not necessarily homeomorphic to $S \times \mathbf{R}$. In general, M_∞ has infinitely many ends. Infinitely many of them may not be topologically tame, that is, any neighborhood of such an end in M is not homeomorphic to the direct product of a surface and $\mathbf{R}$, see [OS].

Suppose that F is a connected o.g.-subsurface of S and $\varphi_n : F(\sigma_n) \setminus \mu_n \to M_n$ ($n \in \mathbf{N}$) are ε-stable pleated maps with ε-thin loci $\mu_n^{(\varepsilon)}$ and $l_{M_n}(\partial F) = l_{\sigma_n}(\partial F) \leq L$ for any $n \in \mathbf{N}$. If necessary passing to a subsequence, we may assume that $F \setminus \mu_n^{(\varepsilon)}$ are homeomorphic to each other. Then there exist homeomorphisms $\eta_n : F(\sigma) \setminus \mu^{(\varepsilon)} \to F_n(\sigma_n) \setminus \mu_n^{(\varepsilon)}$ which are C_n-bi-Lipschitz with $\sup_n\{C_n\} < \infty$ on any compact subset of $F \setminus \mu^{(\varepsilon)}$, where $\mu^{(\varepsilon)} = \mu_1^{(\varepsilon)}$ and $\sigma = \sigma_1$. Let $H_1, \dots, H_m$ be the components of $F \setminus \mu^{(\varepsilon)}$. For each $k \in \{1, \dots, m\}$, suppose that $x_{n,k} = \varphi_n \circ \eta_n(y_k)$ is the base point of M_n for a fixed $y_k \in H_{k\,[\varepsilon,\infty)}$. Let $g_{n,k} : \mathcal{N}_{R_n}(x_{n,k}, M_n) \to \mathcal{N}_{R_n}(x_{\infty,k}, M_{\infty,k})$ be a K_n-bi-Lipschitz map as above. Since the diameters of $\eta_n(H_{k\,[\varepsilon,\infty)})$ are uniformly bounded, for all sufficiently large $n \in \mathbf{N}$, $g_{n,k} \circ \varphi_n \circ \eta_n|_{H_{k\,[\varepsilon,\infty)}} : H_{k\,[\varepsilon,\infty)} \to M_{\infty,k}$ is well defined. By the Ascoli-Arzelà Theorem, $\{g_{n,k} \circ \varphi_n \circ \eta_n|_{H_{k\,[\varepsilon,\infty)}}\}$ have a subsequence converging uniformly to a map from $H_{k\,[\varepsilon,\infty)}$ to $M_{\infty,k}$ which is extended to a pleated map $\psi_k : H_k(\sigma_{\infty,k}) \to M_{\infty,k}$ such that the ε-thin part of $H_k(\sigma_{\infty,k})$ does not contain non-peripheral components. When $\sup_n\{\mathrm{dist}_{M_n}(x_{n,k}, x_{n,l})\} < \infty$, one can identify $M_{\infty,k}$ with $M_{\infty,l}$. Otherwise, we suppose that $M_{\infty,k} \cap M_{\infty,l} = \emptyset$. Let N_∞ be a maximal union of $M_{\infty,k}$'s which are not identified with each other. Then we say that $\{\varphi_n \circ \eta_n\}$ *subconverges geometrically* to the ε-stable pleated map $\psi : F(\sigma_\infty) \setminus \mu^{(\varepsilon)} \to N_\infty$ with $\psi|_{H_k} = \psi_k$, where $F(\sigma_\infty) \setminus \mu^{(\varepsilon)}$ is the disjoint union $H_1(\sigma_{\infty,1}) \cup \cdots \cup H_m(\sigma_{\infty,m})$.

4. Length Upper Bounds Theorem (Non-exceptional case)

Throughout this section, we suppose that F is a connected o.g.-subsurface of S with $\xi(F) \geq 2$ (possibly $F = S$) and M is a hyperbolic 3-manifold with marking $\pi : M \to S$.

The proofs of the following two lemmas are based on those of [Bow2, Theorem 1.1 and Lemmas 8.1, 8.2].

LEMMA 4.1. *Let p be any integer with $p \geq 2$ and L_2 any positive number with $l_M(\partial F) \leq L_2$. Then there exists a constant L_2' depending only on p, L_2, $\xi(F)$ and satisfying the following condition. Suppose that $g = \{v^i\}_{i=0}^q$ is any tight geodesic in $\mathcal{C}_0(F)$ of length $q \leq p$ with $v^0, v^q \in \mathcal{C}_M(F, L_2)$. Then $l_M(v^i) \leq L_2'$ for all $i \in \{0, 1, \ldots, q\}$.*

Before getting to the formal proof, we will explain the special case of $q = 3$ roughly. Suppose that there exists a sequence $\{M_n\}_{n=1}^\infty$ of hyperbolic 3-manifolds with S-markings and tight geodesics $\{v_n^0, v_n^1, v_n^2, v_n^3\}$ in $\mathcal{C}(F)$ with $v_n^0, v_n^3 \in \mathcal{C}_{M_n}(F, L_2)$ and $\lim_{n\to\infty} l_{M_n}(v_n^i) = \infty$ for $i = 1, 2$. If necessary passing to subsequences, we may assume that $\{v_n^0 \cup v_n^1\}$, $\{v_n^1 \cup v_n^2\}$, $\{v_n^2 \cup v_n^3\}$ converge (up to marking) geometrically to laminations $v^0 \cup \underline{\nu}^1$, $\overline{\nu}^1 \cup \underline{\nu}^2$, $\overline{\nu}^2 \cup v^3$ in F respectively which are realized in a geometric limit M_∞ of $\{M_n\}$, where v^0, v^3 are geodesic loops in F and all $\underline{\nu}^i$, $\overline{\nu}^j$ are non-loop laminations. These laminations are pulled back to laminations $v_n^0 \cup \underline{\nu}_n^1$, $\overline{\nu}_n^1 \cup \underline{\nu}_n^2$, $\overline{\nu}_n^2 \cup v_n^3$ realized in M_n via a bi-Lipschitz map g_n as (3.1). By using Lemma 1.4, one can show that $\underline{\nu}_n^1 = \overline{\nu}_n^1 (:= \nu_n^1)$ and $\underline{\nu}_n^2 = \overline{\nu}_n^2 (:= \nu_n^2)$. We assume here that ν_n^1, ν_n^2 are minimal for simplicity. Since ν_n^1, ν_n^2 are sublaminations of $\nu_n^1 \cup \nu_n^2$, the minimality condition implies either $\nu_n^1 = \nu_n^2$ or $\nu_n^1 \cap \nu_n^2 = \emptyset$, and hence the supports F_n^i of ν_n^i $(i = 1, 2)$ satisfy either $F_n^1 = F_n^2$ or $F_n^1 \cap F_n^2 = \emptyset$. Since $v_n^0 \cap \nu_n^1 = \emptyset$ and $\nu_n^2 \cap v_n^3 = \emptyset$, $v_n^0 \cap F_n^1 = \emptyset$ and $F_n^2 \cap v_n^3 = \emptyset$. It is not hard to show that v_n^i $(i = 1, 2)$ is contained in F_n^i for all sufficiently large n. Let β be a simple geodesic loop in F_n^1 crossing v_n^1 transversely. Thus, if $F_n^1 \cap F_n^2 = \emptyset$, then $\beta \cap (v_n^0 \cup v_n^2) = \emptyset$. This contradicts that $\{v_n^i\}$ is a tight geodesic. On the other hand, if $F_n^1 = F_n^2$, then $d_{\mathcal{C}(F)}(v_n^0, \beta) = 1$ and $d_{\mathcal{C}(F)}(\beta, v_n^3) = 1$ and hence $d_{\mathcal{C}(F)}(v_n^0, v_n^3) \leq 2$. This also contradicts that $d_{\mathcal{C}(F)}(v_n^0, v_n^3) = 3$. It follows that at least one of $\{l_{M_n}(v_n^1)\}$ and $\{l_{M_n}(v_n^2)\}$ is bounded. When the former is bounded, by applying a similar argument to the tight geodesics $\{v_n^1, v_n^2, v_n^3\}$ one can show that the latter is also bounded.

PROOF OF LEMMA 4.1. We suppose that the conclusion fails and will derive a contradiction. Then there exist a sequence $\{M_n\}_{n=1}^{\infty}$ of hyperbolic 3-manifolds with markings $\pi_n : M_n \to S$, a sequence $\{F_n\}$ of connected o.g.-subsurfaces of S with $\xi(F_n) = \xi(F)$, $l_{M_n}(\partial F_n) \le L_2$ and tight geodesic sequences $g_n = \{v_n^i\}_{i=0}^{q_n}$ with $q_n \le p$, $l_{M_n}(v_n^0) \le L_2$, $l_{M_n}(v_n^{q_n}) \le L_2$ and $l_{M_n}(v_n^i) \ge n$ for some $i \in \{1, \dots, q_n - 1\}$. If necessary passing to a subsequence, we may assume that $q_n = q$ and that there exist consecutive indices $s, s+1, \dots, t$ of maximal length in $\{1, \dots, q-1\}$ with $\lim_{n\to\infty} l_{M_n}(v_n^i) = \infty$ for any $i \in \{s, s+1, \dots, t\}$. The maximality implies that $\sup_n\{l_{M_n}(v_n^{s-1})\} < \infty$ and $\sup_n\{l_{M_n}(v_n^{t+1})\} < \infty$. We may also assume that, for each i, v_n^i is divided into two unions u_n^i, w_n^i of curves such that the M_n-lengths of u_n^i are uniformly bounded and the M_n-length of every component of w_n^i diverges to infinity as $n \to \infty$. For $i = s, \dots, t$, let $\varphi_n^i : F_n \setminus \mu_n^i \to M_n$ be a pleated map realizing $v_n^{i-1} \cup v_n^i$. By Lemma 2.2, there exists a small constant $\varepsilon > 0$ independent of n such that the φ_n^i are ε-stable for any $i \in \{s, \dots, t\}$. Then the ε-thin locus μ_n^i of φ_n^i can be defined.

If necessary passing to a subsequence again, we may assume that the pairs (F_n, μ_n^i) ($n \in \mathbf{N}$) are all homeomorphic for each $i \in \{s, \dots, t\}$. As was seen in Section 3, there exist homeomorphisms $\eta_n^i : F \setminus \mu^i \to F_n \setminus \mu_n^i$ such that $\{\varphi_n^i \circ \eta_n^i\}$ subconverges geometrically to an ε-stable pleated map $\psi^i : F \setminus \mu^i \to N_\infty^i$. Consider the disjoint union $\underline{w}_n^i$ of simple geodesic loops in $F \setminus \mu^i$ such that $\eta_n^i(\underline{w}_n^i)$ is freely homotopic to w_n^i in $F_n \setminus \mu_n^i$. Similarly, let $\overline{w}_n^{i-1}$ be the disjoint union of simple geodesic loops in $F \setminus \mu^i$ such that $\eta_n^i(\overline{w}_n^{i-1})$ is freely homotopic to w_n^{i-1} in $F_n \setminus \mu_n^i$. The sequence $\{\underline{w}_n^i\}$ (resp. $\{\overline{w}_n^{i-1}\}$) subconverges geometrically to a lamination $\underline{\nu}^i$ (resp. $\overline{\nu}^{i-1}$) in $F \setminus \mu^i$ which is realized by ψ^i. Since $\lim_{n\to\infty} l_{M_n}(w_{n,a}^i) = \infty$ for any components $w_{n,a}^i$ of w_n^i ($n \in \mathbf{N}$), no component of $\overline{\nu}^{i-1} \cup \underline{\nu}^i$ is a loop. Let $\underline{\mathcal{G}}^i$ be the geodesic pattern on $F \setminus \mu^i$ consisting of elements supporting the components of $\underline{\nu}^i$ and $\underline{\mathcal{F}}^i$ the geodesic pattern consisting of elements supporting the components of $\underline{\nu}_{\min}^i$. The geodesic patterns $\overline{\mathcal{G}}^{i-1}$ and $\overline{\mathcal{F}}^{i-1}$ in $F \setminus \mu^i$ are defined similarly. Since $(\overline{\nu}^{i-1} \cup \underline{\nu}^i)_{\min}$ is a lamination containing both $\overline{\nu}_{\min}^{i-1}$ and $\underline{\nu}_{\min}^i$ as unions of components, $\overline{\mathcal{F}}^{i-1}$ and $\underline{\mathcal{F}}^i$ are compatible in $F \setminus \mu^i$. For any $n \in \mathbf{N}$, let $\underline{\mathcal{G}}_n^i$ (resp. $\underline{\mathcal{F}}_n^i$) be the geodesic pattern in $F_n \setminus \mu_n^i$ each element of which is freely homotopic to the η_n^i-image of the corresponding element of $\underline{\mathcal{G}}^i$ (resp. $\underline{\mathcal{F}}^i$). The geodesic patterns $\overline{\mathcal{G}}_n^{i-1}$ and $\overline{\mathcal{F}}_n^{i-1}$ in $F_n \setminus \mu_n^i$ are defined similarly. We note that

$$\bigcup \underline{\mathcal{G}}_n^i \subset F_n \setminus \mu_n^i \quad \text{and} \quad \bigcup \overline{\mathcal{G}}_n^i \subset F_n \setminus \mu_n^{i+1}.$$

Now we will show that $\underline{\mathcal{G}}_n^i = \overline{\mathcal{G}}_n^i$ and $\underline{\mathcal{F}}_n^i = \overline{\mathcal{F}}_n^i$ for any $i \in \{s, \dots, t\}$ and all sufficiently large n. Let $\underline{\lambda}^i$ be any component of $\underline{\nu}^i$ and $\{\underline{c}_n^i\}$ a sequence of unions of components of $\underline{w}_n^i$ converging geometrically to $\underline{\lambda}^i$. Suppose that $\overline{c}_n^i$ is the union of components of $\overline{w}_n^i$ such that $\eta_n^{i+1}(\overline{c}_n^i)$ is freely homotopic to $\eta_n^i(\underline{c}_n^i)$ in F_n. Since the closed geodesic $\eta_n^i(\underline{c}_n^i)^\natural$ in M_n is equal to the closed geodesic $\eta_n^{i+1}(\overline{c}_n^i)^\natural$, $\{\overline{c}_n^i\}$ subconverges geometrically to a sublamination $\overline{\lambda}^i$ of $\overline{\nu}^i$ in $F \setminus \mu^{i+1}$ with $\psi^i(\underline{\lambda}^i) = \psi^{i+1}(\overline{\lambda}^i)$ under the natural identification of the components of N_∞^i and N_∞^{i+1} containing $\psi^i(\underline{\lambda}^i)$ and $\psi^{i+1}(\overline{\lambda}^i)$ respectively. The component is denoted by M_∞^i. Then we have K_n-bi-Lipschitz maps $g_n : \mathcal{N}_{R_n}(x_n, M_n) \to \mathcal{N}_{R_n}(x_\infty, M_\infty)$ as (3.1), where x_n is a point of M_n contained in $\eta_n^i(\underline{c}_n^i)^\natural$. Since $\{g_n \circ \varphi_n^i \circ \eta_n^i\}$ converges uniformly to ψ^i, the composition $g_n \circ \varphi_n^i \circ \eta_n^i$ is homotopic to ψ^i for all sufficiently large n. Let $\underline{G}^i$ (resp. $\overline{G}^i$) be the element of $\underline{\mathcal{G}}^i$ (resp. $\overline{\mathcal{G}}^i$) supporting $\underline{\lambda}^i$ (resp. $\overline{\lambda}^i$). For all sufficiently large n, we have pleated maps $\zeta^i : \eta_n^i(\underline{G}^i) \to M_n$, $\zeta^{i+1} : \eta_n^{i+1}(\overline{G}^i) \to M_n$ realizing $\eta_n^i(\underline{\lambda}_n^i)$ and $\eta_n^{i+1}(\overline{\lambda}_n^i)$ respectively. By the definition of pleated maps, both ζ^i and ζ^{i+1} are marking-preserving. By Lemma 1.3, for any leaf $\underline{l}$ of $\underline{\lambda}_n^i$, there exists a leaf $\bar{l}$ of $\overline{\lambda}^i$ with $\psi^i(\underline{l}) = \psi^{i+1}(\bar{l})$. Since both ζ^k and φ_n^k are marking-preserving maps for $k = i, i+1$, $\zeta^k \circ \eta_n^k|_{G^{(k)}}$ is homotopic to $\varphi_n^k \circ \eta_n^k|_{G^{(k)}}$ and hence to $g_n^{-1} \circ \psi^k|_{G^{(k)}}$, where $G^{(i)} = \underline{G}^i$ and $G^{(i+1)} = \overline{G}^i$. Since $g_n^{-1}(\psi^i(\underline{l})) = g_n^{-1}(\psi^{i+1}(\bar{l}))$, the geodesic lines $\zeta^i(\eta_n^i(\underline{l}))$ and $\zeta^{i+1}(\eta_n^{i+1}(\bar{l}))$ in M_n are equal to each other. This shows that $\zeta^i(\eta_n^i(\underline{\lambda}^i)) \subset \zeta^{i+1}(\eta_n^{i+1}(\overline{\lambda}^i))$, and similarly $\zeta^{i+1}(\eta_n^{i+1}(\overline{\lambda}^i)) \subset \zeta^i(\eta_n^i(\underline{\lambda}^i))$. Then, by Lemma 1.4, $(\eta_n^i(\underline{G}^i), \eta_n^i(\underline{\lambda}^i))$ is isotopic to $(\eta_n^{i+1}(\overline{G}^i), \eta_n^{i+1}(\overline{\lambda}^i))$ in F_n. This implies that $\underline{\mathcal{G}}_n^i \subset \overline{\mathcal{G}}_n^i$ and $\underline{\mathcal{F}}_n^i \subset \overline{\mathcal{F}}_n^i$. Similarly, we have $\overline{\mathcal{G}}_n^i \subset \underline{\mathcal{G}}_n^i$, $\overline{\mathcal{F}}_n^i \subset \underline{\mathcal{F}}_n^i$ and hence $\underline{\mathcal{G}}_n^i = \overline{\mathcal{G}}_n^i (:= \mathcal{G}_n^i)$, $\underline{\mathcal{F}}_n^i = \overline{\mathcal{F}}_n^i (:= \mathcal{F}_n^i)$.

We can see, by Bowditch [Bow2, Section 3, (F8), (F9)], that $\{\mathcal{F}_n^i\}_{i=s}^t$ is a taut sequence for each n and hence in particular $t - s \geq 1$, as follows. In fact, if an element J of $\mathcal{F}_n^i$ did not belong to either $\mathcal{F}_n^{i-1}$ or $\mathcal{F}_n^{i+1}$, then J would be disjoint from $\left(\bigcup \mathcal{G}_n^{i-1}\right) \cup \left(\bigcup \mathcal{G}_n^{i+1}\right)$. From the definition of $\mathcal{F}_n^i$, v_n^i crosses some simple closed geodesic β in J for all sufficiently large n. Since $\{v_n^i\}$ is a tight geodesic, β meets either v_n^{i-1} or v_n^{i+1} non-trivially and hence we have either $\beta \cap \left(\bigcup \mathcal{G}_n^{i-1}\right) \neq \emptyset$ or $\beta \cap \left(\bigcup \mathcal{G}_n^{i+1}\right) \neq \emptyset$, a contradiction.

Since $t - s \geq 1$, 2/3-Lemma implies that

$$d\Big(\bigcup \mathcal{F}_n^s, \bigcup \mathcal{F}_n^t\Big) \leq \left[\frac{2}{3}(t - s + 1)\right] - 1 \leq t - s - 1\,.$$

Since $v_n^{s-1} \cap \mathcal{F}_n^s = \emptyset$ and $v_n^{t+1} \cap \mathcal{F}_n^{t+1} = \emptyset$,

$$d(v_n^{s-1}, v_n^{t+1}) \leq d\Big(\bigcup \mathcal{F}_n^s, \bigcup \mathcal{F}_n^t\Big) + 2 \leq t - s + 1\,.$$

On the other hand, since $\{v_n^i\}$ is a tight geodesic,

$$d(v_n^{s-1}, v_n^{t+1}) = (t+1) - (s-1) = t - s + 2\,,$$

a contradiction. This completes the proof. □

LEMMA 4.2. *Suppose that $l_M(\partial F) \le L$ for a given constant $L > 0$. Let p, r be positive integers with $p \ge 12(r+1)$, and let $L_1 \ge L$ be the uniform constant given in Section 2. Then there exists a constant $L_2 \ge L_1$ depending only on p, L, r, $\xi(S)$ and satisfying the following condition. If $g = \{v^i\}_{i=0}^p$ is any tight geodesic in $\mathcal{C}_0(F)$ of length p with $d(v^k, \mathcal{C}_M(F, L_1)) \le r$ for $k = 0, p$, then $l_M(v^i) \le L_2$ for some $i \in \{0, \dots, p\}$.*

PROOF. Suppose that the conclusion fails. Then there exist a sequence $\{M_n\}_{n=1}^\infty$ of hyperbolic 3-manifolds with markings $\pi_n : M_n \to S$, a sequence $\{F_n\}$ of connected o.g.-subsurfaces of S with $\xi(F_n) = \xi(F)$, $l_{M_n}(\partial F_n) \le L$, and tight geodesic sequences $g_n = \{v_n^i\}_{i=0}^p$ with $d(v_n^k, \mathcal{C}_M(F, L)) \le r$ for $k = 0, p$ and $l_{M_n}(v_n^i) \ge n$ for $i \in \{0, 1, \dots, p\}$. Let $\{\widehat{v}_n^i\}_{i=-s_n}^0$, $\{\widehat{v}_n^i\}_{i=p}^{p+t_n}$ be tight geodesics in $\mathcal{C}_0(F_n)$ such that $l_{M_n}(\widehat{v}_n^{-s_n}) \le L$, $l_{M_n}(\widehat{v}_n^{p+t_n}) \le L$ and $0 \le s_n, t_n \le r$ and $\widehat{v}_n^k$ $(k = 0, p)$ is a vertex of v_n^k with $\lim_{n\to\infty} l_{M_n}(\widehat{v}_n^k) = \infty$. If necessary passing to a subsequence and replacing the markings $\pi_n : M_n \to S$, we may assume that $s_n = s$, $t_n = t$ and $F_n = F$ $(n \in \mathbf{N})$. We may also assume that there exist indices a, b with $-s < a \le 0$, $p \le b < p + t$ and such that $\lim_{n\to\infty} l_{M_n}(\widehat{v}_n^i) = \infty$ for any $i \in \{a, \dots, 0\} \cup \{p, \dots, b\}$ and $\sup_n\{l_{M_n}(\widehat{v}_n^{a-1})\} < \infty$, $\sup_n\{l_{M_n}(\widehat{v}_n^{b+1})\} < \infty$. Applying arguments in the proof of Lemma 4.1 to the tight geodesics $\{\widehat{v}_n^i\}_{i=a-1}^0$, $\{v_n^i\}_{i=0}^p$, $\{\widehat{v}_n^i\}_{i=p}^{b+1}$, one can obtain compatible sequences $\{\widehat{\mathcal{F}}_n^i\}_{i=a}^0$, $\{\mathcal{F}_n^i\}_{i=0}^p$, $\{\widehat{\mathcal{F}}_n^i\}_{i=p}^b$ of geodesic patterns on F with $\widehat{\mathcal{F}}_n^0 \subset \mathcal{F}_n^0$, $\widehat{\mathcal{F}}_n^p \subset \mathcal{F}_n^p$, $\widehat{v}_n^{a-1} \cap \left(\bigcup \widehat{\mathcal{F}}_n^a\right) = \emptyset$, $\widehat{v}_n^{b+1} \cap \left(\bigcup \widehat{\mathcal{F}}_n^b\right) = \emptyset$. Moreover, the extension $\{\mathcal{F}_n^i\}_{i=-1}^{p+1}$ of $\{\mathcal{F}_n^i\}_{i=0}^p$ with $\mathcal{F}_n^{-1} = \mathcal{F}_n^0$ and $\mathcal{F}_n^{p+1} = \mathcal{F}_n^p$ is taut, see [Bow2, Corollary 2.2]. By 2/3-Lemma, there exist simple geodesic loops w_n^k $(k = 0, p)$ contained in elements of $\mathcal{F}_n^k$ with $d(w_n^0, w_n^p) \le [\frac{2}{3}(p+3)] - 1 < \frac{2}{3}p + 2$. Let $\widehat{w}_n^j$ $(j = a-1, b+1)$ be a vertex of $\widehat{v}_n^j$. If w_n^0 is contained in an element of $\widehat{\mathcal{F}}_n^0$, then $d(\widehat{w}_n^{a-1}, w_n^0) \le 1 - a \le r$, and otherwise $d(\widehat{w}_n^{a-1}, w_n^0) \le 2 - a \le r + 1$. Similarly, we have $d(\widehat{w}_n^{b+1}, w_n^p) \le r + 1$. It follows that

$$d(\widehat{w}_n^{a-1}, \widehat{w}_n^{b+1}) \le d(\widehat{w}_n^{a-1}, w_n^0) + d(w_n^0, w_n^p) + d(w_n^p, \widehat{w}_n^{b+1}) < \frac{2}{3}p + 2r + 4\,.$$

On the other hand, since $p \ge 12(r+1)$,

$$d(\widehat{w}_n^{a-1}, \widehat{w}_n^{b+1}) \ge d(\widehat{v}_n^0, \widehat{v}_n^p) - d(\widehat{w}_n^{a-1}, \widehat{v}_n^0) - d(\widehat{v}_n^p, \widehat{w}_n^{b+1}) \ge p - 2r \ge \frac{2}{3}p + 2r + 4\,,$$

a contradiction. This completes the proof. □

An upper bound of the M-lengths of curves in a tight geodesic given in Lemma 4.1 depends on the length of the geodesic. The following Upper Bounds Theorem shows the existence of an upper bound independent of the lengths of tight geodesics.

THEOREM 4.3. *Suppose that $\xi(F) \ge 2$. Let L be any positive number with $l_M(\partial F) \le L$. Then there exists a constant L' depending only on L and $\xi(S)$ such that, for any finite tight* *geodesic* $g = \{v^i\}_{i=0}^p$ *in* $\mathcal{C}_0(F)$ *with* $l_M(v^0), l_M(v^p) \le L_1$, $l_M(v^i)$ *is smaller than* L' *for any* $i \in \{0, 1, \ldots, p\}$.

PROOF. By Theorem 2.1, $\mathcal{C}_M(F, L_1)$ is r-quasi-convex in $\mathcal{C}(F)$ for some uniform constant $r > 0$. Set $d = 12(r+1)$. By Lemma 4.1, it suffices to consider the case of $p \ge d$. We divide $\{v^i\}_{i=0}^p$ into subsequences $\{v^i\}_{i=q(j)}^{q(j+1)}$ with $0 = q(0) < q(1) < \cdots < q(k) = p$ each of which has length at least d and at most $2d$. By Lemma 4.2, each subsequence has a vertex of the M-length at most $L(d)$. These vertices divide $\{v^i\}_{i=0}^p$ into subsequences $\{v^i\}_{i=a(j)}^{a(j+1)}$ with

$$\begin{aligned} 0 = a(0) = q(0) \le a(1) \le q(1) \le a(2) \le q(2) \\ \le \cdots \le q(k-1) \le a(k) \le q(k) = a(k+1) = p \end{aligned}$$

each of which has length at most $4d$ and such that $l_M(v_n^{a(j)}) \le L(d)$ for any $j \in \{0, \ldots, k+1\}$. Applying Lemma 4.1 again to the $\{v^i\}_{i=a(j)}^{a(j+1)}$'s, one can show that the lengths $l_M(v^i)$ ($i \in \{0, 1, \ldots, p\}$) are uniformly bounded. □

5. Length Upper Bounds Theorem (Exceptional case)

Now we consider the case of $\xi(F) = 1$. Then any tight geodesic $\{v^i\}_{i=0}^p$ in $\mathcal{C}_0(F)$ is a usual geodesic. In particular, each entry of which consists of a single vertex. A simple and numerical proof in this case is given by [Bow2, Section 9]. His proof uses trace identities for representations $\pi_1(F) \to \mathrm{PSL}_2(\mathbf{C})$. In this section, we present a proof based on geometric limit arguments.

THEOREM 5.1. *Suppose that* $\xi(F) = 1$. *Let* L *be any positive number with* $l_M(\partial F) \le L$. *Then there exists a constant* L' *depending only on* L *such that, for any finite geodesic* $g = \{v^i\}_{i=0}^p$ *in* $\mathcal{C}_0(F)$ *with* $l_M(v^0) \le L_1, l_M(v^p) \le L_1$, $l_M(v^i)$ *is smaller than* L' *for any* $i \in \{0, 1, \ldots, p\}$.

We give here the proof only in the case that F is a four-holed sphere. The proof in the one-holed torus case is done quite similarly.

LEMMA 5.2. *Under the assumptions as above, for any* $p \in \mathbf{N}$, *there exist a constant* L' *depending only on* L, p *and satisfying the following condition. Suppose that* $g = \{v^i\}_{i=0}^q$ *is a sequence in* $\mathcal{C}_0(F)$ *with* $q \le p$ *and such that*

- $d(v^i, v^{i+1}) = 1$ *for any* $i \in \{0, 1, \ldots, q-1\}$,
- $l_M(v^0) \le L_1, l_M(v^q) \le L_1$,
- $v^i \ne v^j$ *for any* $i, j \in \{0, 1, \ldots, q\}$,

Then $l_M(v^i)$ *is smaller than* L' *for any* $i \in \{0, 1, \ldots, q\}$.

Note that the sequence $\{v^i\}_{i=0}^q$ here is not necessarily a geodesic.

PROOF. We suppose that the conclusion fails. Then there exist a sequence $\{M_n\}_{n=0}^{\infty}$ of hyperbolic 3-manifolds with markings $\pi_n : M_n \to S$, a sequence $\{F_n\}$ of geodesic four-holed spheres in S with $l_{M_n}(\partial F_n) \leq L$, and geodesic sequences $g_n = \{v_n^i\}_{i=0}^{q_n}$ in $\mathcal{C}_0(F_n)$ satisfying the conditions of Lemma 5.2 and $\sup_n\{l_{M_n}(v_i)\} = \infty$ for some $i \in \{1, \dots, q_n - 1\}$. If necessary passing to a subsequence and replacing the makings of M_n, we may assume that $F_n = F$, $q_n = q$ and that there exists s with $1 \leq s \leq q - 1$ and such that $\sup_n\{\max_i\{l_{M_n}(v_n^i)\}\} < \infty$ for any $0 \leq i \leq s - 1$ and $\lim_{n\to\infty} l_{M_n}(v_n^s) = \infty$.

Let $\varphi_n : F(\sigma_n) \to M_n$ be a pleated map realizing v_n^s. We consider the following two cases after passing to a subsequence of $\{\varphi_n\}$ and complete the proof by showing that neither of them occurs.

Case 1. There exists a sequence $\{\varepsilon_n\}$ of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ and such that $F(\sigma_n)_{(0,\varepsilon_n]}$ contains a non-peripheral component A_n.

Let c_n be the geodesic core of A_n. Fix a hyperbolic structure on F and a simple geodesic loop c in F. Note that $F \setminus c$ consists of two components $H_\pm$ each of which is homeomorphic to a three-holed sphere. There exists a homeomorphism $\eta_n : F \setminus c \to F(\sigma_n) \setminus c_n$ which is C_n-bi-Lipschitz with $\sup_n\{C_n\} < \infty$ on any compact subset of $F \setminus c$. For any simple geodesic loops w_n in $F(\sigma_n)$ with $w_n \setminus c_n \neq \emptyset$, let $\widehat{w}_n$ be the union of the geodesic arcs in $F(\widehat{\sigma}_n) \setminus c$ properly homotopic to $\eta_n^{-1}(w_n)$ in $F \setminus c$ without moving the end points, where $\widehat{\sigma}_n$ is an incomplete hyperbolic metric on $F \setminus c$ induced from σ_n via η_n. The sequence $\{\varphi_n \circ \eta_n\}$ subconverges geometrically to a pleated map $\psi : F(\sigma_\infty) \setminus c \to N_\infty$ and $\{\widehat{v}_n^s\}$ does to a lamination consisting of two geodesic lines $\lambda_\pm$ in $H_\pm(\sigma_\infty)$ such that each end of λ_τ ($\tau = \pm$) exits the parabolic cusp of $H_\tau(\sigma_\infty)$ adjacent to c. For any geodesic loop w_n as above, $\{\widehat{w}_n\}$ also converges geometrically to $\lambda_+ \cup \lambda_-$. Fix an arbitrarily small $\varepsilon > 0$ and let B_n be the component of $F(\sigma_n)_{(0,\varepsilon]}$ containing A_n when $\varepsilon \geq \varepsilon_n$. Note that the diameter of B_n diverges to infinity as $n \to \infty$. The loop w_n is divided into geodesic segments $a^1, b^1, \dots, a^m, b^m$ in $F(\sigma_n)$ with $a^j \subset F(\sigma_n) \setminus \mathrm{Int}B_n$ and $b^j \subset B_n$ and such that $\{x^j\} = \partial a^j \cap \partial b^j$, $\{y^j\} = \partial b^j \cap \partial a^{j+1}$ are single point sets, where $a^{m+1} = a^1$. From the geometric convergence of $\{\widehat{w}_n\}$ to $\lambda_+ \cup \lambda_-$, we know that $\varphi_n(a^j)$ is homotopic rel. $\varphi_n(\partial a^j)$ to a geodesic segment α^j in M_n arbitrarily close to a subsegment of $v_n^{s\natural}$. We now consider the case of $d(w_n, v_n^s) = 1$, that is, w_n meets v_n^s in two points. In this case each b^j meets v_n^s at most two points z_k^j, see Fig. 5.1. Since the $\varepsilon > 0$ is taken arbitrarily small, one can suppose that $v_n^s \cap B_n$ consists of mutually close and almost parallel geodesic segments in $F(\sigma_n)$ and hence $\varphi_n(v_n^s \cap B_n)$

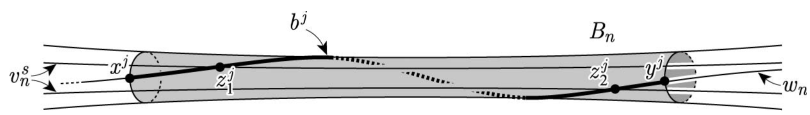

FIGURE 5.1

does so in M_n. It follows that $\varphi_n(b^j)$ is homotopic rel. $\varphi_n(x^j), \varphi_n(y^j), \varphi_n(z_k^j)$ to a polygonal segment β^j consisting of at most three geodesic segments in M_n such that the angle of α^j and β^j at $\varphi_n(x^j)$, that of β^j and α^{j+1} at $\varphi_n(y^j)$ and the internal angles of β^j at $\varphi_n(z_k^j)$ are arbitrarily close to π. This shows that the geodesic loop $w_n^\natural$ is freely homotopic in M_n to the polygonal loop $\omega_n = \alpha^1 \cup \beta^1 \cup \cdots \cup \alpha^m \cup \beta^m$ and is contained in an arbitrarily small neighborhood of $\omega_n^\natural$ in M_n for all sufficiently large n. It follows that $\lim_{n\to\infty} l_{M_n}(w_n) = \infty$. This implies that no such w_n is equal to v_n^{s-1} and hence $v_n^{s-1} = c_n$. Thus $\varphi_n(B_n)$ is contained in the component T_n of $M_{n(0,\varepsilon]}$ with core (or end) $v_n^{s-1\natural}$. By setting $w_n = v_n^{s+1}$, we have $\lim_{n\to\infty} l_{M_n}(v_n^{s+1}) = \infty$ and that $v_n^{s+1\natural} \cap T_n$ is non-empty and consists of almost parallel geodesic segments. So one can repeat similar arguments for $v_n^{s+1}, \dots, v_n^{q-1}$ instead of v_n^s and obtain finally $\lim_{n\to\infty} l_{M_n}(v_n^q) = \infty$, a contradiction. Thus Case 1 does not occur.

Case 2. $F(\sigma_n)_{(0,\varepsilon]}$ contains no non-peripheral components for all $n \in \mathbf{N}$ and some $\varepsilon > 0$.

There exists a homeomorphism $\zeta_n : F \to F(\sigma_n)$ which is C_n'-bi-Lipschitz with $\sup_n\{C_n'\} < \infty$ on any compact subset of F. Then $\{\varphi_n \circ \zeta_n\}$ subconverges geometrically to a pleated map $\chi : F(\sigma_\infty) \to N_\infty$ and $\{v_n^s\}$ does to a lamination ν_∞^s in $F(\sigma_\infty)$ realized by χ. The sequences $\{v_n^{s-1}\}, \{v_n^{s+1}\}$ also subconverge geometrically to laminations ν_∞^{s-1} and ν_∞^{s+1} in $F(\sigma_\infty)$ respectively.

First we show that ν_∞^s contains a compact leaf. It is well known that ν_∞^s has a sub-lamination τ_∞^s which fully supports a transverse invariant measure, for example see [Th1, Proposition 8.10.6]. If τ_∞^s did not have a compact leaf, then each component of $F \setminus \tau_\infty^s$ would be an annulus with just one cusp adjacent to τ_∞^s, see [Th1, Subsection 9.5]. It follows that $\tau_\infty^s = \nu_\infty^s$. Since any transverse invariant measure on a lamination without compact leaves has no atoms, if ν_∞^{s-1} meets ν_∞^s transversely, then any arc α in ν_∞^{s-1} with $\mathrm{Int}(\alpha) \cap \nu_\infty^s \neq \emptyset$ intersects ν_∞^s in infinitely many points. Hence the intersection number $i(v_n^{s-1}, v_n^s)$ would diverge to infinity as $n \to \infty$. This contradicts the fact that the intersection number is two. From this, we know that ν_∞^{s-1} is contained in ν_∞^s. Since $\sup_n\{l_{M_n}(v_n^{s-1})\} < \infty$, ν_∞^{s-1} is a closed geodesic in $\nu_\infty^s = \tau_\infty^s$. This also gives a contradiction. Thus ν_∞^s contains a compact leaf $w(\nu_\infty^s)$, called the *waist* of ν_∞^s.

Note that $\nu_\infty^s \setminus w(\nu_\infty^s)$ consists of two geodesic lines spiraling around $w(\nu_\infty^s)$. For $k = s-1, s+1$, the condition of $i(v_n^s, v_n^k) = 2$ $(n \in \mathbf{N})$ implies that ν_∞^k can not intersect $w(\nu_\infty^s)$ transversely. This shows that ν_∞^k has the waist $w(\nu_\infty^k) = w(\nu_\infty^s)$. In the case that $w(\nu_\infty^k) \neq \nu_\infty^k$, $\nu_\infty^k \setminus w(\nu_\infty^k)$ consists of two spirals. Then, again by $i(v_n^s, v_n^k) = 2$ for any n, one can show that ν_∞^k and ν_∞^s have the same spirals. Thus either $\nu_\infty^k = w(\nu_\infty^k) \subsetneq \nu_\infty^s$ or $\nu_\infty^k = \nu_\infty^s$ necessarily holds. As above, the condition $\sup_n\{l_{M_n}(v_n^{s-1})\} < \infty$ implies $\nu_\infty^{s-1} = w(\nu_\infty^{s-1})$. Since $v_n^{s-1} \neq v_n^{s+1}$ for all n, $\nu_\infty^{s-1} = w(\nu_\infty^{s+1}) \neq \nu_\infty^{s+1}$ and hence $\nu_\infty^{s+1} = \nu_\infty^s$. In particular, this implies $\lim_{n\to\infty} l_{M_n}(v_n^{s+1}) = \infty$. Hence $\chi : F(\sigma_\infty) \to N_\infty$ is also a geometric limit of pleated maps realizing v_n^{s+1} in M_n and realizes ν_∞^{s+1} with $w(\nu_\infty^{s+1}) = \nu_\infty^{s-1}$ in N_∞. By repeating similar arguments for v_n^i $(i = s+2, \dots, q-1)$ instead of v_n^{s+1}, we have $\lim_{n\to\infty} l_{M_n}(v_n^q) = \infty$, a contradiction. Thus Case 2 also does not occur. □

PROOF OF THEOREM 5.1. By Theorem 2.1, $\mathcal{C}_M(F, L_1)$ is r-quasi-convex in $\mathcal{C}(F)$ for some uniform constant $r > 0$. By Lemma 5.2, it suffices to consider the case of $p \geq 3r$. We divide $\{v^i\}_{i=0}^p$ into subsequences $\{v^i\}_{i=q(j)}^{q(j+1)}$ with $0 = q(0) < q(1) < \cdots < q(k) = p$ each of which has length at least $3r$ and at most $6r$. Let w^j $(j = 0, 1, \dots, k)$ be a vertex in $\mathcal{C}_M(F, L_1)$ closest to $v^{q(j)}$ and let $\{x^{j,a}\}_{a=0}^{a_j}$ be a geodesic in $\mathcal{C}_0(F)$ connecting w^j with $v^{q(j)}$. Consider the shortest subgeodesic $\{x^{j,a}\}_{a=0}^{b_j}$ of $\{x^{j,a}\}_{a=0}^{a_j}$ that connects w^j with $\{v^i\}_{i=0}^p$ and suppose the terminal vertex $x^{j,b_j} = v^{\bar{q}(j)}$. Since $q(j+1) - q(j) \geq 3r$, $\bar{q}(j+1) - \bar{q}(j) \geq r$ and $\{x^{j,a}\}_{a=0}^{b_j} \cap \{x^{j+1,a}\}_{a=0}^{b_{j+1}} = \emptyset$. Adding the subgeodesics $\{x^{j,a}\}_{a=0}^{b_j}$, $\{x^{j+1,a}\}_{a=b_{j+1}}^{0}$ to $\{v^i\}_{i=\bar{q}(j)}^{\bar{q}(j+1)}$, we have the extended sequence in $\mathcal{C}_0(F)$ of length at most $8r$ which connects w^j with w^{j+1} and satisfies the conditions of Lemma 5.2. Thus there exists a uniform constant $L' > 0$ satisfying $l_M(v^i) < L'$ for any $i \in \{\bar{q}(j), \dots, \bar{q}(j+1)\}$ with $0 \leq j \leq k-1$. This completes the proof. □

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