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Moderate Killing Helices of Proper Order Four on a Complex Projective Space

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Abstract. In this paper we study Killing helices, which are helices generated by Killing vector fields, of order 4 on a complex projective space whose first and third geodesic curvatures coincide. We construct a natural foliation structure on the set of all congruence classes of these helices. Our study shows that the moduli space of circles is canonically embedded into the moduli space of Killing helices of proper order 4 on a complex projective space.

1. Introduction

A smooth curve γ parameterized by its arclength on a Riemannian manifold M is said to be a *helix of proper order* d if it satisfies the following system of ordinary differential equations

$$\nabla_{\dot\gamma} Y_j = -\kappa_{j-1} Y_{j-1} + \kappa_j Y_{j+1}\,, \quad 1 \leqq j \leqq d\,, \tag{1.1}$$

with positive constants $\kappa_1, \dots, \kappa_{d-1}$ and an orthonormal system $\{Y_1 = \dot\gamma, Y_2, \dots, Y_d\}$ of vector fields along γ. Here we set $\kappa_0 = \kappa_d = 0$ and Y_0, Y_{d+1} to be null vector fields along γ. These constants $\kappa_1, \dots, \kappa_{d-1}$ and the frame field $\{Y_1, \dots, Y_d\}$ are called the *geodesic curvatures* and *Frenet frame* of γ, respectively. Helices of proper order 1 are geodesics and helices of proper order 2 are called circles of positive geodesic curvature.

On real space forms, which are standard spheres, Euclidean spaces and real hyperbolic spaces, it is well-known that all helices are generated by some Killing vector fields on them. But on non-flat complex space forms, which are complex projective spaces and complex hyperbolic spaces, the situation is not the same. Helices on a Kähler manifold (M, J) with complex structure J have another important character. For a helix γ of proper order d on (M, J), we define its complex torsions by $\tau_{ij} = \langle Y_i, JY_j\rangle$ for $1 \leq i < j \leq d$. It is known that a helix on a complex space form is generated by some Killing vector field if and only if all its complex torsions are constant (cf. [9]). We shall call such helices *Killing*. In this paper

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we study a special kind of Killing helices of proper order 4 on a complex projective space whose first and the third geodesic curvatures coincide.

The reason why we focus on such helices of proper order 4 is our interest in the structure of the moduli space of Killing helices. Two smooth curves γ_1, γ_2 on a Riemannian manifold M parameterized by their arclength are said to be *congruent* to each other if there exist an isometry φ of M and a constant t_0 satisfying $\gamma_2(t) = \varphi \circ \gamma_1(t + t_0)$ for all t. The set $\mathcal{K}_d(M)$ of all congruence classes of Killing helices of proper order d on M is called their moduli space. On a real space form $\mathbf{R}M^n$, helices are classified by their geodesic curvatures, and hence the moduli spaces are $\mathcal{K}_1(\mathbf{R}M^n) = \{0\}$ and $\mathcal{K}_d(\mathbf{R}M^n) = (0, \infty)^{d-1}$, the $(d-1)$-product of half lines, for $d \geq 2$. We may hence say that they form a "building structure". On the other hand, on a non-flat complex space form $\mathbf{C}M^n$ circles of positive geodesic curvature are classified by their geodesic curvature and complex torsion, hence their moduli space $\mathcal{K}_2(\mathbf{C}M^n)$ is set theoretically bijective to the band $(0, \infty) \times [0, 1]$. But as we see in [2], from the view point of length spectrum, it is better to consider that it is a disjoint union of the moduli space $\mathcal{EK}_2(\mathbf{C}M^n) \cong (0, \infty)$ of non-geodesic circles of complex torsion ± 1 and the moduli space $\mathcal{K}_2(\mathbf{C}M^n) \setminus \mathcal{EK}_2(\mathbf{C}M^n) \cong (0, \infty) \times [0, 1)$ of other non-geodesic circles. Also, if one carefully reads [8], one finds that the moduli spaces of Killing helices do not form a "building structure" in a trivial sense. In section 2, we study the moduli spaces $\mathcal{K}_3(\mathbf{C}M^n)$ and $\mathcal{K}_4(\mathbf{C}M^n)$ from a set-theoretical point of view, and make clear the above property.

But if we restrict ourselves to helices on a complex projective or hyperbolic plane $\mathbf{C}M^2$, it seems the moduli spaces $\mathcal{EK}_2(\mathbf{C}M^2)$, $\mathcal{K}_3(\mathbf{C}M^2)$ and $\mathcal{K}_4(\mathbf{C}M^2)$ form a "building structure". Therefore it is necessary to interpret the position of $\mathcal{K}_2(\mathbf{C}M^2) \setminus \mathcal{EK}_2(\mathbf{C}M^2)$ in this "building structure". It is known that circles of complex torsion ± 1 induces canonical dynamical systems on the unit tangent bundle, which are called Kähler magnetic flows (see [1]). Inspired by this fact, we consider moderate Killing helices of proper order 4 on a complex projective space. They are helices with complex torsions $\tau_{12} = \tau_{13} = \tau_{24} = \tau_{34} = 0$ and $\tau_{23} = -\tau_{14} = \pm 1$. Since we are interested in the structure of moduli space of helices from the viewpoint of the length spectrum, and as the length spectrum on a complex projective space $\mathbf{C}P^n$ and that on a complex hyperbolic space $\mathbf{C}H^n$ are different, we restrict ourselves to $\mathbf{C}P^n$ in sections 3 and 4. We construct a canonical embedding of $\mathcal{K}_2(\mathbf{C}P^2) \setminus \mathcal{EK}_2(\mathbf{C}P^2)$ into $\mathcal{K}_4(\mathbf{C}P^2)$ whose image is in the moduli space of moderate Killing helices. This gives an explanation for the structure of moduli spaces of Killing helices on $\mathbf{C}P^n$. This also clarifies Chen-Maeda's result on helices of 2-type. In their paper [7], they study curves on $\mathbf{C}P^n$ through the first standard minimal embedding of $\mathbf{C}P^n$ and characterize totally real circles (i.e. circles with $\tau_{12} = 0$) and a special kind of helices of proper order 4. The moduli of the latter is the image of the moduli of the former through our embedding.

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2. Killing helices on a complex space form

We shall start by reviewing Killing helices on a non-flat complex space form $\mathbf{C}M^n$, which is either $\mathbf{C}P^n$ or $\mathbf{C}H^n$. By the equation (1.1), we see that their complex torsions satisfy

$$\tau'_{ij} = -\kappa_{i-1}\tau_{i-1\,j} + \kappa_i\tau_{i+1\,j} - \kappa_{j-1}\tau_{ij-1} + \kappa_j\tau_{ij+1}\,,$$

where we set $\tau_{0k} = \tau_{kk} = \tau_{kd+1} = 0$. Since a helix on $\mathbf{C}M^n$ is Killing if and only if all its complex torsions are constant functions ([9]), we find that all circles are Killing but helices of proper order greater than 2 are not necessarily Killing. Moreover, as we have to choose the Frenet frame to be orthonormal, the degree of freedom for choosing the initial frame depends on the complex dimension of the base manifold. According to [8] we have the following results.

PROPOSITION 1. *We consider a helix of proper order* 3 *on* $\mathbf{C}M^n$.

(1) *When* $n \geq 3$, *it is Killing if and only if its geodesic curvatures and complex torsions satisfy* $\kappa_1\tau_{23} = \kappa_2\tau_{12}$, $\tau_{13} = 0$, $|\tau_{12}| \leq \kappa_1\Big/\sqrt{\kappa_1^2+\kappa_2^2}$.

(2) *When* $n = 2$, *it is Killing if and only if its geodesic curvatures and complex torsions satisfy*

$$\tau_{12} = \pm\frac{\kappa_1}{\sqrt{\kappa_1^2+\kappa_2^2}},\ \tau_{13} = 0,\ \tau_{23} = \pm\frac{\kappa_2}{\sqrt{\kappa_1^2+\kappa_2^2}},$$

where double signs take the same signatures.

It is known that two helices on a complex space form are congruent to each other if and only if they are of the same proper order, have the same series of geodesic curvatures and their series of complex torsions $\tau_{ij}^{(1)}, \tau_{ij}^{(2)}$ satisfy either $\tau_{ij}^{(1)}(t_0) = \tau_{ij}^{(2)}(0)$ for all (i, j) or $\tau_{ij}^{(1)}(t_0) = -\tau_{ij}^{(2)}(0)$ for all (i, j) at some point t_0 (see [9]). Thus the moduli space $\mathcal{K}_3(\mathbf{C}M^n)$ of Killing helices of proper order 3 is bijective to the following sets according to whether $n \geq 3$ or $n = 2$:

$$\begin{cases} \left\{(\kappa_1,\kappa_2,\tau_{12}) \in (0,\infty)\times(0,\infty)\times[0,1)\ \middle|\ \tau_{12} \leq \kappa_1\Big/\sqrt{\kappa_1^2+\kappa_2^2}\right\}, & n \geq 3\,, \\ (0,\infty)\times(0,\infty), & n = 2\,. \end{cases}$$

PROPOSITION 2. *We consider a helix of proper order* 4 *on* $\mathbf{C}M^n$.

(1) *If it is Killing, then its geodesic curvatures and complex torsions satisfy* $\kappa_1\tau_{23} + \kappa_3\tau_{14} = \kappa_2\tau_{12}$, $\kappa_1\tau_{14} + \kappa_3\tau_{23} = \kappa_2\tau_{34}$, $\tau_{13} = \tau_{24} = 0$.

(2) *In particular, when* $n = 2$, *it is Killing if and only if its geodesic curvatures and complex torsions satisfy one of the following:*

i) $\tau_{12} = \tau_{34} = \pm\dfrac{\kappa_1+\kappa_3}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}$, $\tau_{23} = \tau_{14} = \pm\dfrac{\kappa_2}{\sqrt{\kappa_2^2+(\kappa_1+\kappa_3)^2}}$,

$\tau_{13} = \tau_{24} = 0$,

ii) $\tau_{12} = -\tau_{34} = \pm\dfrac{\kappa_1 - \kappa_3}{\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}}$, $\tau_{23} = -\tau_{14} = \pm\dfrac{\kappa_2}{\sqrt{\kappa_2^2 + (\kappa_1 - \kappa_3)^2}}$,

$\tau_{13} = \tau_{24} = 0$.

In each of the above conditions double signs take the same signatures.

Thus the moduli space $\mathcal{K}_4(\mathbf{C}M^n)$ of Killing helices of proper order 4 is a subset of $(0,\infty)^3 \times [-1,1]^2$. When $n = 2$, we can consider this moduli space as $\mathcal{K}_4(\mathbf{C}M^2) = (0,\infty)^2 \times (\mathbf{R} \setminus \{0\})$. Here, for a point $(x_1, x_2, x_3) \in \mathcal{K}_4(\mathbf{C}M^2) \subset \mathbf{R}^3$ with $x_3 > 0$ it corresponds to the congruence class of Killing helices with geodesic curvatures x_1, x_2, x_3 and complex torsions in the condition (2-i) in Proposition 2, and for a point $(x_1, x_2, x_3) \in \mathcal{K}_4(\mathbf{C}M^2)$ with $x_3 < 0$ it corresponds to the congruence class of Killing helices with geodesic curvatures x_1, x_2, x_3 and complex torsions in the condition (2-ii) in Proposition 2. Thus from the set theoretical point of view, these $\mathcal{K}_3(\mathbf{C}M^2)$ and $\mathcal{K}_4(\mathbf{C}M^2)$ are compatible with each other.

We here consider the moduli space $\mathcal{K}_4(\mathbf{C}M^n)$ for the case $n \geq 3$. When $n \geq 3$, a helix of proper order 4 is Killing if and only if its complex torsions satisfy $\tau_{13} = \tau_{24} = 0$,

$$\begin{cases} \kappa_1\tau_{23} + \kappa_3\tau_{14} = \kappa_2\tau_{12}\,, \\ \kappa_3\tau_{23} + \kappa_1\tau_{14} = \kappa_2\tau_{34} \end{cases} \tag{2.1}$$

and

$$\tau_{12}^2 + \tau_{14}^2 \leq 1\,, \quad \tau_{12}^2 + \tau_{23}^2 \leq 1\,, \quad \tau_{23}^2 + \tau_{34}^2 \leq 1\,, \quad \tau_{14}^2 + \tau_{34}^2 \leq 1\,.$$

We hence need to solve the simultaneous system of linear equations (2.1) under the conditions that

$$\begin{aligned} &|\tau_{23}| \leq \min\left(\sqrt{1-\tau_{12}^2}, \sqrt{1-\tau_{34}^2}\right), \quad |\tau_{12}| \leq 1\,, \\ &|\tau_{14}| \leq \min\left(\sqrt{1-\tau_{12}^2}, \sqrt{1-\tau_{34}^2}\right), \quad |\tau_{34}| \leq 1\,. \end{aligned} \tag{2.2}$$

When $\kappa_1 = \kappa_3$, (2.1) has solutions on τ_{23} and τ_{14} satisfying $\tau_{23} + \tau_{14} = (\kappa_2/\kappa_1)\tau_{12}$ if and only if $\tau_{12} = \tau_{34}$. Considering the conditions (2.2), we find that (2.1) has the desired solutions if and only if one of the following conditions holds:

i) $0 \leq \tau_{12} = \tau_{34} < 1$ and $(\kappa_2/\kappa_1)\tau_{12} - \sqrt{1-\tau_{12}^2} \leq \tau_{23}, \tau_{14} \leq \sqrt{1-\tau_{12}^2}$,

ii) $-1 < \tau_{12} = \tau_{34} < 0$ and $-\sqrt{1-\tau_{12}^2} \leq \tau_{23}, \tau_{14} \leq (\kappa_2/\kappa_1)\tau_{12} + \sqrt{1-\tau_{12}^2}$.

In particular, we have $|\tau_{12}| \leq 2\kappa_1 \Big/ \sqrt{4\kappa_1^2 + \kappa_2^2}$. When $\kappa_1 \neq \kappa_3$, the system of equations (2.1) on τ_{23} and τ_{14} has a unique pair of solutions. By the inequalities (2.2), we have to choose τ_{12}

and τ_{34} so that they satisfy

$$
(2.3)\qquad
\begin{aligned}
&|\kappa_1\tau_{12} - \kappa_3\tau_{34}| \le \frac{|\kappa_1^2 - \kappa_3^2|}{\kappa_2} \times \min\left(\sqrt{1-\tau_{12}^2}, \sqrt{1-\tau_{34}^2}\right), \qquad |\tau_{12}| < 1\,,\\
&|-\kappa_3\tau_{12} + \kappa_1\tau_{34}| \le \frac{|\kappa_1^2 - \kappa_3^2|}{\kappa_2} \times \min\left(\sqrt{1-\tau_{12}^2}, \sqrt{1-\tau_{34}^2}\right), \quad |\tau_{34}| \le 1\,.
\end{aligned}
$$

Since these inequalities are symmetric with respect to τ_{12} and τ_{34}, we only treat the case $|\tau_{12}| \ge |\tau_{34}|$. In this case the inequalities (2.3) turn to

$$
\begin{cases}
\dfrac{\kappa_3}{\kappa_1}\tau_{12} - \dfrac{|\kappa_1^2-\kappa_3^2|}{\kappa_1\kappa_2}\sqrt{1-\tau_{12}^2} \le \tau_{34} \le \dfrac{\kappa_3}{\kappa_1}\tau_{12} + \dfrac{|\kappa_1^2-\kappa_3^2|}{\kappa_1\kappa_2}\sqrt{1-\tau_{12}^2}\,,\\
\dfrac{\kappa_1}{\kappa_3}\tau_{12} - \dfrac{|\kappa_1^2-\kappa_3^2|}{\kappa_2\kappa_3}\sqrt{1-\tau_{12}^2} \le \tau_{34} \le \dfrac{\kappa_1}{\kappa_3}\tau_{12} + \dfrac{|\kappa_1^2-\kappa_3^2|}{\kappa_2\kappa_3}\sqrt{1-\tau_{12}^2}\,,
\end{cases}
$$

Comparing both left and right sides of these inequalities with $\pm\tau_{12}$, we obtain the following.

1) When $\kappa_1 > \kappa_3$,

i) if $|\tau_{12}| \le (\kappa_1-\kappa_3)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}$, then $|\tau_{34}| \le |\tau_{12}|$,

ii) if $(\kappa_1-\kappa_3)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2} < \tau_{12} \le (\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}$, then $(\kappa_1/\kappa_3)\tau_{12} - \left((\kappa_1^2-\kappa_3^2)\sqrt{1-\tau_{12}^2}\right)\Big/(\kappa_2\kappa_3) \le \tau_{34} \le \tau_{12}$,

iii) if $-(\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2} \le \tau_{12} < -(\kappa_1-\kappa_3)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}$, then $\tau_{12} \le \tau_{34} \le (\kappa_1/\kappa_3)\tau_{12} + \left((\kappa_1^2-\kappa_3^2)\sqrt{1-\tau_{12}^2}\right)\Big/(\kappa_2\kappa_3)$,

iv) if $|\tau_{12}| \ge (\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}$, then no τ_{34} satisfies (2.3).

2) When $\kappa_1 < \kappa_3$,

i) if $|\tau_{12}| \le (\kappa_3-\kappa_1)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}$, then $|\tau_{34}| \le |\tau_{12}|$,

ii) if $(\kappa_3-\kappa_1)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2} < \tau_{12} \le (\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}$, then $(\kappa_3/\kappa_1)\tau_{12} - \left((\kappa_3^2-\kappa_1^2)\sqrt{1-\tau_{12}^2}\right)\Big/(\kappa_2\kappa_3) \le \tau_{34} \le \tau_{12}$,

iii) if $-(\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2} \le \tau_{12} < -(\kappa_3-\kappa_1)\Big/\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}$, then $\tau_{12} \le \tau_{34} \le (\kappa_3/\kappa_1)\tau_{12} + \left((\kappa_3^2-\kappa_1^2)\sqrt{1-\tau_{12}^2}\right)\Big/(\kappa_2\kappa_3)$,

iv) if $|\tau_{12}| \ge (\kappa_1+\kappa_3)\Big/\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}$, then no τ_{34} satisfies (2.3).

We have to note one more thing: If $\{\tau_{ij}\}$ satisfy the desired conditions, then so do $\{-\tau_{ij}\}$. Thus we can conclude the following:

PROPOSITION 3. *When $n \geq 3$, the moduli space $\mathcal{K}_4(\mathbf{C}M^n)$ is set theoretically bijective to the set $\bigcup_{K\in(0,\infty)^3}\{K\}\times D_K$, where the set $D_K (\subset [-1,1]^2)$ for $K=(\kappa_1,\kappa_2,\kappa_3)\in(0,\infty)^3$ is given as follows*:

i) *When $\kappa_1 > \kappa_3$*,

$$D_K = \left\{(\tau_{12},\tau_{34}) \;\middle|\; |\tau_{34}| \leq \tau_{12},\ 0 \leq \tau_{12} \leq \frac{\kappa_1-\kappa_3}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; \begin{array}{l} \dfrac{\kappa_1}{\kappa_3}\tau_{12} - \dfrac{\kappa_1^2-\kappa_3^2}{\kappa_2\kappa_3}\sqrt{1-\tau_{12}^2} \leq \tau_{34} \leq \tau_{12} \\ \dfrac{\kappa_1-\kappa_3}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}} \leq \tau_{12} \leq \dfrac{\kappa_1+\kappa_3}{\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}} \end{array}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; |\tau_{12}| \leq \tau_{34},\ 0 \leq \tau_{34} \leq \frac{\kappa_1-\kappa_3}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; \begin{array}{l} \dfrac{\kappa_1}{\kappa_3}\tau_{34} - \dfrac{\kappa_1^2-\kappa_3^2}{\kappa_2\kappa_3}\sqrt{1-\tau_{34}^2} \leq \tau_{12} \leq \tau_{34} \\ \dfrac{\kappa_1-\kappa_3}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}} \leq \tau_{34} \leq \dfrac{\kappa_1+\kappa_3}{\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}} \end{array}\right\},$$

ii) *when $\kappa_1 < \kappa_3$*

$$D_K = \left\{(\tau_{12},\tau_{34}) \;\middle|\; |\tau_{34}| \leq \tau_{12},\ 0 \leq \tau_{12} \leq \frac{\kappa_3-\kappa_1}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; \begin{array}{l} \dfrac{\kappa_3}{\kappa_1}\tau_{12} - \dfrac{\kappa_3^2-\kappa_1^2}{\kappa_2\kappa_3}\sqrt{1-\tau_{12}^2} \leq \tau_{34} \leq \tau_{12} \\ \dfrac{\kappa_3-\kappa_1}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}} \leq \tau_{12} \leq \dfrac{\kappa_1+\kappa_3}{\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}} \end{array}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; |\tau_{12}| \leq \tau_{34},\ 0 \leq \tau_{34} \leq \frac{\kappa_3-\kappa_1}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}}\right\}$$

$$\bigcup\left\{(\tau_{12},\tau_{34}) \;\middle|\; \begin{array}{l} \dfrac{\kappa_3}{\kappa_1}\tau_{34} - \dfrac{\kappa_3^2-\kappa_1^2}{\kappa_1\kappa_2}\sqrt{1-\tau_{34}^2} \leq \tau_{12} \leq \tau_{34} \\ \dfrac{\kappa_3-\kappa_1}{\sqrt{(\kappa_1-\kappa_3)^2+\kappa_2^2}} \leq \tau_{34} \leq \dfrac{\kappa_1+\kappa_3}{\sqrt{(\kappa_1+\kappa_3)^2+\kappa_2^2}} \end{array}\right\},$$

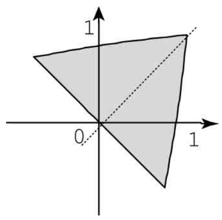

FIGURE 1. D_K in the cases i), ii)

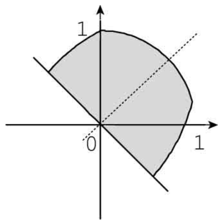

FIGURE 2. D_K in the case iii)

iii) *when* $\kappa_1 = \kappa_3$,

$$D_K = \left\{ (\tau_{12}, \tau_{23}) \,\middle|\, \begin{array}{c} -\tau_{12} \le \tau_{23} \le \dfrac{\kappa_2}{\kappa_1}\tau_{12} + \sqrt{1-\tau_{12}^2}\,, \\ 0 > \tau_{12} \ge -\dfrac{2\kappa_1}{\sqrt{4\kappa_1^2+\kappa_2^2}} \end{array} \right\}$$

$$\bigcup \left\{ (\tau_{12}, \tau_{23}) \,\middle|\, \begin{array}{c} -\tau_{12} \le \tau_{23} \le \sqrt{1-\tau_{12}^2}\,, \\ 0 \le \tau_{12} \le \dfrac{\kappa_1}{\sqrt{(\kappa_1+\kappa_2)^2+\kappa_1^2}} \end{array} \right\}$$

$$\bigcup \left\{ (\tau_{12}, \tau_{23}) \,\middle|\, \begin{array}{c} \dfrac{\kappa_2}{\kappa_1}\tau_{12} + \sqrt{1-\tau_{12}^2} \le \tau_{23} \le \sqrt{1-\tau_{12}^2}\,, \\ \dfrac{\kappa_1}{\sqrt{(\kappa_1+\kappa_2)^2+\kappa_1^2}} \le \tau_{12} \le \dfrac{2\kappa_1}{\sqrt{4\kappa_1^2+\kappa_2^2}} \end{array} \right\}.$$

In view of these results, it seems that in the case $n \ge 3$ the moduli space $\mathcal{K}_3(\mathbf{C}M^n)$ is not compatible with $\mathcal{K}_4(\mathbf{C}M^n)$ even from set theoretical point of view. We shall hence classify Killing helices on $\mathbf{C}M^n$.

We shall call a helix of proper order $2d-1$ or $2d$ on $\mathbf{C}M^n$ *essential* if it lies on some totally geodesic $\mathbf{C}M^d$. A circle γ is essential if and only if its complex torsion is ± 1. Such a circle is interpreted as a trajectory for a Kähler magnetic field (cf. [1]). If we denote by $\mathcal{EK}_d(\mathbf{C}M^n)$ the moduli space of essential Killing helices of proper order d, we see

$$\mathcal{EK}_1(\mathbf{C}M^n) \cong \{0\}\,, \qquad \mathcal{EK}_2(\mathbf{C}M^n) \cong (0,\infty)\,,$$
$$\mathcal{EK}_3(\mathbf{C}M^n) \cong (0,\infty)^2\,,\ \mathcal{EK}_4(\mathbf{C}M^n) \cong (0,\infty)^2 \times (\mathbf{R}\setminus\{0\})\,.$$

and we may say that they form a “building structure”. As we may consider the moduli spaces of essential Killing helices form “frames” of the moduli spaces of Killing helices, we are interested in non-essential parts, especially on $\mathcal{K}_2(\mathbf{C}M^n) \setminus \mathcal{EK}_2(\mathbf{C}M^n) \cong (0,\infty)\times[0,1)$. For this sake we need to investigate some geometric properties of essential Killing helices. Since it is not easy to treat all of them, it is better to restrict ourselves within some special kind of Killing helices of proper order 4. In view of Proposition 2 we find that complex torsions of essential Killing helices of proper order 4 take extremum value when $\kappa_1 = \kappa_3$ under the condition (2-ii) in Proposition 2; $\tau_{12} = \tau_{34} = 0$ and $\tau_{23} = -\tau_{14} = \pm 1$. In this case their Frenet frames satisfy $Y_3 = \mp JY_2,\ Y_4 = \pm J\dot{\gamma}$. Inspired by trajectories for Kähler magnetic fields, we shall consider these helices. We shall call such helices of proper order 4 *moderate*. In the following sections we focus on moderate Killing helices of proper order 4.

3. Moderate Killing helices of proper order four

From now on we restrict ourselves on helices to a complex projective space $\mathbf{C}P^n$. In this section we study when moderate Killing helices on $\mathbf{C}P^n$ are closed. We call a smooth curve γ parameterized by its arc-length *closed* if there is a positive t_c with $\gamma(t + t_c) = \gamma(t)$ for all t. The minimum positive t_c with this property is called the *length* of γ and is denoted by length(γ). When γ is not closed, we say it is *open* and set length$(\gamma) = \infty$. We denote by $\mathbf{C}P^n(c)$ a complex projective space of constant holomorphic sectional curvature c. Since we treat lengths of closed helices, we need to study $\mathbf{C}P^n(c)$. But for the sake of simplicity, we study the case $c = 4$ first.

THEOREM 1. *On* $\mathbf{C}P^n(4)$, *a moderate Killing helix of proper order* 4 *with geodesic curvatures* $\kappa_1, \kappa_2, \kappa_1$ *satisfies the following properties.*

(1) *If* $9\kappa_1^2 + 2\kappa_2^2 = 18$, *then it is closed and has length* $2\sqrt{2}\pi \Big/ \sqrt{8 - \kappa_1^2}$.

(2) *If* $9\kappa_1^2 + 2\kappa_2^2 \neq 18$, *then it is closed if and only if*

$$\frac{\kappa_2|9\kappa_1^2 + 2\kappa_2^2 - 18|}{2(3\kappa_1^2 + \kappa_2^2 + 3)^{3/2}} = \frac{q(9p^2 - q^2)}{(3p^2 + q^2)^{3/2}}$$

for some relatively prime integers p, q *with* $p > q > 0$. *In this case its length is given as* $\pi\delta(p, q)\sqrt{(3p^2 + q^2)/(3\kappa_1^2 + \kappa_2^2 + 3)}$, *where* $\delta(p, q) = 1$ *when the product* pq *is odd and* $\delta(p, q) = 2$ *when the product* pq *is even.*

Since Frenet frame of a moderate Killing helix γ of proper order 4 satisfies $Y_3 = \pm JY_2,\ Y_4 = \mp J\dot{\gamma}$, its system of ordinary differential equations is reduced to

$$\begin{cases} \nabla_{\dot{\gamma}}\dot{\gamma} = \kappa_1 Y_2\,, \\ \nabla_{\dot{\gamma}} Y_2 = -\kappa_1\dot{\gamma} \pm \kappa_2 JY_2\,. \end{cases} \tag{3.1}$$

In order to study Killing helices, it is a basic idea to use a Hopf fibration $\varpi : S^{2n+1}(1) \to \mathbf{C}P^n(4)$. We take a horizontal lift $\hat{\gamma}$ on $S^{2n+1}(1)$ of a moderate Killing helix γ on $\mathbf{C}P^n(4)$ and regard it as a curve in $\mathbf{C}^{n+1}$. The Riemannian connections ∇ on $\mathbf{C}P^n(4)$ and $\overline{\nabla}$ on $\mathbf{C}^{n+1}$ are related by the formula

$$\overline{\nabla}_X Y = \nabla_X Y + \langle X, JY\rangle J\mathcal{N} - \langle X, Y\rangle\mathcal{N} \tag{3.2}$$

for vector fields X, Y on $\mathbf{C}P^n(4)$, which are identified with horizontal vector fields on $S^{2n+1}(1)$. Here $\mathcal{N}$ denotes the outward unit normal of S^{2n+1} in $\mathbf{C}^{n+1}$ and J also denotes the complex structure on $\mathbf{C}^{n+1}$. Therefore the equations (3.1) turn to

$$\begin{cases} \overline{\nabla}_{\dot{\hat{\gamma}}}\dot{\hat{\gamma}} = \kappa_1 Y_2 - \mathcal{N}\,, \\ \overline{\nabla}_{\dot{\hat{\gamma}}} Y_2 = -\kappa_1 \dot{\hat{\gamma}} \ \pm \ \kappa_2 J Y_2\,, \end{cases}$$

which is equivalent to

$$\frac{d^3\hat{\gamma}}{dt^3} \mp \sqrt{-1}\,\kappa_2 \frac{d^2\hat{\gamma}}{dt^2} + (\kappa_1^2+1)\frac{d\hat{\gamma}}{dt} \mp \sqrt{-1}\,\kappa_2\hat{\gamma} = 0\,, \tag{3.3}$$

where the double signs take the same signature.

In order to prove Theorem 1, we need to recall our study on circles on $\mathbf{C}P^n(4)$ in [5]. Let σ be a circle on $\mathbf{C}P^n(4)$ of geodesic curvature $1/\sqrt{2}$ and complex torsion τ $(0 \le |\tau| < 1)$. By use of (3.2) we find a horizontal lift $\hat{\sigma}$ of σ with respect to a Hopf fibration $\varpi : S^{2n+1}(1) \to \mathbf{C}P^n(4)$ satisfies $\hat{\sigma}''' + (3/2)\,\hat{\sigma}' - \sqrt{-1}\,(\tau/\sqrt{2})\,\hat{\sigma} = 0$ as a curve in $\mathbf{C}^{n+1}$. Its characteristic equation $\lambda^3 + (3/2)\lambda - \sqrt{-1}\tau/\sqrt{2} = 0$ should have three distinct pure imaginary solutions. By putting $\lambda = \sqrt{-1}\,\theta$, we denote by a_τ, b_τ and d_τ the solutions of

$$\theta^3 - (3/2)\,\theta + (\tau/\sqrt{2}) = 0 \tag{3.4}$$

satisfying $a_\tau < b_\tau < d_\tau$.

PROPOSITION 4 ([5]). *Let σ be a circle on $\mathbf{C}P^n(4)$ of geodesic curvature $1/\sqrt{2}$ and complex torsion τ $(0 \le |\tau| < 1)$.*

(1) *If $\tau = 0$, then it is closed and has length $2\sqrt{6}\pi/3$.*
(2) *If one of (hence all of) the ratios $a_\tau/b_\tau, b_\tau/d_\tau, d_\tau/a_\tau$ is (are) rational, it is closed and has length $2\pi \times$L.C.M.$\{(b_\tau - a_\tau)^{-1},\ (d_\tau - a_\tau)^{-1}\}$, where L.C.M$(\alpha, \beta)$ denotes the least common multiple of α and β.*
(3) *It is closed if and only if $\tau = \pm q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with some relatively prime integers p, q satisfying $p > q > 0$. In this case its length is given as $(\sqrt{2}/3)\pi\delta(p,q)\sqrt{3p^2+q^2}$ by use of $\delta(p,q)$ given in Theorem 1.*

We are now in a position to prove Theorem 1. The characteristic equation for (3.3) is

$$\lambda^3 \mp \sqrt{-1}\,\kappa_2\lambda^2 + (\kappa_1^2+1)\lambda \mp \sqrt{-1}\,\kappa_2 = 0\,,$$

which should have three distinct pure imaginary solutions. If we put $\lambda = \sqrt{-1}(\Lambda \pm (\kappa_2/3))$, it becomes

$$\Lambda^3 - \frac{1}{3}(3\kappa_1^2 + \kappa_2^2 + 3)\,\Lambda \mp \frac{\kappa_2}{27}\,(9\kappa_1^2 + 2\kappa_2^2 - 18) = 0\,. \tag{3.5}$$

We denote by $a_{(\kappa_1,\kappa_2)}$, $b_{(\kappa_1,\kappa_2)}$, $d_{(\kappa_1,\kappa_2)}$ the solutions of this cubic equation satisfying $a_{(\kappa_1,\kappa_2)} < b_{(\kappa_1,\kappa_2)} < d_{(\kappa_1,\kappa_2)}$. We then have

$$\hat{\gamma}(t) = e^{\pm\sqrt{-1}\,\kappa_2 t/3}\big\{A\exp(\sqrt{-1}\,a_{(\kappa_1,\kappa_2)}t) + B\exp(\sqrt{-1}\,b_{(\kappa_1,\kappa_2)}t) + D\exp(\sqrt{-1}\,d_{(\kappa_1,\kappa_2)}t)\big\}$$

with some $A, B, D \in \mathbf{C}^{n+1}$. This guarantees that γ is closed if and only if one of (hence all of) the ratios $a_{(\kappa_1,\kappa_2)}/b_{(\kappa_1,\kappa_2)}$, $b_{(\kappa_1,\kappa_2)}/d_{(\kappa_1,\kappa_2)}$, $d_{(\kappa_1,\kappa_2)}/a_{(\kappa_1,\kappa_2)}$ is (are) rational, and its length in this case is

$$2\pi \times \text{L.C.M}\{(b_{(\kappa_1,\kappa_2)} - a_{(\kappa_1,\kappa_2)})^{-1}, (d_{(\kappa_1,\kappa_2)} - a_{(\kappa_1,\kappa_2)})^{-1}\}.$$

We now compare two characteristic cubic equations (3.4), (3.5) for circles and for moderate Killing helices. If we put $\Theta = \big(3\big/\sqrt{2(3\kappa_1^2+\kappa_2^2+3)}\big)\Lambda$, then the equation (3.5) becomes

$$\Theta^3 - \frac{3}{2}\,\Theta - \frac{\kappa_2(9\kappa_1^2+2\kappa_2^2-18)}{2\sqrt{2}(3\kappa_1^2+\kappa_2^2+3)^{3/2}} = 0\,. \tag{3.6}$$

Since the function $f_{\kappa_2}(\kappa_1) = \kappa_2^2(9\kappa_1^2+2\kappa_2^2-18)^2 - 4(3\kappa_1^2+\kappa_2^2+3)^3$ is monotone decreasing when $\kappa_1 > 0$ and $f_{\kappa_2}(0) = -108(\kappa_2^2-1)^2$ for each positive κ_2, we see that

$$\tau_{(\kappa_1,\kappa_2)} = \frac{\kappa_2(9\kappa_1^2+2\kappa_2^2-18)}{2(3\kappa_1^2+\kappa_2^2+3)^{3/2}}$$

satisfies $|\tau_{(\kappa_1,\kappa_2)}| < 1$. Thus a moderate Killing helix γ of proper order 4 with geodesic curvatures $\kappa_1, \kappa_2, \kappa_1$ has quite similar properties to a circle σ of geodesic curvature $1/\sqrt{2}$ and of complex torsion $\tau_{(\kappa_1,\kappa_2)}$. The helix γ is closed if and only if σ is closed. When it is closed, as we have $a_{(\kappa_1,\kappa_2)} = \big(\sqrt{2(3\kappa_1^2+\kappa_2^2+3)}\big/3\big)a_{\tau_{(\kappa_1,\kappa_2)}}$ and so on for $b_{(\kappa_1,\kappa_2)}$, $d_{(\kappa_1,\kappa_2)}$, we find that the lengths of γ and σ satisfy

$$\text{length}(\gamma) = \frac{3}{\sqrt{2(3\kappa_1^2+\kappa_2^2+3)}} \times \text{length}(\sigma)\,.$$

With the aid of Proposition 4, this demonstrates the following properties for moderate Killing helices on $\mathbf{C}P^n(4)$. When $9\kappa_1^2 + 2\kappa_2^2 = 18$, then $\tau_{(\kappa_1,\kappa_2)} = 0$, hence a moderate Killing helix of geodesic curvatures $\kappa_1, \kappa_2, \kappa_1$ is closed and has length $2\sqrt{3}\pi\big/\sqrt{3\kappa_1^2+\kappa_2^2+3} = 2\sqrt{2}\pi\big/\sqrt{8-\kappa_1^2}$. When $9\kappa_1^2+2\kappa_2^2 \neq 18$, then it is closed if and only if $\tau_{(\kappa_1,\kappa_2)} = \pm q(9p^2 - q^2)(3p^2+q^2)^{-3/2}$ with some relatively prime integers p, q satisfying $p > q > 0$. In this case its length is given as $\pi\delta(p,q)\sqrt{(3p^2+q^2)/(3\kappa_1^2+\kappa_2^2+3)}$. These complete the proof of Theorem 1.

We here consider the influence of a homothetic change of metrics. We take a helix γ of geodesic curvatures κ_i $(i=1,\dots,d)$ on a Riemannian manifold (M,g). If we change the metric g on M homothetically to $\tilde{g}=\lambda^2 g$ with a positive λ, then sectional curvatures of $(M,\tilde{g})$ are λ^{-2}-times of those of (M,g). Under this operation on metrics, the curve $\tilde{\gamma}(t)=\gamma(t/\lambda)$ becomes a helix of geodesic curvatures κ_i/λ $(i=1,\dots,d)$ on $(M,\tilde{g})$, and hence length$(\tilde{\gamma})=\lambda\times$ length(γ) when γ is closed. When $(\mathbf{C}P^n,g)$ has constant holomorphic sectional curvature c, we consider the metric $\tilde{g}=(c/4)g$. Then $(\mathbf{C}P^n,\tilde{g})$ has constant holomorphic sectional curvature 4. Thus we obtain the following:

THEOREM 1'. *On $\mathbf{C}P^n(c)$, a moderate Killing helix of proper order* 4 *with geodesic curvatures $\kappa_1,\kappa_2,\kappa_1$ satisfies the following properties.*

(1) *If $18\kappa_1^2+4\kappa_2^2=9c$, then it is closed and has length $2\sqrt{2}\pi\Big/\sqrt{2c-\kappa_1^2}$.*

(2) *If $18\kappa_1^2+4\kappa_2^2\neq 9c$, then it is closed if and only if*

$$\frac{2\kappa_2|18\kappa_1^2+4\kappa_2^2-9c|}{(12\kappa_1^2+4\kappa_2^2+3c)^{3/2}}=\frac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}}$$

for some relatively prime integers p,q with $p>q>0$. In this case its length is given as $2\pi\delta(p,q)\sqrt{(3p^2+q^2)/(12\kappa_1^2+4\kappa_2^2+3c)}$, where $\delta(p,q)$ is given in Theorem 1.

COROLLARY 1. *On $\mathbf{C}P^n(c)$, a moderate Killing helix of proper order* 4 *with geodesic curvatures $(\sqrt{2c},\kappa_2)$ is closed if and only if $\kappa_2=\sqrt{c}\,q(9p^2-q^2)/\{2p(p^2-q^2)\}$ with some pair of relatively prime positive integers p,q satisfying $p>q$. In this case its length is $2\pi\delta(p,q)p(p^2-q^2)/\{\sqrt{c}\,(3p^2+q^2)\}$.*

For moderate Killing helices on $\mathbf{C}P^n(c)$ whose geodesic curvatures satisfy $18\kappa_1^2+4\kappa_2^2=9c$, Chen-Maeda[7] also study them from the viewpoint of type numbers. Generally, a Riemannian submanifold M of $\mathbf{R}^m$ is said to be of k-type if the isometric immersion $\varphi: M\to\mathbf{R}^m$ regarding it as a position vector has the spectral decomposition $\varphi=\varphi_0+\sum_{i=1}^k\varphi_i$, where φ_0 is constant and φ_i satisfies $\Delta\varphi_i=\lambda_i\varphi_i$ for $i=1,2,\dots,k$ with the Laplacian Δ and mutually different eigenvalues $\lambda_1,\dots,\lambda_k$ (see [6] for details). In [7], they studied helices on $\mathbf{C}P^n(4)$ through the embedding F given by the composition of the minimal embedding $\mathbf{C}P^n(4)\to S^{n(n+2)-1}(2(n+1)/n)$ and a totally geodesic embedding $S^{n(n+2)-1}(2(n+1)/n)\to\mathbf{R}^{n(n+2)}$. In this paper we say a helix γ is of k-type if the curve $F(\gamma)$ is of k-type. As was pointed out in [7], moderate Killing helices whose geodesic curvatures satisfy $18\kappa_1^2+4\kappa_2^2=9c$ are helices of 2-type because $b_{(\kappa_1,\kappa_2)}=0$ and $a_{(\kappa_1,\kappa_2)}=-d_{(\kappa_1,\kappa_2)}$. They gave a characterization of these helices and circles with null structure torsion, which are also of 2-type.

4. Embeddings of the moduli space of non-essential circles

On the moduli space $\mathcal{K}(\mathbf{C}P^n) = \bigcup_{d=1}^{\infty} \mathcal{K}_d(\mathbf{C}P^n)$ of Killing helices on $\mathbf{C}P^n$, we define the *length spectrum* $\mathcal{L} : \mathcal{K}(\mathbf{C}P^n) \to (0, \infty) \cup \{\infty\}$ of Killing helices by $\mathcal{L}([\gamma]) = \text{length}(\gamma)$, where $[\gamma] \in \mathcal{K}(\mathbf{C}P^n)$ denotes the congruence class containing a Killing helix γ. We also denote by $\mathcal{L}$ the restriction of $\mathcal{L}$ onto each stage $\mathcal{K}_d(\mathbf{C}P^n)$. In this section we construct an injection of $\mathcal{K}_2(\mathbf{C}P^n) \setminus \mathcal{EK}_2(\mathbf{C}P^n)$ into the moduli space $\mathcal{MK}_4(\mathbf{C}P^n)$ $(\subset \mathcal{EK}_4(\mathbf{C}P^n))$ of moderate Killing helices which is compatible with the length spectrum.

It is well known that the length spectrum of circles on a real space form $\mathbf{R}M^n(c)$ of constant sectional curvature c is given as

$$\mathcal{K}_2(\mathbf{R}M^n(c)) \cong (0, \infty) \ni \kappa \mapsto 2\pi \Big/ \sqrt{\kappa^2 + c} \in (0, \infty]\,,$$

where we regard $2\pi/\sqrt{\kappa^2 + c}$ as infinity when $\kappa^2 + c \leq 0$. It is hence continuous with respect to the canonical Euclidean topology on $\mathcal{K}_2(\mathbf{R}M^n(c)) \cong (0, \infty)$. For the length spectrum of circles on a complex space form, the situation is quite different from this. In the preceding paper [2] we studied the length spectrum of circles. The moduli space $\mathcal{K}_2(\mathbf{C}P^n)$ admits a natural lamination structure $\{\mathcal{F}_\mu\}_{\mu \in [0,1]}$. If we induce the canonical Euclidean topology and the differential structure on $\mathcal{K}_2(\mathbf{C}P^n(c)) \cong (0, \infty) \times [0, 1]$, we see

i) the length spectrum $\mathcal{L} : \mathcal{K}_2(\mathbf{C}P^n(c)) \to (0, \infty]$ of circles is smooth on each leaf,
ii) each leaf is set-theoretically maximal with respect to this property.

More precisely, leaves are given in the following manner:

$$\mathcal{F}_\mu = \begin{cases} \big\{[\gamma_{\kappa,0}] \bigm| \kappa > 0\big\}\,, & \text{if } \mu = 0\,, \\ \big\{[\gamma_{\kappa,\tau}] \bigm| 3\sqrt{3}c\kappa\tau(4\kappa^2 + c)^{-3/2} = \mu\big\}\,, & \text{if } 0 < \mu < 1\,, \\ \big\{[\gamma_{\kappa,1}] \bigm| \kappa > 0\big\}\,, & \text{if } \mu = 1, \end{cases}$$

where $[\gamma_{\kappa,\tau}] \in \mathcal{K}_2(\mathbf{C}P^n)$ denotes the congruence class of circles of geodesic curvature κ and of complex torsion τ (≥ 0) on $\mathbf{C}P^n(c)$. When $\mu = q(9p^2 - q^2)(3p^2 + q^2)^{-3/2}$ with a pair of relatively prime positive integers p, q satisfying $p > q$, each point $[\gamma_{\kappa,\tau}] \in \mathcal{F}_\mu$ is a congruence class of closed circles whose lengths are given as $2\delta(p, q)\pi\sqrt{(3p^2 + q^2)/\{3(4\kappa^2 + c)\}}$, and in other cases all points on $\mathcal{F}_\mu$ corresponds to open circles. Since $\mathcal{F}_1 = \mathcal{EK}_2(\mathbf{C}P^n)$ and the topological closure of $\mathcal{F}_\mu$ with $0 < \mu < 1$ have common two points (see Fig. 3), we find that $\mathcal{EK}_2(\mathbf{C}P^n)$ is quite different from other part of $\mathcal{K}_2(\mathbf{C}P^n)$.

In view of the proof of Theorem 1 we find that the moduli space $\mathcal{MK}_4(\mathbf{C}P^n)$ admits a natural foliation structure $\{\mathcal{G}_\mu\}_{\mu \in (-1,1)}$. We should note that $\mathcal{MK}_4(\mathbf{C}P^n)$ is a part of plane $\{(\kappa_1, \kappa_2, -\kappa_1) | \kappa_1, \kappa_2 > 0\}$ in $(0, \infty)^2 \times (\mathbf{R} \setminus \{0\}) \cong \mathcal{EK}_4(\mathbf{C}P^n)$. We induce on $\mathcal{MK}_4(\mathbf{C}P^n) \cong (0, \infty)^2$ the Euclidean topology and the differential structure. Let $[\gamma_{(\kappa_1,\kappa_2)}]$ denote the congruence class containing a moderate Killing helix with geodesic curvatures

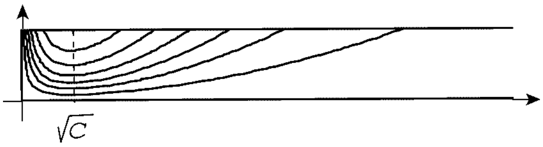

FIGURE 3. Lamination on $\mathcal{K}_2(\mathbf{C}P^n)$

$\kappa_1, \kappa_2, \kappa_1$ on $\mathbf{C}P^n(c)$. We define leaves $\mathcal{G}_\mu$ $(-1 < \mu < 1)$ on $\mathcal{MK}_4(\mathbf{C}P^n(c))$ by

$$\mathcal{G}_\mu = \left\{ [\gamma_{(\kappa_1,\kappa_2)}] \;\middle|\; \frac{2\kappa_2(18\kappa_1^2 + 4\kappa_2^2 - 9c)}{(12\kappa_1^2 + 4\kappa_2^2 + 3c)^{3/2}} = \mu \right\}.$$

We should note that when $\mu \leq 0$ the leaf $\mathcal{G}_\mu$ contains $[\gamma_{(\kappa_1^0,\,\sqrt{c}/2)}]$ for some κ_1^0 with $0 < \kappa_1^0 \leq 2\sqrt{c}/3$ and when $\mu > 0$ the leaf $\mathcal{G}_\mu$ contains $[\gamma_{(\sqrt{2c},\,\kappa_2^0)}]$ for some κ_2^0. In the latter case we see $\mu = 2\kappa_2^0(4(\kappa_2^0)^2 + 27c)^{-1/2}$.

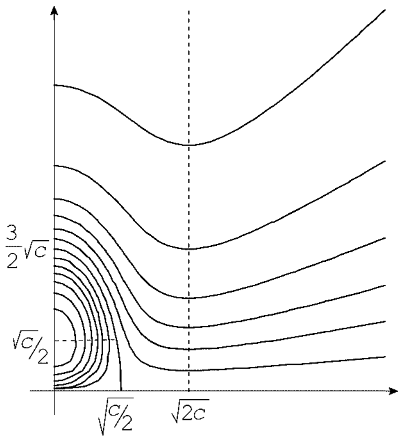

FIGURE 4. Foliation on $\mathcal{MK}_4(\mathbf{C}P^n)$

By the equation (3.6), we obtain the following:

THEOREM 2. *There exists a unique foliation structure* $\{\mathcal{G}_\mu\}_{\mu\in(-1,1)}$ *on the moduli space* $\mathcal{MK}_4(\mathbf{C}P^n)$ *of moderate Killing helices on* $\mathbf{C}P^n$ *which satisfies the following properties*:

i) *the length spectrum* $\mathcal{L} : \mathcal{MK}_4(\mathbf{C}P^n) \to (0, \infty]$ *of moderate Killing helices is smooth on each leaf*,

ii) *each leaf is set-theoretically maximal with respect to this property.*

We now define an injection $\Phi : \mathcal{K}_2(\mathbf{C}P^n(c)) \setminus \mathcal{EK}_2(\mathbf{C}P^n(c)) \to \mathcal{MK}_4(\mathbf{C}P^n(c))$. Let $K : (0,\infty) \to (0,3)$ be a smooth function satisfying

i) $K(1/\sqrt{2}) = 1$,

ii) $K(\kappa) < 1$ if $0 < \kappa < 1/\sqrt{2}$ and $K(\kappa) > 1$ if $\kappa > 1/\sqrt{2}$,

iii) $2K(\kappa)(9 - K(\kappa)^2)(K(\kappa)^2 + 3)^{-3/2} = 3\sqrt{3}\kappa(\kappa^2+1)^{-3/2}$.

This function K is monotone increasing and satisfies $\lim_{\kappa\to\infty} K(\kappa) = 3$ and $\lim_{\kappa\downarrow 0} K(\kappa) = 0$. We set $\Phi([\gamma_{\kappa,\tau}]) = [\gamma_{(\kappa_1,\kappa_2)}]$, where $\kappa_2 = \kappa_2(\kappa)$ is given as $\kappa_2 = (\sqrt{c}/2)\,K(2\kappa/\sqrt{c})$ and $\kappa_1 = \kappa_1(\kappa,\tau)$ satisfies $18\kappa_1^2 + 4\kappa_2^2 \le 9c$ and

$$\frac{2\kappa_2(9c - 18\kappa_1^2 - 4\kappa_2^2)}{(12\kappa_1^2 + 4\kappa_2^2 + 3c)^{3/2}} = \frac{3\sqrt{3}\,c\kappa\tau}{(4\kappa^2+c)^{3/2}}\,.$$

If we represent $\mathcal{MK}_4(\mathbf{C}P^n(c))$ as $(0,\infty)^2$, the image of Φ is $\{(\kappa_1,\kappa_2) | 18\kappa_1^2 + 4\kappa_2^2 \le 9c\}$. As we see the function f_{κ_2} in the previous section is monotone decreasing for each κ_2, we find that this map Φ is an embedding with respect to the induced smooth structures.

THEOREM 3. *The embedding*

$$\Phi : \mathcal{K}_2(\mathbf{C}P^n(c)) \setminus \mathcal{EK}_2(\mathbf{C}P^n(c)) \to \mathcal{MK}_4(\mathbf{C}P^n(c)) \ \ (\subset \mathcal{EK}_4(\mathbf{C}P^n(c)))$$

maps each leaf $\mathcal{F}_\mu$ $(0 \le \mu < 1)$ *onto a leaf* $\mathcal{G}_{-\mu}$, *and satisfies*

$$\mathcal{L}\circ\Phi([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3(4\kappa^2+c)}{12\kappa_1^2(\kappa,\tau) + 4\kappa_2^2(\kappa) + 3c}} \times \mathcal{L}([\gamma_{\kappa,\tau}])\,.$$

PROOF. The foliation structure $\{\mathcal{F}_\mu\}_{\mu\in[0,1)}$ on $\mathcal{K}_2(\mathbf{C}P^n(c)) \setminus \mathcal{EK}_2(\mathbf{C}P^n(c))$ is constructed by use of canonical maps of normalization which are given by

$$[\gamma_{\kappa,\tau}] \mapsto \big[\gamma_{\sqrt{2c}/4,\ 3\sqrt{3}\,c\kappa\tau(4\kappa^2+c)^{-3/2}}\big]$$

for each κ. By this we see

$$\mathcal{L}([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3c}{2(4\kappa^2+c)}} \times \mathcal{L}\big(\big[\gamma_{\sqrt{2c}/4,\ 3\sqrt{3}\,c\kappa\tau(4\kappa^2+c)^{-3/2}}\big]\big)\,.$$

hence we get the conclusion. □

REMARK 1. Through this embedding Φ, the moduli space $\mathcal{F}_0 = \{[\gamma_{\kappa,0}] | \kappa > 0\}$ of totally real circles is mapped onto the moduli space $\mathcal{G}_0 = \{[\gamma_{(\kappa_1,\kappa_2)}] | 18\kappa_1^2 + 4\kappa_2^2 = 9c\}$ of Killing helices of proper order 4 and of 2-type. Moreover, this embedding Φ maps congruence classes of circles of 3-type to congruence classes of Killing helices of 3-type.

Here we study a bit more on length spectrum on the image of Φ. When $\mu = -q(9p^2 - q^2)(3p^2+q^2)^{-3/2}$, each point $[\gamma_{(\kappa_1,\kappa_2)}] \in \mathcal{G}_\mu$ is a congruence class of closed helices. At an end point of this leaf we see

$$\lim_{(\kappa_1,\kappa_2)\to(0,\,\sqrt{c}\,q/(2p))} \mathcal{L}([\gamma_{(\kappa_1,\,\kappa_2)}]) = 2\pi\delta(p,q)p/\sqrt{c}\,,$$

where $[\gamma_{(\kappa_1,\,\kappa_2)}]$ runs on the leaf $\mathcal{G}_\mu$. This value is $\delta(p,q)p$ times of the length of a geodesic. Also on the leaf $[\gamma_{(\kappa_1,\,\kappa_2)}] \in \mathcal{G}_0$, we see

$$\lim_{(\kappa_1,\kappa_2)\to(\sqrt{c/2},\,0)} \mathcal{L}\big([\gamma_{(\kappa_1,\,\kappa_2)}]\big) = 4\pi/\sqrt{3c}\,,$$

$$\lim_{(\kappa_1,\kappa_2)\to(0,\,3\sqrt{c}/2)} \mathcal{L}\big([\gamma_{(\kappa_1,\,\kappa_2)}]\big) = 2\pi/\sqrt{c}\,,$$

which are the length of a circle with geodesic curvature $\sqrt{c/2}$ and null complex torsion and the length of a geodesic.

There is another embedding Ψ of

$$\mathcal{K}_2^*(\mathbf{C}P^n(c)) := \mathcal{K}_2(\mathbf{C}P^n(c)) \setminus \Big(\mathcal{E}\mathcal{K}_2(\mathbf{C}P^n(c)) \cup \big\{[\gamma_{\kappa,0}] \in \mathcal{K}_2(\mathbf{C}P^n(c)) \;\big|\; \kappa > 0\big\}\Big)$$

of the moduli space of circles of positive geodesic curvature and of complex torsion $0 < |\tau| < 1$ into $\mathcal{M}\mathcal{K}_4(\mathbf{C}P^n(c))$ which preserves the foliation structure. Let $\widehat{K} : (0,\infty) \to (0,\infty)$ be a smooth function satisfying

i) $\widehat{K}(1/\sqrt{2}) = 2\sqrt{2}$,

ii) $\widehat{K}(\kappa) < 2\sqrt{2}$ if $0 < \kappa < 1/\sqrt{2}$ and $\widehat{K}(\kappa) > 2\sqrt{2}$ if $\kappa > 1/\sqrt{2}$,

iii) $2\widehat{K}(\kappa)^2(\widehat{K}(\kappa)^2 + 4)^{-3/2} = \kappa(\kappa^2+1)^{-3/2}$.

This function $\widehat{K}$ is monotone increasing and satisfies $\lim_{\kappa\to\infty} \widehat{K}(\kappa) = \infty$ and $\lim_{\kappa\downarrow 0} \widehat{K}(\kappa) = 0$. We set $\Psi([\gamma_{\kappa,\tau}]) = [\gamma_{(\kappa_1,\kappa_2)}]$, where $\kappa_1 = (\sqrt{c}/2)\,\widehat{K}(2\kappa/\sqrt{c})$ and $\kappa_2 = \kappa_2(\kappa,\tau)$ satisfies $18\kappa_1^2 + 4\kappa_2^2 > 9c$ and

$$\frac{2\kappa_2(18\kappa_1^2 + 4\kappa_2^2 - 9c)}{(12\kappa_1^2 + 4\kappa_2^2 + 3c)^{3/2}} = \frac{3\sqrt{3}\,c\kappa\tau}{(4\kappa^2 + c)^{3/2}}\,.$$

The image of Ψ is $\{[\gamma_{(\kappa_1,\kappa_2)}] \in \mathcal{M}\mathcal{K}_4(\mathbf{C}P^n(c)) | 18\kappa_1^2 + 4\kappa_2^2 > 9c\}$. One can easily see that it is an embedding.

PROPOSITION 5. *The embedding* $\Psi : \mathcal{K}_2^*(\mathbf{C}P^n(c)) \to \mathcal{M}\mathcal{K}_4(\mathbf{C}P^n(c))$ *maps each leaf* $\mathcal{F}_\mu$ $(0 < \mu < 1)$ *onto a leaf* $\mathcal{G}_\mu$, *and satisfies*

$$\mathcal{L}\circ\Psi([\gamma_{\kappa,\tau}]) = \sqrt{\frac{3(4\kappa^2 + c)}{12\kappa_1^2(\kappa) + 4\kappa_2^2(\kappa,\tau) + 3c}} \times \mathcal{L}([\gamma_{\kappa,\tau}])\,.$$

REMARK 2. If we consider $[\gamma_{(\kappa_1,0)}]$ to be the congruence class of circles with geodesic curvature κ_1 and null complex torsion for $\kappa_1 \geqq \sqrt{c/2}$, we find the map Ψ extends continuously to $\mathcal{K}_2(\mathbf{C}P^n(c)) \setminus \mathcal{E}\mathcal{K}_2(\mathbf{C}P^n(c))$ which satisfies the length spectrum relation. The image of this extended map is $\{[\gamma_{(\kappa_1,\kappa_2)}] | 18\kappa_1^2 + 4\kappa_2^2 \geq 9c\}$.

With the aid of these maps Φ and Ψ we may say that we can describe the structure of the moduli space $\mathcal{K}_1(\mathbf{C}P^n) \cup \mathcal{K}_2(\mathbf{C}P^n) \cup \mathcal{E}\mathcal{K}_3(\mathbf{C}P^n) \cup \mathcal{E}\mathcal{K}_4(\mathbf{C}P^n)$ of helices of low order.

Finally, we mention some properties of the length spectrum of moderate Killing helices on $\mathbf{C}P^n$. For a positive number ℓ, we call the cardinality of the set $\mathcal{L}^{-1}(\ell)$ the multiplicity of $\mathcal{L}$ at ℓ.

PROPOSITION 6. *The length spectrum* $\mathcal{L} : \mathcal{M}\mathcal{K}_4(\mathbf{C}P^n(c)) \to (0, \infty]$ *of moderate Killing helices of proper order* 4 *is surjective and its multiplicity is infinite at each point in* $(0, \infty)$. *Even if we consider a restriction* $\mathcal{L}|_{\mathcal{G}_\mu}$ *on each leaf* $\mathcal{G}_\mu$ *with* $\mu \in (0, 1/8)$ *which consists of congruence classes of closed helices*, *it is not injective.*

PROOF. By Theorem 1', the image of $\mathcal{L}$ is the set

$$\left\{ \frac{2\sqrt{2}\pi}{\sqrt{2c-\kappa_1^2}} \,\middle|\, 0 < \kappa_1 < \sqrt{2c} \right\}$$

$$\bigcup \left\{ 2\pi\delta(p,q)\sqrt{\frac{3p^2+q^2}{12\kappa_1^2+4\kappa_2^2+3c}} \,\middle|\, \begin{array}{l} p > q > 0 \text{ and } p, q \text{ are relatively prime,} \\ \text{positive } \kappa_1, \kappa_2 \text{ satisfy} \\ \dfrac{2\kappa_2|18\kappa_1^2+4\kappa_2^2-9c|}{(12\kappa_1^2+4\kappa_2^2+3c)^{3/2}} = \dfrac{q(9p^2-q^2)}{(3p^2+q^2)^{3/2}} \end{array} \right\}$$

$$\bigcup \{\infty\} \,.$$

We first show $\mathcal{L}$ is surjective. We take an arbitrary positive number α. We choose a pair of relatively prime odd integers (p_0, q_0) satisfying $p_0 > q_0 > 0$ and $3p_0^2 + q_0^2 - q_0^{2/3}(9p_0^2 - q_0^2)^{2/3} > 12c\alpha^{-2}$. For these α, p_0, q_0, we consider the following simultaneous equations

$$\left\{ \begin{array}{l} 12\kappa_1^2 + 4\kappa_2^2 + 3c = \alpha^2(3p_0^2 + q_0^2)\,, \\ 4\kappa_2^2(18\kappa_1^2 + 4\kappa_2^2 - 9c)^2 = \alpha^6 q_0^2(9p_0^2 - q_0^2)^2\,, \end{array} \right. \tag{4.1}$$

on κ_1 and κ_2. If a pair (κ_1, κ_2) satisfies these equations, then a moderate Killing helix of geodesic curvatures $\kappa_1, \kappa_2, \kappa_1$ is closed and is of length $2\pi/\alpha$. Substituting the former equation in (4.1) into the latter we have

$$16\kappa_2^6 - 24\{\alpha^2(3p_0^2+q_0^2) - 9c\}\kappa_2^4 + 9\{\alpha^2(3p_0^2+q_0^2) - 9c\}^2\kappa_2^2 - \alpha^6 q_0^2(9p_0^2-q_0^2)^2 = 0\,.$$

We denote the left hand side of the above by $h(\kappa_2^2) = h_{p_0,q_0,\alpha}(\kappa_2^2)$ and consider the cubic equation $h(x) = 0$. Since $h(0) < 0$ and

$$\begin{aligned} &h(\{\alpha^2(3p_0^2+q_0^2) - 3c\}/4) \\ &\quad = (\alpha^2(3p_0^2+q_0^2) - 3c)(\alpha^2(3p_0^2+q_0^2) - 12c)^2 - \alpha^6 q_0^2(9p_0^2-q_0^2)^2 > 0 \end{aligned}$$

by the assumption on (p_0, q_0), we see the cubic equation $h(x) = 0$ has a solution in the interval $(0, \{\alpha^2(3p_0^2 + q_0^2) - 3c\}/4)$. Thus we find (4.1) has a solution. This shows that $\mathcal{L}$ takes the value $2\pi/\alpha$. We deduce that $\mathcal{L}$ is surjective.

The above proof also shows that the multiplicity is infinite at each point. For a positive integer α there are infinitely many odd integers p (> 1) satisfying $3p^2 - (9p^2 - 1)^{2/3} > 12c\alpha^{-2} - 1$. For each pair $(p, 1)$ we have a solution $(\kappa_1(p), \kappa_2(p))$ for (4.1) corresponding to this pair. This show $\mathcal{L}(\gamma_{(\kappa_1(p),\kappa_2(p))}) = 2\pi/\alpha$. By the first equality in (4.1) we see $(\kappa_1(p), \kappa_2(p)) \neq (\kappa_1(p'), \kappa_2(p'))$ if $p \neq p'$, hence the multiplicity of $\mathcal{L}$ at $2\pi/\alpha$ is infinite.

We next show that $\mathcal{L}|_{\mathcal{G}_\mu}$ is not injective. We suppose $\mu = q_0(9p_0^2 - q_0^2)(3p_0^2 + q_0^2)^{-3/2}$ with a pair of relatively prime integers (p_0, q_0) with $p_0 > q_0 > 0$. Since $0 < \mu < 1/8$ we have $(3p_0^2 + q_0^2)^3 > 2^6 q_0^2(9p_0^2 - q_0^2)^2$, hence we can choose a positive number α satisfying $\{3p_0^2 + q_0^2 - q_0^{2/3}(9p_0^2 - q_0^2)^{2/3}\}/9 > c\alpha^{-2} > (3p_0^2 + q_0^2)/12$. For this α, the cubic function $h(x)$ takes its maximum at $x = \{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4$ and its minimum at $x = 3\{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4$ because $\alpha^2(3p_0^2 + q_0^2) > 9c$. Since $h(0) < 0$ and

$$h(\{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4) = \{\alpha^2(3p_0^2 + q_0^2) - 9c\}^3 - \alpha^6 q_0^2(9p_0^2 - q_0^2)^2 > 0\,,$$
$$h(3\{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4) = -\alpha^6 q_0^2(9p_0^2 - q_0^2)^2 < 0\,,$$

the cubic equation $h(x) = 0$ has solutions s_1, s_2, s_3 satisfying

$$0 < s_1 < \{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4 < s_2 < 3\{\alpha^2(3p_0^2 + q_0^2) - 9c\}/4 < s_3\,.$$

As we have $\alpha^2(3p_0^2 + q_0^2) - 3c - 4s_2 > 24c - 2\alpha^2(3p_0^2 + q_0^2) > 0$, we find (4.1) has two pairs of positive solutions. Moreover, for these pairs (κ_1, κ_2) we have

$$18\kappa_1^2 + 4\kappa_2^2 - 9c = \frac{3}{2}\{\alpha^2(3p_0^2 + q_0^2) - 9c\} - 2\kappa_2 > 0\,.$$

Hence $\mathcal{L}|_{\mathcal{G}_\mu}$ is not injective. □

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