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Information Geometry of Poisson Kernels on Damek-Ricci Spaces

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Abstract. For Damek-Ricci spaces (X, g) we compute the exact form of the Busemann function which is needed to represent the Poisson kernel of (X, g) in exponential form in terms of the Busemann function and the volume entropy. From this fact, we show that the Poisson kernel map $\varphi : (X, g) \to (\mathcal{P}(\partial X), G)$ is a homothetic embedding. Here $\mathcal{P}(\partial X)$ is the space of probability measures having positive density function on the ideal boundary ∂X of X, and G is the Fisher information metric on $\mathcal{P}(\partial X)$.

1. Introduction

Let (X^n, g) be a Hadamard manifold and ∂X its ideal boundary, that is, the space of all geodesic rays up to asymptotic equivalence, which is identified with an $(n-1)$-dimensional sphere. Then, we can consider the Dirichlet problem at infinity in a similar way to the classical Dirichlet problem on a bounded domain in $\mathbb{R}^n$, that is, for a given $f \in C^0(\partial X)$, finding a function u on $X \cup \partial X$ satisfying

$$\Delta_X u = 0 \quad \text{and} \quad u|_{\partial X} = f \tag{1}$$

where Δ_X is the Laplace-Beltrami operator on X. We call the fundamental solution to the Dirichlet problem at infinity the *Poisson kernel*, denoted by $P(x, \theta)$, on X. Namely, the solution to (1) is given by the Poisson integral representation $u(x) = \int_{\theta \in \partial X} P(x, \theta) f(\theta) d\theta$. In particular, for each $x \in X$ we have $\int_{\partial X} P(x, \theta) d\theta = 1$.

The first author and Y. Shishido [14] considered the map $\varphi : X \ni x \mapsto P(x, \theta) d\theta \in \mathcal{P}(\partial X)$ called the *Poisson kernel map*. Here $\mathcal{P}(\partial X)$ is the set of all probability measures on ∂X with positive density function. $\mathcal{P}(\partial X)$ carries a Riemannian metric G which is a natural extension of the Fisher information matrix on statistical models, and we can regard $(\mathcal{P}(\partial X), G)$ as an infinite dimensional Riemannian manifold. We call this metric G the *Fisher information metric*. The first author and Y. Shishido proved the following:

THEOREM 1 ([14, Theorem A]). *If (X, g) is an n-dimensional rank one symmetric space of non-compact type, then the Poisson kernel map φ is a homothetic embedding. More*

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explicitly,

$$\varphi^* G = \frac{\rho^2}{n} g \,. \tag{2}$$

Here ρ *is the volume entropy of* (X, g), *which is a geometric quantity measuring the rate of increase of volume*, *defined by*

$$\rho = \lim_{r \to \infty} \frac{1}{r} \log \operatorname{Vol} B(x; r) \,.$$

REMARK 1. The first author and Y. Shishido note that the Poisson kernel map on a rank one symmetric space of non-compact type is a minimal embedding [14, Theorem B]. However, in order to prove the minimality of the Poisson kernel map we need deeper considerations (see [13]).

It is an interesting problem whether the converse statement of Theorem 1 is true or not. That is, do there exist non-symmetric Hadamard manifolds such that the Poisson kernel map is homothetic? Our aim is the following theorem concerning this question:

THEOREM 2. *Let* (X, g) *be an* n*-dimensional Damek-Ricci space. Then*, *the Poisson kernel map* φ *is a homothetic embedding. In particular*, φ *satisfies* (2).

A *Damek-Ricci space* is a one-dimensional extension of a generalized Heisenberg group. It is a solvable Lie group with a left invariant metric, and is a Riemannian homogeneous Hadamard manifold which is harmonic (see [4] for details). A Riemannian manifold (X, g) is *harmonic* if every sufficiently small geodesic sphere in X has constant mean curvature (see [5, Chapter 6] and [4, p.11] for details). Damek-Ricci spaces are generalizations of rank one symmetric spaces of non-compact type. Thus $H^N(\mathbb{C})$, $H^N(\mathbb{H})$, $H^2(\mathbb{O})$ and $H^N(\mathbb{R})$ are examples of Damek-Ricci spaces. From Dotti's theorem [8] it is known that a Damek-Ricci space is of strictly negative curvature if and only if it is rank one symmetric. A Damek-Ricci space satisfies in general the *visibility axiom*, namely any points θ, θ' of its ideal boundary can be joined by a geodesic (see [9, Theorem 2.3] and for the notion of visibility see [3]).

If we suppose that a Hadamard manifold (X, g) satisfies the strictly negative curvature condition $-b^2 \leq K_X \leq -a^2 < 0$ where K_X is the sectional curvature of X, then the existence and uniqueness of the Poisson kernel on (X, g) is guaranteed [18, Chapter II]. Furthermore, when (X, g) is a rank one symmetric space of non-compact type, the Poisson kernel is explicitly written in an exponential form as

$$P(x, \theta) = \exp\left(-\rho B(x, \theta)\right) \tag{3}$$

in terms of the volume entropy ρ and the Busemann function $B(x, \theta)$ on X. This fact is crucial for the proof of Theorem 1, which can be generalized as follows:

PROPOSITION 1. *Let* (X, g) *is an* n*-dimensional Hadamard manifold.*

Assume (X, g) is Riemannian homogeneous and suppose that the Poisson kernel on X is given by

$$P(x,\theta) = \exp\left(-cB(x,\theta)\right) \tag{4}$$

for a positive constant c. Then φ satisfies $\varphi^* G = \frac{c^2}{n} g$.

The geometric meaning of the formula (4), namely, the Poisson kernels thus analytically defined coincide with the exponential form of the Busemann functions defined geodesically, is as follows. For $\theta \in \partial X$ let $H_x(\theta)$ be the horosphere, namely the level hypersurface of the Busemann function $B(\cdot,\theta)$ through $x \in X$ and centered at θ. Then the harmonicity of $\exp(-cB(x,\theta))$, that is, $\Delta B(x,\theta) = -c$, is equivalent to the fact that the mean curvature of any horosphere $H_x(\theta)$ takes a common constant value c, since the Hessian of $B(x,\theta)$ is the second fundamental form of the $H_x(\theta)$. This gives rise to a Hadamard manifold (X, g) being asymptotically harmonic (see [15]). As is well known as a counterexample of the Lichnerowicz conjecture, a Damek-Ricci space is a harmonic, Hadamard manifold. Thus, a Damek-Ricci space is also asymptotically harmonic so that the harmonicity of $\exp(-cB(x,\theta))$ holds. In order to have the formula (4) we need a further analytical criterion, that is, the Dirac delta condition [11]: $\lim_{x\to\theta'} \exp(-cB(x,\theta))d\theta = \delta_\theta(\theta')$ for $\theta' \in \partial X$ which is equivalent to the visibility axiom together with the integral normality condition

$$\int_{\partial X} \exp(-cB(x,\theta))d\theta = 1 \tag{5}$$

for any x. The following lemma is derived from an exact form of the Busemann function of Damek-Ricci spaces.

LEMMA 1. *The Busemann function on Damek-Ricci spaces satisfies condition* (5) *for $c = \rho$.*

Hence, we can assert that (4) holds not only for rank one symmetric spaces but also for Damek-Ricci spaces. From Proposition 1 and Lemma 1, we obtain Theorem 2.

REMARK 2. E. Damek [6] gives for Damek-Ricci spaces a form of the Poisson kernel explicitly from harmonic analysis of Lie groups, which coincides with our construction of $P(x,\theta)$. The details will be given in section 6.

REMARK 3. We also remark that when (X, g) admits a compact Riemannian quotient, (4) implies that (X, g) is a rank one symmetric space of non-compact type [14, Theorem C].

REMARK 4. As in the case of the Poisson kernel map, by using the heat kernel we can define the *heat kernel map* $\varphi_t : X \to \mathcal{P}(X)$ parametrized by $t > 0$. If (X, g) is a harmonic Hadamard manifold, φ_t is a homothetic embedding (see [12]).

The paper is organized as follows. In section 2 we introduce the space of all probability measures with positive density function and the Fisher information metric on it. We recall in section 3 some basic facts about the Poisson kernel and the Busemann function on a Hadamard manifold. In section 4 we consider the condition under which the Poisson kernel is described in the form (3) and give a proof of Proposition 1. We present in section 5 some necessary material about the Damek-Ricci spaces. In the final section we give the Busemann function of a Damek-Ricci space in terms of the group structure in order to prove Lemma 1.

2. The space of probability measures and the Fisher information metric

Let M be a compact, oriented, n-dimensional Riemannian manifold and dv_M the canonical Riemannian volume form with unit volume.

A probability measure $\rho = p dv_M$ over M is an n-form over M satisfying $\int_M \rho = 1$. Each probability measure is assumed to have a density function p which is everywhere positive. We define the space of probability measures having positive density function as

$$\mathcal{P}(M) = \left\{ \rho = p dv_M \,\middle|\, p \in L^2_k(M),\ p(x) > 0,\ \int_M \rho = 1 \right\},$$

where k is an integer satisfying $k > n/2$. The Sobolev space $L^2_k(M)$ is needed to ensure that a certain Sobolev inequality argument works. Any probability measure with positive density function is considered as a point of the space $\mathcal{P}(M)$. We remark that $\mathcal{P}(M) \subset \Gamma(M, \wedge^n(M))$ which is the space of n-forms on M.

By using a Sobolev space argument, one can show that this space is an infinite dimensional manifold whose tangent space $T_\rho\mathcal{P}(M)$ at a point ρ is identified with

$$T_\rho\mathcal{P}(M) \simeq \left\{ \tau = q dv_M \,\middle|\, q \in L^2_k(M),\ \int_M \tau = 0 \right\}.$$

Take a point $\rho \in \mathcal{P}(M)$ and a tangent vector $\tau \in T_\rho\mathcal{P}(M)$. Then, it is easily shown that in the space $\mathcal{P}(M)$ there exists a parametrized curve $\rho(s) = \rho + s\tau$ in $s \in (-\varepsilon, \varepsilon)$ for a sufficiently small ε such that $\rho(0) = \rho$ and the velocity vector $\rho'(0) = \tau$.

On the space $\mathcal{P}(M)$ we introduce a Riemannian metric G.

DEFINITION 1. For $\rho \in \mathcal{P}(M)$, we define an inner product G_ρ on $T_\rho\mathcal{P}(M)$ by

$$G_\rho(\tau_1, \tau_2) = \int_M \frac{d\tau_1}{d\rho}\frac{d\tau_2}{d\rho}\rho \tag{6}$$

where $\tau_i = q_i dv_M \in T_\rho\mathcal{P}(M)$, $i = 1, 2$ are tangent vectors at ρ and

$$\frac{d\tau_i}{d\rho}(x) = \frac{q_i(x)}{p(x)}$$

are the Radon-Nikodym derivative of τ_i with respect to ρ. So

$$G_\rho(\tau_1, \tau_2) = \int_{x\in M} \frac{q_1(x)}{p(x)}\frac{q_2(x)}{p(x)} p(x) dv_M(x) = \mathbf{E}_\rho\left[\frac{q_1(x)}{p(x)}\frac{q_2(x)}{p(x)}\right]$$

where $\mathbf{E}_\rho[\,\cdot\,]$ denotes the expectation with respect to the probability measure ρ. We call $G = \{G_\rho\}_{\rho\in\mathcal{P}(M)}$ the *Fisher information metric* on $\mathcal{P}(M)$.

The Fisher information metric G on $\mathcal{P}(M)$ is a generalization of the Fisher information matrices on statistical models. We refer to [1] for more information on statistical models.

THEOREM 3 ([10]). *The Riemannian structure G on $\mathcal{P}(M)$ enjoys the following properties:*

(1) *the Levi-Civita connection ∇ of G is*

$$\nabla_{\tau_1}\tau = -\frac{1}{2}\left(\frac{d\tau}{d\rho}\frac{d\tau_1}{d\rho} - \int_M \frac{d\tau}{d\rho}\frac{d\tau_1}{d\rho}\rho\right)\rho\,, \quad \tau,\ \ \tau_1 \in T_\rho\mathcal{P}(M)\,, \tag{7}$$

where τ is considered as a vector field extended by parallel translation,

(2) *it has constant sectional curvature equal to* $1/4$,

(3) $\mathrm{Diff}_+(M)$, *the group of all orientation preserving diffeomorphisms of M, acts on $\mathcal{P}(M)$ by pull-back transitively and isometrically*,

(4) *it is not geodesically complete.*

REMARK 5. The property (3) in Theorem 3 stems from the well known Moser's theorem on volume forms [16]. This fact means $\mathcal{P}(M) \simeq \mathrm{Diff}_+(M)/\mathcal{K}$ where $\mathcal{K}$ is the isotropy subgroup of $\mathrm{Diff}_+(M)$ fixing a certain point $\rho \in \mathcal{P}(M)$.

3. Hadamard manifolds, the ideal boundary and Poisson kernels

Let X be a simply connected complete n-dimensional Riemannian manifold whose sectional curvature K_X is non-positive. We call such a manifold a *Hadamard manifold*. Hadamard manifolds are diffeomorphic, via the exponential map at any point, to the n-dimensional Euclidean space.

Let $\mathcal{G}(X)$ be the set of all smooth half-open geodesics on X parametrized by arc length. Two smooth half-open geodesics $\gamma_i = \gamma_i(t) \in \mathcal{G}(X)$ $(i = 1, 2,\ 0 \le t < \infty)$ are *asymptotically equivalent*, denoted by $\gamma_1 \sim \gamma_2$, if the distance $d(\gamma_1(t), \gamma_2(t))$ $(t \ge 0)$ is bounded from above. The quotient space of $\mathcal{G}(X)$ by the equivalence relation $\sim$ is called the *ideal boundary* of X, denoted by ∂X.

REMARK 6. Fix a point $x_0 \in X$. Then we can identify the ideal boundary ∂X with the unit sphere in the tangent space $T_{x_0}X$ because for an unit vector $v \in T_{x_0}X$ there exists a unique smooth half-open geodesic $\gamma(t)$ such that $\gamma(0) = x_0$ and $\gamma'(0) = v$. Hence we can regard ∂X as an $(n-1)$-dimensional sphere $S^{n-1}(1) \subset T_{x_0}X$.

Fix a base point $x_0 \in X$. For $\theta \in \partial X$, we define a function $B(\cdot, \theta)$ on X by

$$B(x, \theta) = \lim_{t\to\infty} (d(\gamma(t), x) - t) \tag{8}$$

where γ is the half-open geodesic parametrized by arc length with $\gamma(0) = x_0$ and $\lim_{t\to\infty} \gamma(t) = \theta$. We call the above function B the *Busemann function* on X with base point x_0. We immediately get $B(x_0, \theta) = 1$ for any $\theta \in \partial X$.

REMARK 7. If for any $\theta, \theta' \in \partial X$ there exists a geodesic $\gamma : \mathbb{R} \to X$ such that $\lim_{t\to\infty} \gamma(t) = \theta$ and $\lim_{t\to-\infty} \gamma(t) = \theta'$, then we say that X satisfies *visibility axiom*. This condition is equivalent to $\lim_{x\to\theta'} B(x,\theta) = \infty$ $(\theta \neq \theta')$ (see [3, p. 54, Lemma]).

The Busemann function $B(\cdot, \theta)$ is C^1-class and satisfies

$$\mathrm{grad}_X\, B(x,\theta) = -\gamma'_{x,\theta}(0) \tag{9}$$

where $\gamma_{x,\theta} \in \mathcal{G}(X)$ starts $x \in X$ converging asymptotically to $\theta \in \partial X$. In particular, $|\,\mathrm{grad}_X\, B| = 1$. Moreover, for $\theta \in \partial X$, the difference

$$B_p(\cdot,\theta) - B_q(\cdot,\theta) = B_p(q,\theta) \tag{10}$$

is a constant function (see [17, p. 213]). Here $B_p(x,\theta)$ is the Busemann function with base point $p \in X$.

We denote by $\mathrm{Isom}_+(X, g)$ the group of all orientation preserving isometries of (X, g). Let $\psi \in \mathrm{Isom}_+(X, g)$. Then ψ induces naturally an action on ∂X as follows: for $\theta \in \partial X$, $\psi(\theta) = \lim_{t\to\infty} \psi(\gamma(t))$ where $\gamma \in \mathcal{G}(X)$ converges asymptotically to θ. Under this action on $X \cup \partial X$, we obtain the transition formula of the Busemann function.

LEMMA 2. *Let $B(x,\theta)$ be the Busemann function on a Hadamard manifold (X, g) with base point x_0. Then, for $\psi \in \mathrm{Isom}_+(X, g)$ we have*

$$B(\psi(x),\theta) = B(x, \psi^{-1}(\theta)) + B(\psi(x_0),\theta)\,.$$

PROOF. By using (10), we have

$$\begin{aligned}
B_{x_0}(\psi(x),\theta) &= \lim_{t\to\infty}\{d(\psi(x), \gamma_{x_0,\theta}(t)) - d(x_0, \gamma_{x_0,\theta}(t))\}\\
&= \lim_{t\to\infty}\{d(x, \psi^{-1}\gamma_{x_0,\theta}(t)) - d(\psi^{-1}(x_0), \psi^{-1}\gamma_{x_0,\theta}(t))\\
&\qquad - d(x_0, \psi^{-1}\gamma_{x_0,\theta}(t)) + d(\psi^{-1}(x_0), \psi^{-1}\gamma_{x_0,\theta}(t))\\
&\qquad + d(\psi(x_0), \gamma_{x_0,\theta}(t)) - d(x_0, \gamma_{x_0,\theta}(t))\}\\
&= B_{\psi^{-1}(x_0)}(x, \psi^{-1}\theta) - B_{\psi^{-1}(x_0)}(x_0, \psi^{-1}\theta) + B_{x_0}(\psi(x_0),\theta)\\
&= B_{x_0}(x, \psi^{-1}\theta) + B_{x_0}(\psi(x_0),\theta)\,.
\end{aligned}$$

□

On the manifold $X \cup \partial X$ with boundary ∂X, we can consider the *Dirichlet problem* at infinity(1) with given boundary values. We call its fundamental solution $P(x,\theta)$ the *Poisson kernel* on M, that is, for $\theta \in \partial X$ $P(\cdot,\theta) \in C^0(X \cup (\partial X \setminus \{\theta\}))$ and the solution u to (1) is given by

$$u(x) = \int_{\theta\in\partial X} P(x,\theta) f(\theta) d\theta\,. \tag{11}$$

Here $d\theta$ is a probability measure on ∂X, identified with the standard measure on $S^{n-1}(1) \subset T_{x_0}X$ with unit volume. It is clear that $u \equiv 1$ is the harmonic function with boundary value $f \equiv 1$. Hence we have

$$\int_{\theta\in\partial X} P(x,\theta)d\theta = 1$$

for any $x \in X$. Thus, we can define the *Poisson kernel map*

$$\varphi : X \to \mathcal{P}(\partial X);\ x \mapsto P(x,\theta)d\theta\,.$$

REMARK 8. If X is a Hadamard manifold satisfying $-b^2 \le K_X \le -a^2 < 0$, then a Poisson kernel uniquely exists for any $\theta \in X$. Moreover, P is characterized by the following condition [18, p. 45]:

(1) $P(\cdot,\theta)$ is a positive, harmonic function on X,
(2) There exists a unique point $x_0 \in M$ such that $P(x_0,\theta) = 1$ for any $\theta \in \partial M$,
(3) for $\theta \in \partial X$, $P(\cdot,\theta) \in C^0(X \cup \partial X \setminus \{\theta\})$ and $P(\cdot,\theta)|_{\partial X\setminus\{\theta\}} = 0$.

In particular, when X is a rank one symmetric space of noncompact type, $\exp(-\rho B(x,\theta))$ satisfies above three conditions. Hence the Poisson kernel is explicitly given as the exponential form of the Busemann function.

4. Proof of Proposition 1

Let (X,g) be a Hadamard manifold. Assume that the Poisson kernel of X normalized at $x_0 \in X$ is given in the form (4). Then, from Lemma 2 we immediately get the transition formula of the Poisson kernel:

$$P(\psi(x),\theta) = P(x,\psi^{-1}(\theta))P(\psi(x_0),\theta) \tag{12}$$

where $\psi \in \mathrm{Isom}_+(X,g)$. Moreover, the standard measure $d\theta$ on ∂X in (11) satisfies the following:

LEMMA 3.

$$(\psi^{-1})^*d\theta = P(\psi(x_0),\theta)d\theta\,. \tag{13}$$

PROOF. For $\psi \in \mathrm{Isom}_+(X,g)$, let u, v be the solutions to the Dirichlet problem with boundary values f, $f\circ\psi \in C^0(\partial X)$, respectively, that is,

$$u(x) = \int_{\partial X} P(x,\theta)f(\theta)d\theta\,, \tag{14}$$

$$v(x) = \int_{\partial X} P(x,\theta)f(\psi(\theta))d\theta = \int_{\partial X} P(x,\psi^{-1}(\theta))f(\theta)(\psi^{-1})^*d\theta\,. \tag{15}$$

On the other hand, since $v(\theta) = f(\psi(\theta)) = u(\psi(\theta))$, we have from (12)

$$v(x) = \int_{\partial X} P(\psi(x),\theta)f(\theta)d\theta = \int_{\partial X} P(x,\psi^{-1}(\theta))P(\psi(x_0),\theta)f(\theta)d\theta\,. \tag{16}$$

From (15) and (16), for any $f \in C^0(\partial X)$ we have

$$\int_{\partial X} P(x, \psi^{-1}(\theta)) f(\theta)\{(\psi^{-1})^* d\theta - P(\psi(x_0), \theta) d\theta\} = 0$$

from which we get (13). □

PROOF OF PROPOSITION 1. From (12) and (13), we find that the Poisson kernel map φ and $\psi \in \mathrm{Isom}_+(X, g)$ satisfy

$$(\psi^{-1})^* \circ \varphi = \varphi \circ \psi \, . \tag{17}$$

Since X is a Riemannian homogeneous space and $\mathrm{Isom}_+(X, g)$ acts on $(\mathcal{P}(\partial X), G)$ isometrically, it is sufficient to compute $\varphi^* G$ at the base point x_0. For $v \in T_{x_0}X, |v| = 1$, we have

$$\begin{aligned}
\varphi^* G(v, v) &= \int_{\partial X} \{v \log P(\cdot, \theta)\}^2 \, P(x_0, \theta) d\theta \\
&= c^2 \int_{\partial X} \langle v, \mathrm{grad}_X \, B(x_0, \theta)\rangle^2 d\theta \\
&= c^2 \int_{\partial X} \langle v, \gamma'_{x_0,\theta}(0)\rangle^2 d\theta \\
&= c^2 \int_{u \in S^{n-1}(1)} \langle v, u\rangle^2 d\mu_{S^{n-1}(1)} \\
&= \frac{c^2}{n} \, .
\end{aligned}$$

□

REMARK 9. Under the assumption in Proposition 1, the Poisson kernel map is a minimal embedding. To prove this fact, we need the consideration of the harmonicity of the map $\varphi : X \to \mathcal{P}(\partial X)$. See [13] for details.

In general $\exp(-cB(x, \theta))$ does not necessarily give a Poisson kernel. Nevertheless we have the following lemma which states a criterion for the Dirac delta condition of $\exp(-cB(x, \theta))$.

LEMMA 4. *Let X be a Hadamard manifold. If X satisfies the visibility axiom and* $\int_{\partial X} \exp(-cB(x, \theta)) d\theta = 1$, *then* $\exp(-cB(x, \theta)) d\theta$ *satisfies the Dirac delta condition*, *that is*, *for* $f \in C^0(\partial X)$

$$\lim_{x \to \theta_0} \int_{\theta \in \partial X} f(\theta) \exp(-cB(x, \theta)) d\theta = f(\theta_0) \, . \tag{18}$$

PROOF. We set

$$u(x) = \int_{\theta \in \partial X} f(\theta) \exp(-cB(x, \theta)) d\theta \, .$$

Fix a point $\theta_0 \in \partial X$. Then, we have

$$u(x) - f(\theta_0) = \int_{\theta \in \partial X} (f(\theta) - f(\theta_0)) \exp(-cB(x, \theta)) d\theta \,.$$

Since ∂X is compact and $\lim_{x \to \theta_0} B(x, \theta) = \infty (\theta_0 \neq \theta)$, for any $\varepsilon > 0$ there exist a neighborhood $U_\varepsilon \subset \partial X$ of θ_0 and $t_\varepsilon > 0$ such that

$$\max_{\theta \in U_\varepsilon} |f(\theta) - f(\theta_0)| < \varepsilon \,, \quad \text{and} \quad \exp(-cB(\gamma(t), \theta)) < \varepsilon \quad \text{for any } \ t > t_\varepsilon \,,$$

where $\gamma(t)$ is the half-open geodesic from x converging asymptotically to θ_0. Here we make use of the cone topology of $X \cup \partial X$. Let $M \in \mathbb{R}$ be the maximum value of $|f - f(\theta_0)|$ on $\partial X \setminus U_\varepsilon$. Then, for all $t > t_\varepsilon$ we have

$$\begin{aligned} |u(\gamma(t)) - f(\theta_0)| &\leq \int_{U_\varepsilon} |f(\theta) - f(\theta_0)| \exp(-cB(\gamma(t), \theta)) d\theta \\ &\quad + \int_{\partial X \setminus U_\varepsilon} |f(\theta) - f(\theta_0)| \exp(-cB(\gamma(t), \theta)) d\theta \\ &< \varepsilon \int_{U_\varepsilon} \exp(-cB(\gamma(t), \theta)) d\theta + \varepsilon M \int_{\partial X \setminus U_\varepsilon} d\theta \\ &\leq \varepsilon \int_{\partial X} \exp(-cB(\gamma(t), \theta)) d\theta + \varepsilon M \int_{\partial X} d\theta \\ &= (1 + M)\varepsilon \end{aligned}$$

from which we obtain (18). □

5. Damek-Ricci spaces

Let $(\mathfrak{n}, [\cdot, \cdot]_{\mathfrak{n}})$ be a 2-step nilpotent algebra with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$. Denote by $\mathfrak{z}$ the center of $\mathfrak{n}$ and by $\mathfrak{v}$ the orthogonal complement to $\mathfrak{z}$. For $Z \in \mathfrak{z}$ we define the linear map $J_Z : \mathfrak{v} \to \mathfrak{v}$ by

$$\langle J_Z V, V' \rangle_{\mathfrak{n}} = \langle Z, [V, V']_{\mathfrak{n}} \rangle_{\mathfrak{n}} \quad (V, V' \in \mathfrak{v}) \,.$$

If for every $Z \in \mathfrak{z}$

$$(J_Z)^2 = -|Z|^2 \, \mathrm{id}_{\mathfrak{v}} \,,$$

then we say that $\mathfrak{n}$ is the *generalized Heisenberg algebra*.

The corresponding simply connected Lie group N is called the *generalized Heisenberg group*. When we identify N with its Lie algebra $\mathfrak{n}$ via the exponential map, the multiplication in $N \simeq \mathfrak{n} = \mathfrak{v} \oplus \mathfrak{z}$ reads

$$(V, Z) \cdot (V', Z') = \left(V + V', Z + Z' + \frac{1}{2}[V, V']_{\mathfrak{n}} \right) .$$

Let $\mathfrak{n}$ be a generalized Heisenberg algebra $\mathfrak{n}$. We set $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}$ and define the bracket product $[\cdot,\cdot]_{\mathfrak{s}}$ by

$$[(V, Z, l), (V', Z', l')]_{\mathfrak{s}} = \left(\frac{l}{2}V' - \frac{l'}{2}V, lZ' - l'Z + [V, V']_{\mathfrak{n}}, 0\right) \tag{19}$$

and an inner product $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ by

$$\langle(V, Z, l), (V', Z', l')\rangle_{\mathfrak{s}} = \langle V, V'\rangle + \langle Z, Z'\rangle + ll' \tag{20}$$

on $\mathfrak{s}$. We call the simply connected solvable Lie group S whose Lie algebra is $\mathfrak{s}$ with the left invariant Riemannian metric induced from $\langle\cdot,\cdot\rangle_{\mathfrak{s}}$ the *Damek-Ricci space*. When we regard $S \simeq \mathfrak{v} \times \mathfrak{z} \times \mathbb{R}_+$, the group structure on S is given by

$$(V, Z, a) \cdot (V', Z', a') = \left(V + \sqrt{a}V', Z + aZ' + \frac{\sqrt{a}}{2}[V, V']_{\mathfrak{s}}, aa'\right). \tag{21}$$

REMARK 10. Every Damek-Ricci space is a harmonic, Hadamard manifold [4].

The left Haar measure on S is written by $dv_S = a^{-Q-1}dadXdZ$. Here $Q = \frac{\dim \mathfrak{v}}{2} + \dim \mathfrak{z}$, called the *homogeneous dimension* of N. In terms of the geodesic coordinate, dv_S is written in the form

$$dv_S = 2^{\dim \mathfrak{v} + \dim \mathfrak{z}} \left(\sinh \frac{r}{2}\right)^{\dim \mathfrak{v} + \dim \mathfrak{z}} \left(\cosh \frac{r}{2}\right)^{\dim \mathfrak{z}} dr d\mu_{S^{n-1}(1)}$$

from which the volume entropy is $\rho = Q$ [2, 7].

LEMMA 5 ([4, Theorem 1, p. 93]). *Let $(U, Y, l) \in \mathfrak{s}$ be a unit vector. Then, the geodesic $\gamma : \mathbb{R} \to S$ parametrized by arc length with $\gamma(0) = (\mathbf{0}_{\mathfrak{v}}, \mathbf{0}_{\mathfrak{z}}, 1)$ and $\gamma'(0) = (U, Y, l)$ is written by*

$$\gamma(t) = \left(\frac{2\theta(1 - l\theta)}{\chi}U + \frac{2\theta^2}{\chi}J_Y U, \frac{2\theta}{\chi}Y, \frac{1 - \theta^2}{\chi}\right) \tag{22}$$

where $\theta = \theta(t) := \tanh(t/2)$ and $\chi = \chi(t)$ is a function of t, $\chi = \chi_{(Y,l)}(t) := (1 - l\theta(t))^2 + |Y|^2\theta(t)^2$.

From this lemma, we get immediately

$$\lim_{t\to\infty} \gamma(t) = \begin{cases} \dfrac{2}{(1 - l)^2 + |Y|^2}((1 - l)U + J_Y U, Y, 0) & \text{if } l \neq 1\,, \\ (\mathbf{0}_{\mathfrak{v}}, \mathbf{0}_{\mathfrak{z}}, \infty) & \text{if } l = 1\,, \end{cases} \tag{23}$$

which means that we can regard $\partial S \simeq N \cup \{\infty\}$.

6. The Busemann function and the Poisson kernel on Damek-Ricci spaces

The aim of this section is to show that if X is an n-dimensional Damek-Ricci space S, then $\int_{\partial X} \exp(-\rho B(x,\theta))\, d\theta = 1$ as will be given in Lemma 8. This, along with the visibility of Damek-Ricci spaces, means that $\exp(-\rho B(x,\theta))\, d\theta$ is the Poisson kernel on S (Lemma 4). Thus, from Proposition 1, we obtain Theorem 2.

In order to compute the Busemann function on a Damek-Ricci space S with base point $e_S := (\mathbf{0}_{\mathfrak{v}}, \mathbf{0}_{\mathfrak{z}}, 1)$ the identity element of S, we state the explicit expression of the distance function on S.

LEMMA 6 ([4, Section 4.4]). *The distance $d(e_S, x)$ from e_S to $x = (V, Z, a) \in S$ is*

$$d(e_S, x) = \log\left(\frac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda}}{2}\right) \tag{24}$$

where

$$\lambda = \lambda(x) := \frac{1}{a}\left\{\left(1 + a + \frac{1}{4}|V|^2\right)^2 + |Z|^2\right\}.$$

By using Lemmas 5 and 6, we obtain the following:

THEOREM 4. *Let $B(x,\theta)$ be the Busemann function on S with base point e_S for $\theta \in \partial S \simeq N \cup \{\infty\}$. Then, we have*

$$B(x,\theta) = \begin{cases} -\log\left(\dfrac{a\big((1+\frac{1}{4}|v|^2)^2 + |z|^2\big)}{\big(a + \frac{1}{4}|v - V|^2\big)^2 + \big|z - Z - \frac{1}{2}[V, v]_{\mathfrak{n}}\big|^2}\right) & \text{if } \theta = (v, z) \in N \\ -\log a & \text{if } \theta = \infty \end{cases}$$

where $x = (V, Z, a) \in S$.

PROOF. Let $\gamma(t)$ be the half-open geodesic from $\gamma(0) = e_S$ with velocity vector $\gamma'(0) = (U, Y, l)$. Then we have

$$\begin{aligned} B(x, \gamma(\infty)) &= \lim_{t\to\infty}(d(x, \gamma(t)) - t) \\ &= \lim_{t\to\infty}(d(e_S, x^{-1}\cdot\gamma(t)) - t) \\ &= \lim_{t\to\infty}\log\left(\frac{\lambda - 2 + \sqrt{\lambda^2 - 4\lambda}}{2e^t}\right) \\ &= \lim_{t\to\infty}\log\left(\frac{\lambda - 2}{e^t}\right) \end{aligned}$$

where $\lambda = \lambda(x^{-1}\cdot\gamma(t))$. We set $x^{-1}\cdot\gamma(t) = (V(t), Z(t), a(t))$. In fact, from Lemma 5, we have

$$\begin{cases} V(t) = \dfrac{1}{\sqrt{a}}\left(-V + \dfrac{2\theta(1-l\theta)}{\chi}U + \dfrac{2\theta^2}{\chi}J_Y U \right), \\ Z(t) = \dfrac{1}{a}\left(-Z + \dfrac{2\theta}{\chi}Y - \dfrac{1}{2}\left[V, \dfrac{2\theta(1-l\theta)}{\chi}U + \dfrac{2\theta^2}{\chi}J_Y U \right]_{\mathfrak{n}} \right), \\ a(t) = \dfrac{1-\theta^2}{a\chi}, \end{cases}$$

where $\theta = \tanh(t/2)$ and $\chi = (1-l\theta)^2 + |Y|^2\theta^2$. Here

$$\begin{aligned} \frac{\lambda - 2}{e^t} &= \frac{1}{e^t}\left\{ \frac{1}{a(t)}\left(\left(a(t) + 1 + \frac{1}{4}|V(t)|^2 \right)^2 + |Z(t)|^2 \right) - 2 \right\} \\ &= \frac{a(t)}{e^t} + \frac{|V(t)|^2}{2e^t} + \frac{1}{a(t)e^t}\left\{ \left(1 + \frac{1}{4}|V(t)|^2 \right)^2 + |Z(t)|^2 \right\}. \end{aligned} \tag{25}$$

If $l = 1$, that is, $U = \mathbf{0}_{\mathfrak{v}}$, $Y = \mathbf{0}_{\mathfrak{z}}$, then $\chi = (1-\theta)^2$ so that

$$\begin{cases} V(t) = -\dfrac{1}{\sqrt{a}}V, \\ Z(t) = -\dfrac{1}{a}Z, \\ a(t) = \dfrac{1+\theta}{a(1-\theta)} = \dfrac{e^t}{a}. \end{cases}$$

Hence, we have

$$B(x, \infty) = \log\left(\frac{1}{a}\right). \tag{26}$$

If $l \neq 1$, then $0 < \chi < +\infty$ for any $t \geq 0$. Hence, $|V(t)|^2$ and $|Z(t)|^2$ are finite for any $t \geq 0$. Moreover, since

$$a(t) = \frac{4}{a\chi} \cdot \frac{1}{e^t + e^{-t} + 2}, \tag{27}$$

from (25) we have

$$\lim_{t\to\infty} \frac{\lambda - 2}{e^t} = \frac{a\chi_\infty}{4}\left\{ \left(1 + \frac{1}{4}|V_\infty|^2 \right)^2 + |Z_\infty|^2 \right\}, \tag{28}$$

where $V_\infty = \lim_{t\to\infty} V(t)$, $Z_\infty = \lim_{t\to\infty} Z(t)$ and $\chi_\infty = (1-l)^2 + |Y|^2$. We set $\gamma(\infty) =: m = (v, z) \in N$. From (23), we have $V_\infty = \frac{1}{\sqrt{a}}(v - V)$ and $Z_\infty = z - Z - \frac{1}{2}[v, V]_{\mathfrak{n}}$. Hence, we have

$$\left(1+\frac{1}{4}|V_\infty|^2\right)^2+|Z_\infty|^2=\frac{1}{a^2}\left\{\left(a+\frac{1}{4}|v-V|^2\right)^2+\left|z-Z-\frac{1}{2}[V,v]_{\mathfrak{n}}\right|^2\right\}. \tag{29}$$

Here,

$$\begin{aligned}\left(1+\frac{1}{4}|x|^2\right)^2+|z|^2&=\left\{1+\frac{|U|^2}{\chi_\infty^2}((1-l)^2+|Y|^2)\right\}^2+\frac{4}{\chi_\infty^2}|Y|^2\\&=\left(1+\frac{|U|^2}{\chi_\infty}\right)^2+\frac{4}{\chi_\infty^2}|Y|^2\\&=\left(1+\frac{2(1-l)-\chi_\infty}{\chi_\infty}\right)^2+\frac{4}{\chi_\infty^2}|Y|^2\\&=\frac{4(1-l)^2}{\chi_\infty^2}+\frac{4}{\chi_\infty^2}|Y|^2\\&=\frac{4}{\chi_\infty}.\end{aligned} \tag{30}$$

From (28), (29) and (30), together with (26), we get our lemma. □

Next, we give the explicit form of the standard measure $d\theta$ on $\partial S\simeq N$ with unit volume of (11).

LEMMA 7. *The standard measure on $S^{n-1}(1)\subset T_{e_S}S\simeq\mathfrak{s}$ is given in the form*

$$d\theta=\frac{c}{\left\{\left(1+\frac{1}{4}|v|^2\right)^2+|z|^2\right\}^Q}dvdz\,,\quad(\theta=(v,z))$$

where c is the constant, given as

$$c=2^{k-1}\pi^{-n/2}\Gamma\left(\frac{n}{2}\right),\quad k=\dim\mathfrak{v} \tag{31}$$

(*see* [2]).

PROOF. It suffices to write down the standard volume form $d\mu_{S^{n-1}(1)}$ on $S^{n-1}(1)$ with unit volume in term of the coordinate $(v,z)\in N$ under the identification

$$N\simeq\partial S\simeq S^{n-1}(1)\subset T_{e_S}S\simeq\mathfrak{s}\,.$$

From (23), the identification $N\simeq S^{n-1}(1)\subset\mathfrak{s}$ is given by the map $\partial C:N\to\mathfrak{s}$: $(v,z)\mapsto$

(U, Y, l) defined by

$$\begin{cases} U = U(v,z) = \dfrac{\left(1 + \frac{1}{4}|v|^2 - J_z\right)v}{\left(1 + \frac{1}{4}|v|^2\right)^2 + |z|^2}\,, \\ Y = Y(v,z) = \dfrac{2z}{\left(1 + \frac{1}{4}|v|^2\right)^2 + |z|^2}\,, \\ l = l(v,z) = \dfrac{-1 + \left(\frac{1}{4}|v|^2\right)^2 + |z|^2}{\left(1 + \frac{1}{4}|v|^2\right)^2 + |z|^2}\,. \end{cases}$$

Let $\pi : \mathfrak{s} \supset S^{n-1}(1) \to N$ be the stereographic projection with the projection point e_S, that is,

$$\pi(U, Y, l) = (\widetilde{v}, \widetilde{z}) = \frac{1}{1-l}(V, W)\,.$$

As is well known, we have

$$\begin{aligned} (\pi^{-1})^* d\mu_{S^{n-1}(1)} &= \frac{c'}{\left(|\widetilde{v}|^2 + |\widetilde{z}|^2 + 1\right)^{n-1}} d\widetilde{v} d\widetilde{z} \\ &= c'\left(\frac{1 + \frac{1}{4}|v|^2}{\left(1 + \frac{1}{4}|v|^2\right)^2 + |z|^2}\right)^{n-1} d\widetilde{v} d\widetilde{z}\,. \end{aligned} \tag{32}$$

Here c' is the constant satisfying $\int (\pi^{-1})^* d\mu_{S^{n-1}(1)} = 1$. Now we define two maps $\alpha_i : N \to N$ $(i = 1, 2)$ by

$$\alpha_1(v,z) = (\overline{v}, \overline{z}) = \left(v, \frac{1}{\left(1 + \frac{1}{4}|v|^2\right)} z\right),$$

$$\alpha_2(\overline{v}, \overline{z}) = (\widetilde{v}, \widetilde{z}) = \left(\frac{1}{2}(1 - J_{\overline{z}})\overline{v}, \overline{z}\right).$$

Then, we find that $\pi \circ \partial C = \alpha_2 \circ \alpha_1$. Hence we have

$$d\widetilde{v} d\widetilde{z} = \det d(\pi \circ \partial C)_{(v,z)} dv dz = \det d\alpha_{2(\overline{v},\overline{z})} \cdot \det d\alpha_{1(v,z)} dv dz\,. \tag{33}$$

By using the argument in [7] we get

$$\begin{aligned} \det d\alpha_{1(v,z)} &= \left(\frac{1}{1 + \frac{1}{4}|v|^2}\right)^{\dim \mathfrak{z}}, \\ \det d\alpha_{2(\overline{v},\overline{z})} &= \frac{\left(\left(1 + \frac{1}{4}|v|^2\right)^2 + |z|^2\right)^{\dim \mathfrak{v}/2}}{2^{\dim \mathfrak{v}} \cdot \left(1 + \frac{1}{4}|v|^2\right)^{\dim \mathfrak{v}}}\,. \end{aligned} \tag{34}$$

From (32), (33) and (34), we obtain our lemma. □

LEMMA 8.

$$\int_{\theta \in N} \exp(-QB(x,\theta))\, d\theta = 1\,.$$

PROOF. From Theorem 4 and Lemma 7, we have

$$\exp(-QB(x,\theta))\, d\theta = \frac{ca^Q}{\left\{\left(a+\frac{1}{4}|v-V|^2\right)^2 + \left|z - Z - \frac{1}{2}[V,v]_{\mathfrak{n}}\right|^2\right\}^Q} dvdz \tag{35}$$

where $x = (V, Z, a) \in S$ and $\theta = (v, z) \in N$. An easy computation shows that the integration of the right hand side of (35) over N is independent of any $(V, Z, a) \in S$ and equal to 1. □

REMARK 11. E. Damek [6] defines the function $P_a(n)$ on S by

$$P_a(n) = \frac{ca^Q}{\left\{\left(a+\frac{1}{4}|V|^2\right)^2 + |Z|^2\right\}^Q} \quad (n = (V, Z) \in N)\,,$$

and shows

$$\Delta P = 0\,,$$

$$\lim_{a\to 0} f * P_a(n) = f(n) \quad (f \in L^p(N))\,,$$

where the convolution is defined by using the group structure

$$f * P_a(n) = \int_{m\in N} P_a(nm^{-1}) f(m) dm\,.$$

This means that P is the fundamental solution to the Dirichlet problem on $S \cup N$. In fact, we have

$$P_a(nm^{-1})dm = \exp(-QB(x,\theta))d\theta$$

where $x = (n, a) = (V, Z, a) \in S$ and $\theta = m \in N$.

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