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Sectional Invariants of Hyperquadric Fibrations over a Smooth Projective Curve

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Abstract. Let (X, L) be a hyperquadric fibration over a smooth curve with $\dim X = n \geq 3$. In this paper we will calculate the ith sectional Euler number $e_i(X, L)$ of (X, L). Using this, we will study a lower bound for the ith sectional Betti number $b_i(X, L)$ with $i \leq 4$. In particular we will prove that $b_2(X, L) \geq 3$, $b_3(X, L) \geq 0$ and $b_4(X, L) \geq 3$.

1. Introduction

Let X be a projective variety of dimension n defined over the field of complex numbers, and let L be an ample line bundle on X. Then (X, L) is called a *polarized variety*. If X is smooth, then we say that (X, L) is a polarized *manifold*.

In [6], we introduced some ith sectional invariants of (X, L), that is, the ith sectional Euler number $e_i(X, L)$ and the ith sectional Betti number $b_i(X, L)$ for every integer i with $0 \leq i \leq n$, and we investigated some properties of these.

Here we explain the meaning of these invariants if X is smooth, L is very ample, and i is an integer with $1 \leq i \leq n-1$. Let $H_1, \ldots, H_{n-i}$ be general members of $|L|$. We put $X_{n-i} := H_1 \cap \cdots \cap H_{n-i}$. Then X_{n-i} is smooth with $\dim X_{n-i} = i$, and we can show that $e_i(X, L) = e(X_{n-i})$ and $b_i(X, L) = b_i(X_{n-i})$, where $e(X_{n-i})$ is the Euler number of X_{n-i} and $b_i(X_{n-i})$ is the ith Betti number $\dim H^i(X_{n-i}, \mathbf{C})$ of X_{n-i}.

We think that we will be able to investigate the structure of polarized manifolds more deeply by using these sectional invariants. But in general, it is very hard to calculate these sectional invariants concretely. So, in order to make a study of a general theory of these sectional invariants, as the first step, it is important to calculate these for the case of some special polarized manifolds, for example, polarized manifolds arising from the adjunction theory.

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In [7] we calculated and studied the ith sectional Euler number and the ith sectional Betti number of $(\mathbf{P}_X(\mathcal{E}), H(\mathcal{E}))$, where $\mathcal{E}$ is an ample vector bundle of rank r on a smooth projective variety X of dimension n, $\mathbf{P}_X(\mathcal{E})$ is the projective space bundle associated with $\mathcal{E}$ over X and $H(\mathcal{E})$ is its tautological line bundle.

As the next step, in this paper, we will study the ith sectional Euler number and the ith sectional Betti number of hyperquadric fibrations over a smooth curve. (For the definition of a hyperquadric fibration over a smooth curve, see Definition 2.3 below.) First we give a formula of the ith sectional Euler number $e_i(X, L)$ of hyperquadric fibrations over a smooth curve C (Theorem 3.1). Theorem 3.1 shows that $e_i(X, L)$ can be calculated by using $e = \deg \mathcal{E}$, $b = \deg B$ and the genus $g(C)$ of C, which are fundamental quantity of hyperquadric fibrations over C (see Remark 2.2). Next we consider a lower bound for the ith sectional Betti number $b_i(X, L)$ of hyperquadric fibrations over a smooth curve. In general, the following conjecture was given in [6, Conjecture 3.1 (2)].

CONJECTURE 1.1. *Let (X, L) be a polarized manifold of dimension n. Then $b_i(X, L) \geq h^i(X, \mathbf{C})$ holds for every integer i with $1 \leq i \leq n$.*

Here we note that if $i = 1$, then this conjecture is a famous conjecture of Fujita (see [4, (13.7) Remark] or [1, Question 7.2.11]) because $b_1(X, L) = 2g(X, L)$ (see [6, Remark 3.1 (2)]) and $h^1(X, \mathbf{C}) = 2h^1(\mathcal{O}_X)$, where $g(X, L)$ is the sectional genus of (X, L). So it is interesting to consider this conjecture. If L is base point free, then this conjecture is true (see [6, Proposition 3.3 (2)]). But it seems to be too difficult at the moment to solve this conjecture in general. So, as the first step, we study the non-negativity of $b_i(X, L)$. If $i = 1$ (resp. $i = 2$ and $\kappa(X) \geq 0$), then by [4, (12.1) Theorem], [9, Lemma 7] and [6, Remark 3.1 (2)] (resp. [6, Theorem 4.4 (4.4.2)]) we get $b_1(X, L) \geq 0$ (resp. $b_2(X, L) \geq 0$). But in general it is also unknown whether $b_i(X, L)$ is non-negative or not. So we think that it is meaningful to investigate the non-negativity of $b_i(X, L)$ for the case where (X, L) is a special polarized manifold. If (X, L) is a hyperquadric fibration over a smooth curve, then by virtue of Theorem 3.1, we can study the non-negativity of $b_i(X, L)$ for $i \leq 4$. In this paper we will prove the following:

(1) $b_2(X, L) \geq 3$ if $n \geq 3$.
(2) $b_3(X, L) \geq 0$ if $n \geq 4$.
(3) $b_4(X, L) \geq 3$ if $n \geq 5$.

Moreover we will consider a classification of hyperquadric fibrations over a smooth curve with the following:

(i) $n \geq 3$ and $b_2(X, L) = 3$.
(ii) $n \geq 4$ and $b_3(X, L) = 0$.
(iii) $n \geq 5$ and $b_4(X, L) = 3$.

2. Preliminaries

DEFINITION 2.1 (See [6, Definition 3.1]). Let (X, L) be a polarized manifold of dimension n. For every integer i with $0 \leq i \leq n$ we define the following:

(1) The *ith sectional Euler number* $e_i(X, L)$ *of* (X, L).

$$e_i(X, L) := \sum_{l=0}^{i}(-1)^l \binom{n-i+l-1}{l} c_{i-l}(X)L^{n-i+l} .$$

(2) The *ith sectional Betti number* $b_i(X, L)$ *of* (X, L).

$$b_i(X, L) := \begin{cases} e_0(X, L) & \text{if } i = 0 , \\ (-1)^i(e_i(X, L) - \sum_{j=0}^{i-1} 2(-1)^j b_j(X)) & \text{if } 1 \leq i \leq n , \end{cases}$$

where $b_j(X) := h^j(X, \mathbf{C})$.

DEFINITION 2.2. Let (X, L) be a polarized manifold of dimension n. We say that (X, L) is a *quadric fibration over a normal projective variety* Y *with* $\dim Y = m$ if there exists a surjective morphism with connected fibers $f : X \to Y$ such that $1 \leq m < n$ and $K_X + (n-m)L = f^*A$ for some ample line bundle A on Y.

DEFINITION 2.3. (X, L) is called a *hyperquadric fibration over a smooth curve* C if (X, L) is a quadric fibration over C such that every fiber is irreducible and reduced.

REMARK 2.1. Assume that (X, L) is a quadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $f : X \to C$ be its morphism. By [2, (3.2.6) Theorem] and the proof of [9, Lemma (c) in Section 1], we see that (X, L) is one of the following:

(a) f is the contraction morphism of an extremal ray, and every fiber of f is irreducible and reduced. Moreover $\rho(X) = \rho(C) + 1 = 2$. Therefore $b_2(X) = 2$ in this case.

(b) X is a $\mathbf{P}^1$-bundle over a smooth surface and $L|_F = \mathcal{O}_{\mathbf{P}^1}(1)$ for every fiber F.

So if (X, L) is a hyperquadric fibration over a smooth curve with $\dim X \geq 4$, then $b_2(X) = 2$. But if $\dim X = 3$, then $b_2(X) \neq 2$ is possible.

REMARK 2.2. Let (X, L) be a hyperquadric fibration over a smooth curve C and let $f : X \to C$ be its morphism.

(1) We put $\mathcal{E} := f_*(L)$. Then $\mathcal{E}$ is a locally free sheaf of rank $n+1$ on C. Let $\pi : \mathbf{P}_C(\mathcal{E}) \to C$ be the projection. Then there exists an embedding $i : X \hookrightarrow \mathbf{P}_C(\mathcal{E})$ such that $f = \pi \circ i$, $X \in |2H(\mathcal{E}) + \pi^*(B)|$ for some $B \in \mathrm{Pic}(C)$, and $L = H(\mathcal{E})|_X$, where $\mathbf{P}_C(\mathcal{E})$ is the projective space bundle associated with $\mathcal{E}$ over C and $H(\mathcal{E})$ is its tautological line bundle.

(2) Since $K_X + (n-1)L = f^*(A)$ for some ample line bundle A on C, we have $g(X, L) \geq 2$, where $g(X, L)$ is the sectional genus of L.

DEFINITION 2.4. Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Then for every integer j with $j \geq 0$, the *jth Segre class* $s_j(\mathcal{F})$ *of* $\mathcal{F}$ is defined by the following equation: $c_t(\mathcal{F}^\vee)s_t(\mathcal{F}) = 1$, where $\mathcal{F}^\vee := Hom_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, $c_t(\mathcal{F}^\vee)$ is the Chern polynomial of $\mathcal{F}^\vee$ and $s_t(\mathcal{F}) = \sum_{j\geq 0} s_j(\mathcal{F})t^j$.

REMARK 2.3. (1) Let X be a smooth projective variety and let $\mathcal{F}$ be a vector bundle on X. Let $\tilde{s}_j(\mathcal{F})$ be the Segre class which is defined in [8, Chapter 3]. Then $s_j(\mathcal{F}) = \tilde{s}_j(\mathcal{F}^\vee)$.

(2) For every integer i with $1 \leq i$, $s_i(\mathcal{F})$ can be written by using the Chern classes $c_j(\mathcal{F})$ with $1 \leq j \leq i$. (For example, $s_1(\mathcal{F}) = c_1(\mathcal{F})$, $s_2(\mathcal{F}) = c_1(\mathcal{F})^2 - c_2(\mathcal{F})$, and so on.)

3. Calculations of the sectional Euler number of hyperquadric fibrations over a smooth curve

Let (X, L) be a hyperquadric fibration over a smooth curve C with $\dim X = n \geq 3$. Let $\mathcal{E}$ and B be as in Remark 2.2 (1), and set $e := \deg(\mathcal{E})$ and $b := \deg B$. Then we are going to calculate $e_i(X, L)$.

THEOREM 3.1. *Let (X, L) be a hyperquadric fibration over a smooth curve C with* $\dim X = n \geq 3$, *and let i be an integer with* $0 \leq i \leq n$. *Then*

$$e_i(X, L) = (-1)^i(2e + (i+1)b) + \begin{cases} 2(i+1)(1-g(C)) & \text{if } i \text{ is odd}, \\ 2i(1-g(C)) & \text{if } i \text{ is even}. \end{cases}$$

PROOF. Here we use notation in Remark 2.2 (1).

If $i = 0$, then by [6, Remark 3.1 (1)] and [3, (3.4)], we have $e_0(X, L) = L^n = 2e + b$. Hence this is true. So we may assume that $i \geq 1$.

By Definition 2.1 we have

$$e_i(X, L) = \sum_{l=0}^{i} (-1)^l \binom{n-i+l-1}{l} c_{i-l}(X)L^{n-i+l}. \tag{1}$$

Here we will calculate $c_{i-l}(X)L^{n-i+l}$. First we note that by [8, Example 3.2.12]

$$c_{i-l}(X) = \sum_{j=0}^{i-l} c_j(\mathcal{T}_{\mathbf{P}_C(\mathcal{E})}|_X)s_{i-l-j}(\mathcal{O}(-X|_X)),$$

where $\mathcal{T}_{\mathbf{P}_C(\mathcal{E})}$ is the tangent bundle of $\mathbf{P}_C(\mathcal{E})$. On the other hand we have

$$s_{i-l-j}(\mathcal{O}(-X|_X)) = (-1)^{i-l-j}c_1(\mathcal{O}(X|_X))^{i-l-j}$$

and by [8, Example 3.2.11]

$$\begin{aligned} c_j(\mathcal{T}_{\mathbf{P}_C(\mathcal{E})}) &= \sum_{k=0}^{j} c_k(\pi^*\mathcal{T}_C)c_{j-k}(\pi^*(\check{\mathcal{E}}) \otimes H(\mathcal{E})) \\ &= c_j(\pi^*(\check{\mathcal{E}}) \otimes H(\mathcal{E})) + c_1(\pi^*\mathcal{T}_C)c_{j-1}(\pi^*(\check{\mathcal{E}}) \otimes H(\mathcal{E})) \end{aligned}$$

$$= \sum_{k=0}^{j} \binom{n+1-k}{j-k} c_k(\pi^*(\check{\mathcal{E}})) H(\mathcal{E})^{j-k}$$
$$+ c_1(\pi^* \mathcal{T}_C) \sum_{k=0}^{j-1} \binom{n+1-k}{j-1-k} c_k(\pi^*(\check{\mathcal{E}})) H(\mathcal{E})^{j-1-k} .$$

Therefore

$$\begin{aligned}(2)\quad & c_{i-l}(X) L^{n-i+l} \\ &= s_{i-l}(\mathcal{O}(-X|_X)) L^{n-i+l} + \left(\sum_{j=1}^{i-l} c_j(\mathcal{T}_{\mathbf{P}_C(\mathcal{E})}|_X) s_{i-l-j}(\mathcal{O}(-X|_X)) \right) L^{n-i+l} \\ &= (-1)^{i-l} \mathcal{O}(X|_X)^{i-l} L^{n-i+l} \\ &\quad + \left(\sum_{j=1}^{i-l} (-1)^{i-l-j} \sum_{k=0}^{j} \binom{n+1-k}{j-k} c_k(f^*(\check{\mathcal{E}})) L^{j-k} \mathcal{O}(X|_X)^{i-l-j} \right) L^{n-i+l} \\ &\quad + \left(\sum_{j=1}^{i-l} (-1)^{i-l-j} c_1(f^* \mathcal{T}_C) \sum_{k=0}^{j-1} \binom{n+1-k}{j-1-k} \right. \\ &\quad \left. \times c_k(f^*(\check{\mathcal{E}})) L^{j-1-k} \mathcal{O}(X|_X)^{i-l-j} \right) L^{n-i+l} .\end{aligned}$$

Here we note that

$$\begin{aligned}\mathcal{O}(X|_X)^{i-l-j} &= (2H(\mathcal{E}) + \pi^*(B))|_X^{i-l-j} \\ &= (2L + f^*(B))^{i-l-j} \\ &= 2^{i-l-j} L^{i-l-j} + (i-l-j) 2^{i-l-j-1} L^{i-l-j-1} f^*(B) ,\end{aligned}$$

and

$$\begin{aligned}& c_k(f^*(\check{\mathcal{E}})) L^{j-k} \mathcal{O}(X|_X)^{i-l-j} \\ &\quad = c_k(f^*(\check{\mathcal{E}})) L^{j-k} (2^{i-l-j} L^{i-l-j} + (i-l-j) 2^{i-l-j-1} L^{i-l-j-1} f^*(B)) \\ &\quad = 2^{i-l-j} c_k(f^*(\check{\mathcal{E}})) L^{i-l-k} + (i-l-j) 2^{i-l-j-1} c_k(f^*(\check{\mathcal{E}})) L^{i-l-1-k} f^*(B).\end{aligned}$$

Hence

$$\begin{aligned}(3)\quad & \left(\sum_{j=1}^{i-l} (-1)^{i-l-j} \sum_{k=0}^{j} \binom{n+1-k}{j-k} c_k(f^*(\check{\mathcal{E}})) L^{j-k} \mathcal{O}(X|_X)^{i-l-j} \right) L^{n-i+l} \\ &= \sum_{j=1}^{i-l} (-1)^{i-l-j} \sum_{k=0}^{j} \binom{n+1-k}{j-k} c_k(f^*(\check{\mathcal{E}})) (2^{i-l-j} L^{n-k}\end{aligned}$$

$$+ (i-l-j)2^{i-l-j-1}L^{n-k-1}f^*(B))$$

$$= \sum_{j=1}^{i-l}(-1)^{i-l-j}\left\{\binom{n+1}{j}2^{i-l-j}L^n + \binom{n+1}{j}(i-l-j)2^{i-l-j-1}L^{n-1}f^*(B)\right.$$

$$\left. + \binom{n}{j-1}2^{i-l-j}c_1(f^*(\check{\mathcal{E}}))L^{n-1}\right\}.$$

On the other hand,

$$(4)\quad \left(\sum_{j=1}^{i-l}(-1)^{i-l-j}c_1(f^*\mathcal{T}_C)\sum_{k=0}^{j-1}\binom{n+1-k}{j-1-k}c_k(f^*(\check{\mathcal{E}}))L^{j-1-k}\mathcal{O}(X|_X)^{i-l-j}\right)L^{n-i+l}$$

$$= \sum_{j=1}^{i-l}(-1)^{i-l-j}c_1(f^*\mathcal{T}_C)$$

$$\times \sum_{k=0}^{j-1}\binom{n+1-k}{j-1-k}c_k(f^*(\check{\mathcal{E}}))L^{j-1-k}(2^{i-l-j}L^{n-j}$$

$$+ (i-l-j)2^{i-l-j-1}L^{n-j-1}f^*(B))$$

$$= \sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\left(\sum_{k=0}^{j-1}\binom{n+1-k}{j-1-k}c_k(f^*(\check{\mathcal{E}}))c_1(f^*\mathcal{T}_C)L^{n-1-k}\right)$$

$$= \sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j-1}c_1(f^*\mathcal{T}_C)L^{n-1}.$$

Therefore by (2), (3) and (4) we have

$$c_{i-l}(X)L^{n-i+l} = (-1)^{i-l}(2L+f^*(B))^{i-l}L^{n-i+l}$$

$$+ \sum_{j=1}^{i-l}(-1)^{i-l-j}\left\{\binom{n+1}{j}2^{i-l-j}L^n + \binom{n+1}{j}(i-l-j)2^{i-l-j-1}L^{n-1}f^*(B)\right.$$

$$\left. + \binom{n}{j-1}2^{i-l-j}c_1(f^*(\check{\mathcal{E}}))L^{n-1}\right\}$$

$$+ \sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j-1}c_1(f^*\mathcal{T}_C)L^{n-1}$$

$$= \left(\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j}\right)L^n$$

$$+\left(\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j-1}\binom{n+1}{j}(i-l-j)\right)L^{n-1}f^*(B)$$
$$+\left(\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n}{j-1}\right)c_1(f^*(\check{\mathcal{E}}))L^{n-1}$$
$$+\left(\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j-1}\right)c_1(f^*\mathcal{T}_C)L^{n-1}.$$

Here we note that $L^n = 2e + b$, $L^{n-1}f^*(B) = 2b$, $c_1(f^*(\check{\mathcal{E}}))L^{n-1} = -2e$ and $c_1(f^*\mathcal{T}_C)L^{n-1} = 4 - 4g(C)$. Hence

$$\begin{aligned}
&c_{i-l}(X)L^{n-i+l}\\
&=\left(\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j}\right)(2e+b)\\
&\quad+\left(\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j-1}\binom{n+1}{j}(i-l-j)\right)(2b)\\
&\quad+\left(\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n}{j-1}\right)(-2e)\\
&\quad+\left(\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j-1}\right)(4-4g(C))\\
&=\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j+1}\binom{n}{j}e+\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j}(i-l-j+1)b\\
&\quad+\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j+2}\binom{n+1}{j-1}(1-g(C)).
\end{aligned}$$

Therefore by (1)

(5)
$$\begin{aligned}
&e_i(X,L)\\
&=\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}c_{i-l}(X)L^{n-i+l}\\
&=\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j+1}\binom{n}{j}\right\}e
\end{aligned}$$

$$+\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-l}(-1)^{i-l-j}2^{i-l-j}\binom{n+1}{j}(i-l-j+1)\right\}b$$

$$+\sum_{l=0}^{i-1}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=1}^{i-l}(-1)^{i-l-j}2^{i-l-j+2}\binom{n+1}{j-1}\right\}(1-g(C)).$$

Next we prove the following:

CLAIM 3.1. (a) *Let i be a non-negative integer. Then*

$$\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-l}(-2)^{i-l-j}\binom{n}{j}\right\}=(-1)^i.$$

(b) *Let i be a non-negative integer. Then*

$$\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-l}(-2)^{i-l-j}\binom{n+1}{j}(i-l-j+1)\right\}=(-1)^i(i+1).$$

(c) *Let i be a positive integer. Then*

$$\sum_{l=0}^{i-1}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=1}^{i-l}(-2)^{i-l-j}\binom{n+1}{j-1}\right\}=\begin{cases}(i+1)/2 & \text{if } i \text{ is odd},\\ i/2 & \text{if } i \text{ is even}.\end{cases}$$

PROOF. First we consider (a). Let x be a variable with $|x|<1$. Then the following holds.

$$\begin{aligned}(1+x)^i &= (1+x)^n(1+x)^{-n+i}\\ &=\left\{\sum_{j=0}^{n}\binom{n}{j}x^j\right\}\left\{\sum_{l\geq 0}\binom{-n+i}{l}x^l\right\}\\ &=\left\{\sum_{j=0}^{n}\binom{n}{j}x^j\right\}\left\{\sum_{l\geq 0}(-1)^l\binom{n-i+l-1}{l}x^l\right\}\\ &=\sum_{l=0}^{i}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-l}\binom{n}{j}x^{j+l}\right\}.\end{aligned}\tag{6}$$

By substituting $-1/2$ for x in (6) and multiplying it by $(-2)^i$, we get the assertion of (a).

Next we consider (b). By the same argument as above we get the following equality.

$$\begin{aligned}(1+x)^{i+1} &= (1+x)^{n+1}(1+x)^{-n+i}\\ &=\left\{\sum_{j=0}^{n+1}\binom{n+1}{j}x^j\right\}\left\{\sum_{l\geq 0}\binom{-n+i}{l}x^l\right\}\end{aligned}\tag{7}$$

$$= \left\{ \sum_{j=0}^{n+1} \binom{n+1}{j} x^j \right\} \left\{ \sum_{l \geq 0} (-1)^l \binom{n-i+l-1}{l} x^l \right\}$$

$$= \sum_{l=0}^{i+1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i+1-l} \binom{n+1}{j} x^{j+l} \right\}.$$

By multiplying (7) by $(1/x)^{i+1}$ we have

$$\left(1+\frac{1}{x}\right)^{i+1} = \sum_{l=0}^{i+1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i+1-l} \binom{n+1}{j} x^{j+l-i-1} \right\}.$$

By differentiating the both sides with respect to x, we see that

$$(i+1)\left(1+\frac{1}{x}\right)^{i}\left(-\frac{1}{x^2}\right)$$

$$= \sum_{l=0}^{i+1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i+1-l} \binom{n+1}{j} (j+l-i-1) x^{j+l-i-2} \right\}$$

$$= \sum_{l=0}^{i} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i-l} \binom{n+1}{j} (j+l-i-1) x^{j+l-i-2} \right\}.$$

By substituting $-1/2$ for x, we get

$$(8) \qquad (i+1)(-1)^i(-4)$$
$$= \sum_{l=0}^{i} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i-l} \binom{n+1}{j} (j+l-i-1)(-2)^{i-j-l+2} \right\}.$$

By multiplying (8) by $-1/4$, we get the assertion of (b).

Finally we consider (c). Here we use (7) again. We note that by using (7) we see

$$(9) \qquad \sum_{l=0}^{i+1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i+1-l} \binom{n+1}{j} x^{j+l} \right\}$$
$$= x^{i+1} + (i+1)x^i + \sum_{l=0}^{i-1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i-1-l} \binom{n+1}{j} x^{j+l} \right\}.$$

Hence by (7) and (9) we have

$$(10) \qquad (1+x)^{i+1} - x^{i+1} - (i+1)x^i$$
$$= \sum_{l=0}^{i-1} (-1)^l \binom{n-i+l-1}{l} \left\{ \sum_{j=0}^{i-1-l} \binom{n+1}{j} x^{j+l} \right\}.$$

By substituting $1/x$ for x in (10) and multiplying it to x^{i-1}, we have

$$\tag{11} \begin{aligned} &\frac{(1+x)^{i+1}-1-(i+1)x}{x^2} \\ &\quad = \sum_{l=0}^{i-1}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=0}^{i-1-l}\binom{n+1}{j}x^{i-1-j-l}\right\} \\ &\quad = \sum_{l=0}^{i-1}(-1)^l\binom{n-i+l-1}{l}\left\{\sum_{j=1}^{i-l}\binom{n+1}{j-1}x^{i-j-l}\right\}. \end{aligned}$$

By substituting -2 for x in (11), we get the assertion (c). □

By using this claim and (5), we have

$$e_i(X,L) = (-1)^i(2e+(i+1)b) + \begin{cases} 2(i+1)(1-g(C)) & \text{if } i \text{ is odd}, \\ 2i(1-g(C)) & \text{if } i \text{ is even}. \end{cases}$$

Therefore we get the assertion of Theorem 3.1. □

REMARK 3.1. (1) Let $\mathbf{Q}^m$ be a quadric hypersurface in $\mathbf{P}^{m+1}$. Then by [11, Example (1.5), (iv)] and using the Lefschetz theorem on hyperplane sections we see that for $0 \le j \le m-1$

$$b_j(\mathbf{Q}^m) = \begin{cases} 0 & \text{if } j \text{ is odd}, \\ 1 & \text{if } j \text{ is even}, \end{cases}$$

and

$$b_m(\mathbf{Q}^m) = \begin{cases} 2 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}. \end{cases}$$

(2) Let (X, L) be a hyperquadric fibration over a smooth curve C with $\dim X = n \ge 3$, let i be an integer with $1 \le i \le n$ and let F be a general fiber of it. Then $(F, L|_F) \cong (\mathbf{Q}^{n-1}, \mathcal{O}_{\mathbf{Q}^{n-1}}(1))$. By (1) above and [6, Proposition 3.2 (3.2.2)] we have

$$b_{i-1}(F, L|_F) = b_{i-1}(\mathbf{Q}^{i-1}) = \begin{cases} 2 & \text{if } i \text{ is odd}, \\ 0 & \text{if } i \text{ is even}. \end{cases}$$

Moreover if $i \ge 2$, then by (1) above we have

$$2\sum_{j=0}^{i-2}(-1)^j b_j(F) = \begin{cases} i-1 & \text{if } i \text{ is odd}, \\ i & \text{if } i \text{ is even}. \end{cases}$$

Hence by Definition 2.1 we have

$$e_{i-1}(F, L|_F) = \begin{cases} (L|_F)^{n-1} & \text{if } i = 1\,, \\ 2\sum_{j=0}^{i-2}(-1)^j b_j(F) + (-1)^{i-1} b_{i-1}(F, L|_F) & \text{if } i \geq 2\,, \end{cases}$$

$$= \begin{cases} i+1 & \text{if } i \text{ is odd}\,, \\ i & \text{if } i \text{ is even}\,. \end{cases}$$

Therefore by Theorem 3.1 we have

$$e_i(X, L) - e_{i-1}(F, L|_F)e(C) = (-1)^i(2e + (i+1)b)\,.$$

(Here $e(C)$ is the Euler number of C.)

REMARK 3.2. Let (X, L) be a hyperquadric fibration over a smooth curve C with $\dim X = n$. Then we see that the degree L^n is equal to $2e+b$ and the sectional genus $g(X, L)$ is equal to $2g(C) - 1 + e + b$. So if we are able to know the value of L^n, $g(X, L)$ and $g(C)$, then we can calculate b and e, and we can also calculate $e_i(X, L)$ by Theorem 3.1.

4. A lower bound for the sectional Betti numbers of hyperquadric fibrations over a smooth curve

In this section, we consider a lower bound for the sectional Betti numbers by using Theorem 3.1.

THEOREM 4.1. *Let (X, L) be an n-dimensional hyperquadric fibration over a smooth curve C.*

(1) *Assume that $n \geq 3$. Then $b_2(X, L) \geq 3$. Moreover if this equality holds, then $n = 3$, $e = 2$, $b = -1$, $g(C) \geq 1$ and every fiber is smooth.*

(2) *Assume that $n \geq 4$. Then $b_3(X, L) \geq 0$. Moreover if this equality holds, then $X \cong \mathbf{P}^1 \times \mathbf{Q}^3$ and $L = p_1^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbf{Q}^3}(1)$, where p_i is the ith projection.*

(3) *Assume that $n \geq 5$. Then $b_4(X, L) \geq 3$. Moreover if this equality holds, then $n = 5$, $e = 3$, $b = -1$, $g(C) \geq 1$ and every fiber is smooth.*

PROOF. (1) By Theorem 3.1 we have $e_2(X, L) = 2e + 3b + 4(1 - g(C))$. Hence

$$\begin{aligned} b_2(X, L) &= e_2(X, L) - 2(b_0(X) - b_1(X)) \\ &= 2e + 3b + 4(1 - g(C)) - 2(1 - 2g(C)) \\ &= 2e + 3b + 2\,. \end{aligned}$$

Here we show the following:

CLAIM 4.1. $2e + 3b \geq 1$ *holds. Moreover if this equality holds, then we have $n = 3$, $e = 2$ and $b = -1$.*

PROOF. Here we note that the following inequalities hold (see [3, (3.3) and (3.4)]):

(a) $2e + b > 0$.

(b) $2e + (n + 1)b \geq 0$.

If $b \geq 0$, then by (a) we have $2e + 2b > 0$. Since $2e + 2b$ is an even number, we get $2e + 2b \geq 2$. Therefore $2e + 3b \geq 2$.

If $b < 0$, then by (b) we have $2e + 3b \geq 1$. Therefore we get the first assertion.

If $2e + 3b = 1$, then by the above argument we have $n = 3$ and $2e + 4b = 0$. Hence we see that $b = -1$ and $e = 2$. So we get the second assertion. □

By this claim we have $b_2(X, L) \geq 3$. Moreover if this equality holds, then by Claim 4.1 we see that $2e + 3b = 1$. Hence we have $n = 3$, $e = 2$ and $b = -1$. Here we note that $g(X, L) = 2g(C)$ in this case. Hence by Remark 2.2 (2) we have $g(C) \geq 1$. Since $2e + (n + 1)b = 2e + 4b = 0$, we see that every fiber is smooth by [3, (3.3)].

(2) By Theorem 3.1 we have $e_3(X, L) = -2e - 4b + 8(1 - g(C))$. We also note that $b_2(X) = 2$ in this case because $n \geq 4$ (see Remark 2.1). Hence

$$\begin{aligned} b_3(X, L) &= -e_3(X, L) + 2(b_0(X) - b_1(X) + b_2(X)) \\ &= 2e + 4b - 8(1 - g(C)) + 2(1 - 2g(C) + 2) \\ &= 2e + 4b - 2 + 4g(C)\,. \end{aligned}$$

Here we show the following:

CLAIM 4.2. *$2e+4b \geq 2$ holds. Moreover if this equality holds, then we have $(e, b) = (1, 0)$, $(3, -1)$ or $(5, -2)$. Moreover if $(e, b) = (3, -1)$ (resp. $(5, -2)$), then $n = 4$ or 5 (resp. $n = 4$).*

PROOF. (I) If $b \geq 0$, then by (a) in the proof of Claim 4.1 we have $2e + 4b > 0$. Since $2e + 4b$ is an even number, we have $2e + 4b \geq 2$.

If $b < 0$, then by (b) in the proof of Claim 4.1 we have $2e + 4b > 0$. (Here we note that $n \geq 4$.) So we get the first assertion because $2e + 4b$ is even.

(II) Assume that $2e + 4b = 2$.

(II.1) If $b \geq 0$, then we see that $b = 0$ because $2e + b > 0$. Hence $e = 1$.

(II.2) If $b < 0$, then we see that $n = 4$ or 5 because $2e + (n + 1)b \geq 0$ and $n \geq 4$.

(II.2.1) Assume that $n = 4$. Then $2e + 5b \geq 0$. So we have $b = -1$ or -2. If $b = -1$ (resp. -2), then $e = 3$ (resp. 5).

(II.2.2) Assume that $n = 5$. Then $2e+6b \geq 0$. So we get $b = -1$ and $e = 3$. Therefore we get the second assertion. □

By this claim we have $b_3(X, L) \geq 0$. Moreover we assume that $b_3(X, L) = 0$. Then $2e + 4b = 2$ and $g(C) = 0$. By Claim 4.2, we have $(e, b) = (1, 0)$, $(3, -1)$ or $(5, -2)$. Here we note that $g(X, L) = 2g(C) - 1 + e + b$. So if $(e, b) = (1, 0)$ (resp. $(3, -1)$, $(5, -2)$), then $g(X, L) = 0$ (resp. 1, 2). Here we also note that by Remark 2.2 (2) we have $g(X, L) \geq 2$. Therefore there exists only $(n, e, b) = (4, 5, -2)$. Since $g(X, L) = 2$, by [3, (3.30) Theorem] we see that $X \cong \mathbf{P}^1 \times \mathbf{Q}^3$ and $L = p_1^*\mathcal{O}_{\mathbf{P}^1}(1) \otimes p_2^*\mathcal{O}_{\mathbf{Q}^3}(1)$.

(3) By Theorem 3.1 we see $e_4(X, L) = 2e + 5b + 8(1 - g(C))$. By [12, (2.74) Corollary], we have $b_3(X) \geq b_1(X) = 2g(C)$. We also note that $b_2(X) = 2$ in this case because $n \geq 5$. Hence

$$\begin{aligned} b_4(X, L) &= e_4(X, L) - 2\sum_{j=0}^{3}(-1)^j b_j(X) \\ &\geq 2e + 5b + 8(1 - g(C)) + 2(-1 + 2g(C) - 2 + 2g(C)) \\ &= 2e + 5b + 2\,. \end{aligned}$$

By the same argument as in the proof of Claim 4.1 we get $2e + 5b \geq 1$ and we see that if this equality holds, then $n = 5$, $e = 3$ and $b = -1$. Since $g(X, L) = 2g(C) - 1 + e + b = 2g(C) + 1$, we have $g(C) \geq 1$ by Remark 2.2 (2). Since $2e + (n + 1)b = 2e + 6b = 0$, we see that every fiber is smooth by [3, (3.3)].

This completes the proof. □

REMARK 4.1. (1) Here we note that if (X, L) is a hyperquadric fibration over a smooth curve, then $h_i^{i,0}(X, L) = h_i^{0,i}(X, L) = g_i(X, L) = 0$ for $i = 2, 3$ by [5, Example 2.10 (9)] and [6, Theorem 3.1 (3.1.4)], where $h_i^{j,i-j}(X, L)$ is the ith sectional Hodge number of type $(j, i - j)$ (see [6, Definition 3.1 (3)]). Hence by [6, Theorem 3.1 (3.1.1) and (3.1.3)], we see that $b_2(X, L) = h_2^{1,1}(X, L)$ and $b_3(X, L) = 2h_3^{1,2}(X, L)$.

(2) There exists an example of a hyperquadric fibration over a smooth curve C such that $b_2(X, L) = 3$ and $g(C) = 1$. See [3, Example (3.14)].

PROBLEM 4.1. (1) Does there exist an example of a hyperquadric fibration (X, L) over a smooth curve C with $b_2(X, L) = 3$ and $g(C) \geq 2$?

(2) Does there exist an example of a hyperquadric fibration (X, L) over a smooth curve C with $b_4(X, L) = 3$ and $g(C) = 1$? (If this (X, L) exists, then this is also an example of a hyperquadric fibration over a smooth elliptic curve (X, L) with $g(X, L) = 3$ and $L^n = 5$. See [10, (2.4)].)

Theorem 4.1 suggests the following conjecture.

CONJECTURE 4.1. *Let (X, L) be a hyperquadric fibration over a smooth curve C with* $\dim X = n$. *Then*

$$b_i(X, L) \geq \begin{cases} 0 & \text{if } i \text{ is odd}, \\ 3 & \text{if } i \text{ is even with } i \geq 2\,. \end{cases}$$

REMARK 4.2. Let (X, L) be a hyperquadric fibration over a smooth curve C and let $f : X \to C$ be its morphism. We consider the case where $i = 1$. Then by Remark 2.2 (2) we have $g(X, L) \geq 2$. Therefore $b_1(X, L) = 2g(X, L) \geq 4$ (see [6, Remark 3.1 (2)]).

EXAMPLE 4.1. (1) Let $X = C \times \mathbf{Q}^{2k+1}$ and $L = p_1^*(A) + p_2^*(\mathcal{O}_{\mathbf{Q}^{2k+1}}(1))$, where C is a smooth curve, A is an ample divisor on C with $\deg A = 1$, k is a positive integer and p_i

is the ith projection for $i = 1$ and 2. Then (X, L) is a hyperquadric fibration over C via the first projection $X \to C$. In this case, we see that $e = 2k + 3$ and $b = -2$. Hence by Theorem 3.1 we have $e_{2k+1}(X, L) = 4(k + 1)(1 - g(C)) - 2$. On the other hand, by the formula of Künneth (see [13, Theorem 11.38]), we have

$$b_j(X) = \begin{cases} 1 & \text{if } j = 0, \\ 2 & \text{if } j \text{ is even with } 2 \le j \le 2k, \\ 2g(C) & \text{if } j \text{ is odd with } 1 \le j \le 2k-1. \end{cases}$$

Hence

$$2\sum_{j=0}^{2k}(-1)^j b_j(X) = 4k + 2 - 4kg(C)\,.$$

Therefore we have $b_{2k+1}(X, L) = 4g(C)$. In particular, if $g(C) = 0$, then $b_{2k+1}(X, L) = 0$.

(2) Let $X = C \times \mathbf{Q}^{2k}$ and $L = p_1^*(A) + p_2^*(\mathcal{O}_{\mathbf{Q}^{2k}}(1))$, where C is a smooth curve, A is an ample divisor on C with $\deg A = 1$, k is a positive integer and p_i is the ith projection for $i = 1$ and 2. Then (X, L) is a hyperquadric fibration over C via the first projection $X \to C$. In this case, we see that $e = 2k + 2$ and $b = -2$. Hence by Theorem 3.1 we have $e_{2k}(X, L) = 4k(1 - g(C)) + 2$. On the other hand, by the formula of Künneth, we have

$$b_j(X) = \begin{cases} 1 & \text{if } j = 0\,, \\ 2 & \text{if } j \text{ is even with } 2 \le j \le 2k-2\,, \\ 2g(C) & \text{if } j \text{ is odd with } 1 \le j \le 2k-1\,. \end{cases}$$

Hence

$$2\sum_{j=0}^{2k-1}(-1)^j b_j(X) = 4k(1 - g(C)) - 2\,.$$

Therefore we have $b_{2k}(X, L) = 4$.

Finally we note the following:

PROPOSITION 4.1. *Let (X, L) be a hyperquadric fibration over a smooth curve C with $\dim X = n$, and let F be a general fiber of it. For every integer i with $1 \le i \le n$, we have*

$$e_i(X, L) - e_{i-1}(F, L|_F)e(C) \begin{cases} \le 0 & \textit{if } i \textit{ is odd}, \\ \ge 0 & \textit{if } i \textit{ is even}. \end{cases}$$

PROOF. By Remark 3.1 (2) we have

$$e_i(X, L) - e_{i-1}(F, L|_F)e(C) = (-1)^i(2e + (i + 1)b)\,.$$

By the same argument as in the proof of Claim 4.1 we can prove that $2e + (i + 1)b \ge 0$. So we get the assertion. □

References

[1] M. C. BELTRAMETTI and A. J. SOMMESE, *The adjunction theory of complex projective varieties*, de Gruyter Expositions in Math. **16**, Walter de Gruyter, Berlin, NewYork, (1995).

[2] M. C. BELTRAMETTI, A. J. SOMMESE and J. A. WIŚNIEWSKI, Results on varieties with many lines and their applications to adjunction theory (with an appendix by M. C. Beltrametti and A. J. Sommese), in *Complex Algebraic Varieties*, Bayreuth 1990, ed. by K. Hulek, T. Peternell, M. Schneider, and F.-O. Schreyer, Lecture Notes in Math., **1507** (1992), 16–38, Springer-Verlag, New York.

[3] T. FUJITA, Classification of polarized manifolds of sectional genus two, the Proceedings of "Algebraic Geometry and Commutative Algebra" in Honor of Masayoshi Nagata, 1987, 73–98.

[4] T. FUJITA, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Ser. **155**, Cambridge University Press, (1990).

[5] Y. FUKUMA, *On the sectional geometric genus of quasi-polarized varieties, I*, Comm. Algebra **32** (2004), 1069–1100.

[6] Y. FUKUMA, On the sectional invariants of polarized manifolds, J. Pure Appl. Alg. **209** (2007), 99–117.

[7] Y. FUKUMA, Sectional invariants of scrolls over a smooth projective variety, Rend. Sem. Mat. Univ. Padova **121** (2009), 93–119.

[8] W. FULTON, *Intersection Theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete **2** (1984), Springer-Verlag.

[9] P. IONESCU, Generalized adjunction and applications, Math. Proc. Camb. Phil. Soc. **99** (1986), 457–472.

[10] H. ISHIHARA, On polarized manifolds of sectional genus three, Kodai Math. J. **18** (1995), 328–343.

[11] A. LANTERI and D. STRUPPA, Projective manifolds whose topology is strongly reflected in their hyperplane sections, Geom. Dedicata **21** (1986), 357–374.

[12] B. SHIFFMAN and A. J. SOMMESE, *Vanishing Theorems on Complex Manifolds*, Progress in Mathematics **56** (1985), Birkhäuser.

[13] C. VOISIN, *Hodge Theory and Complex Algebraic Geometry, I*, Cambridge studies in advanced mathematics **96** (2002), Cambridge University Press.

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