The Homogeneous Slice Theorem for the Complete Complexification of a Proper Complex Equifocal Submanifold

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Abstract. The notion of a complex equifocal submanifold in a Riemannian symmetric space of non-compact type has been recently introduced as a generalization of isoparametric hypersurfaces in the hyperbolic space. As its subclass, the notion of a proper complex equifocal submanifold has been introduced. Some results for a proper complex equifocal submanifold have been recently obtained by investigating the lift of its complete complexification to some path space. In this paper, we give a new construction of the complete complexification of a proper complex equifocal submanifold and, by using the construction, show that leaves of focal distributions of the complete complexification are the images by the normal exponential map of principal orbits of a certain kind of pseudo-orthogonal representation on the normal space of the corresponding focal submanifold.

1. Introduction

C. L. Terng and G. Thorbergsson [TT1] introduced the notion of an equifocal submanifold in a Riemannian symmetric space, which is defined as a compact submanifold with globally flat and abelian normal bundle such that the focal radii for each parallel normal vector field are constant. This notion is a generalization of isoparametric submanifolds in the Euclidean space and isoparametric hypersurfaces in the sphere or the hyperbolic space. For (not necessarily compact) submanifolds in a Riemannian symmetric space of non-compact type, the equifocality is a rather weak property. So, we [K1, 2] introduced the notion of a complex focal radius as a general notion of a focal radius and defined the notion of a complex equifocal submanifold as a submanifold with globally flat and abelian normal bundle such that the complex focal radii for each parallel normal vector field are constant with constant multiplicities. E. Heintze, X. Liu and C. Olmos [HLO] defined the notion of an isoparametric submanifold with flat section as a submanifold with globally flat and abelian normal bundle such that sufficiently close parallel submanifolds are of constant mean curvature with respect to the radial direction. The following fact is known (see Theorem 15 of [K2]):

All isoparametric submanifolds with flat section are complex equifocal and, conversely, all curvature-adapted and complex equifocal submanifolds are isoparametric submanifolds with flat section.

Furthermore, as its subclass, we [K1, 2] introduced the notion of a proper complex equifocal submanifold. For a proper complex equifocal submanifold, the following fact is known ([K3]):

Principal orbits of Hermann type actions on a Riemannian symmetric space of non-compact type are curvature-adapted and proper complex equifocal.

For a (general) submanifold in a Riemannian symmetric space of non-compact type, the (non-real) complex focal radii are defined algebraically. We needed to find their geometrical essence. For its purpose, we defined the complexification of the ambient Riemannian symmetric space and defined the extrinsic complexification of the submanifold as a certain kind of submanifold in the complexified symmetric space, where the original submanifold needs to be assumed to be complete and real analytic. We assume that all submanifolds in the Riemannian symmetric space are complete and real analytic. We [K2] showed that the complex focal radii of the original submanifold indicate the positions of the focal points of the complexified submanifold. If the original submanifold is complex equifocal, then the extrinsic complexification is an anti-Kaehlerian equifocal submanifold in the sense of [K2]. Also, if the original submanifold is proper complex equifocal, then the complexified one is a proper anti-Kaehlerian equifocal submanifold in the sense of this paper. Thus, the study of an anti-Kaehlerian equifocal (resp. proper anti-Kaehlerian equifocal) submanifold leads to that of a complex equifocal (resp. proper complex equifocal) submanifold. The complexified submanifold is not necessarily complete. In the global problem, we need to extend the complexified submanifold to a complete one. In [K2], we obtained the complete extension of the complexified submanifold by the following method. We first lifted the complexified submanifold to some path space (which is an infinite dimensional anti-Kaehlerian space) through some submersion, extended the lifted submanifold to the complete one by repeating some kind of extension infinitely many times and obtained the complete extension of the original complexified submanifold as the image of the complete one by the submersion. In this paper, we give a new construction of the complete extension of the complexified submanifold without the infinite process (see the proof of Theorem B) and investigate the detailed structure of the complete extension in terms of the new construction. First we prove the following fact for an anti-Kaehlerian equifocal submanifold.

THEOREM A. *Let M be an anti-Kaehlerian equifocal submanifold in a semi-simple anti-Kaehlerian symmetric space of non-positive (or non-negative) curvature having a focal submanifold F. If the sections of M are properly embedded, then M is an open portion of a partial tube over F whose each fibre is the image by the normal exponential map of a principal orbit of a pseudo-orthogonal representation on the normal space of F which is equivalent to the direct sum representation of an aks-representation and a trivial representation.*

REMARK 1.1. (i) For a focal submanifold F of M, we call $(\exp^{\perp}|_{T_x^{\perp}F})^{-1}$

$(\exp^{\perp}(T_x^{\perp}F) \cap M)$ (rather than $\exp^{\perp}(T_x^{\perp}F) \cap M$) the slice of M. This theorem states that slices of a complete anti-Kaehlerian equifocal submanifold are homogeneous.

(ii)   The dual action H^* of a Hermann type action H on a Riemannian symmetric space G/K of non-compact type is a Hermann action on the compact dual G^*/K of G/K, where G is assumed to be a connected semi-simple Lie group admitting a faithful real representation. Note that the existence of the dual action H^* is assured by replacing H by the conjugate group if necessary. Hence the sections of the H^*-action are flat tori. From this fact, we see that the sections of the $H^{\mathbf{c}}$-action on $G^{\mathbf{c}}/K^{\mathbf{c}}$ are properly embedded, where $H^{\mathbf{c}}$ is the complexification of H and $G^{\mathbf{c}}/K^{\mathbf{c}}$ is the anti-Kaehlerian symmetric space associated with G/K. On the other hand, the principal orbits of the $H^{\mathbf{c}}$-action are proper anti-Kaehlerian equifocal. Thus the principal orbits are submanifolds as in the statement of Theorem A.

(iii)   This result is an analogy of that of M. Brück [B] for an equifocal submanifold in a simply connected Riemannian symmetric space of compact type.

In [K4, 5], we proved some global results for a proper complex equifocal submanifold by investigating the lift of the complete complexification of the submanifold to some path space through some submersion. Thus, in the global study of a proper complex equifocal submanifold, it is important to investigate the detailed structure of its complete complexification. By using Theorem A, we obtain a new construction of the complete complexification of a proper complex equifocal submanifold (see the proof of Theorem B). From the construction and Theorem A, we obtain the following homogeneous slice theorem for the complete complexification of a proper complex equifocal submanifold.

THEOREM B.   *Assume that the sections of the complexification of a proper complex equifocal submanifold M in a Riemannian symmetric space G/K of non-compact type are properly embedded. Then the following statements* (i) *and* (ii) *hold*:

(i)   *Each leaf of any focal distribution of the complete complexification $\widehat{M^{\mathbf{c}}}$ of M is the image by the normal exponential map of a principal orbit of a pseudo-orthogonal representation on the normal space of a focal submanifold which is equivalent to the direct sum representation of an aks-representation and a trivial representation.*

(ii)   *Let E_0 be the distribution on $\widehat{M^{\mathbf{c}}}$ defined by $(E_0)_x := \bigcap\limits_{v \in T_x^{\perp}\widehat{M^{\mathbf{c}}}} (\mathrm{Ker}\, R^{\mathbf{c}}(\cdot, v)v \cap$*

$\mathrm{Ker}\, A_v^{\mathbf{c}})$ *$(x \in \widehat{M^{\mathbf{c}}})$, where $R^{\mathbf{c}}$ is the curvature tensor of $G^{\mathbf{c}}/K^{\mathbf{c}}$ and $A^{\mathbf{c}}$ is the shape tensor of $\widehat{M^{\mathbf{c}}}$. Then there exists a family $\{E_i \,|\, i = 1, \ldots, k\}$ of focal distributions of $\widehat{M^{\mathbf{c}}}$ such that the leaves of E_i $(i = 1, \ldots, k)$ are the images by the normal exponential map of complex spheres in the normal spaces of focal submanifolds and that $E_0 \oplus \sum\limits_{i=1}^{k} E_i = T\widehat{M^{\mathbf{c}}}$ holds.*

For a curvature-adapted and proper complex equifocal submanifold, we obtain the following fact in terms of Theorem B.

THEOREM C.   *Let M be a proper complex equifocal submanifold in a Riemannian symmetric space of non-compact type as in Theorem B and $\{E_0, \ldots, E_k\}$ be as in the state-*

ment (ii) *of Theorem B. Assume that M is curvature-adapted. Then $E_i^{\mathbf{R}} := E_i|_M \cap TM$ ($i = 0, \ldots, k$) are integrable distributions on M, leaves of $E_i^{\mathbf{R}}$ are half-dimensional totally real submanifolds of leaves of E_i and $TM = E_0^{\mathbf{R}} \oplus \sum_{i=1}^{k} E_i^{\mathbf{R}}$, where $E_i|_M$ is the restriction of E_i to M.*

REMARK 1.2.   B. Wu ([W2]) showed that leaves of curvature distributions of a complete isoparametric submanifold in a hyperbolic space are totally umbilic spheres, totally umbilic hyperbolic spaces or horospheres, where we note that the complexifications of a totally umbilic sphere and a totally umbilic hyperbolic space are totally anti-Kaehlerian umbilic complex spheres in the complexification (which is a complex sphere) of the ambient hyperbolic space. See [K2] for the definition of the totally anti-Kaehlerian umbilicity. Thus the statement of Theorem C is interpreted as an analogy of this result by B. Wu.

As future research, by using Theorem B, we will investigate whether the complete complexifications of proper complex equifocal submanifolds are homogeneous. By using Theorems B and C, we will investigate whether curvature-adapted and proper complex equifocal submanifolds are homogeneous.

2.   Basic notions

In this section, we recall basic notions introduced in [K1∼3]. We first recall the notion of a complex equifocal submanifold introduced in [K1]. Let M be an immersed submanifold with abelian normal bundle (i.e., the sectional curvature for each 2-plane in the normal space is equal to zero) of a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T_x^{\perp} M$ and $X \in T_x M$ ($x = gK$). Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The strongly M-Jacobi field Y along γ_v with $Y(0) = X$ (hence $Y'(0) = -A_v X$) is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D_{sv}^{co} - s D_{sv}^{si} \circ A_v))(X)\,,$$

where $Y'(0) = \widetilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$ and D_{sv}^{co} (resp. D_{sv}^{si}) is given by

$$D_{sv}^{co} = g_* \circ \cos(\sqrt{-1}\mathrm{ad}(s g_*^{-1} v)) \circ g_*^{-1}$$
$$\left(\text{resp.}\quad D_{sv}^{si} = g_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}(s g_*^{-1} v))}{\sqrt{-1}\mathrm{ad}(s g_*^{-1} v)} \circ g_*^{-1} \right).$$

Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. All focal radii of M along γ_v are obtained as real numbers s_0 with $\mathrm{Ker}(D_{s_0 v}^{co} - s_0 D_{s_0 v}^{si} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D_{z_0 v}^{co} - z_0 D_{z_0 v}^{si} \circ A_v^{\mathbf{c}}) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D_{z_0 v}^{co} - z_0 D_{z_0 v}^{si} \circ A_v^{\mathbf{c}})$ the *multiplicity* of the complex focal radius z_0, where $D_{z_0 v}^{co}$

(resp. $D^{si}_{z_0 v}$) is a **C**-linear transformation of $(T_x N)^{\mathbf{c}}$ defined by

$$D^{co}_{z_0 v} = g^{\mathbf{c}}_* \circ \cos(\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g^{-1}_* v)) \circ (g^{\mathbf{c}}_*)^{-1}$$

$$\left(\text{resp.} \quad D^{si}_{sv} = g^{\mathbf{c}}_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g^{-1}_* v))}{\sqrt{-1}\mathrm{ad}^{\mathbf{c}}(z_0 g^{-1}_* v)} \circ (g^{\mathbf{c}}_*)^{-1} \right),$$

where $g^{\mathbf{c}}_*$ (resp. $\mathrm{ad}^{\mathbf{c}}$) is the complexification of g_* (resp. ad). Here we note that, in the case where M is of class C^ω, complex focal radii along γ_v indicate the positions of focal points of the extrinsic complexification $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}})$ of M along the complexified geodesic $\gamma^{\mathbf{c}}_{\iota_* v}$, where $G^{\mathbf{c}}/K^{\mathbf{c}}$ is the anti-Kaehlerian symmetric space associated with G/K and ι is the natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$. See the following paragraph for the definitions of $G^{\mathbf{c}}/K^{\mathbf{c}}$, $M^{\mathbf{c}}(\hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}})$ and $\gamma^{\mathbf{c}}_{\iota_* v}$. Also, for a complex focal radius z_0 of M along γ_v, we call $z_0 v \ (\in (T^\perp_x M)^{\mathbf{c}})$ a *complex focal normal vector of M at x*. Furthermore, assume that M has globally flat normal bundle (i.e., the normal holonomy group of M is trivial). Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Let $\{r_{i,x} \,|\, i = 1, 2, \dots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} < 0$". Let $r_i \ (i = 1, 2, \dots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions $r_i \ (i = 1, 2, \dots)$ *complex focal radius functions for $\tilde{v}$*. We call $r_i \tilde{v}$ a *complex focal normal vector field for $\tilde{v}$*. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*. Let $\phi : H^0([0,1], \mathfrak{g}) \to G$ be the parallel transport map for G. See Section 4 of [K1] for the definition of the parallel transport map. This map ϕ is a pseudo-Riemannian submersion. Let $\pi : G \to G/K$ be the natural projection. It follows from Theorem 1 of [K2] that, M is complex equifocal if and only if each component of $(\pi \circ \phi)^{-1}(M)$ is complex isoparametric. See Section 2 of [K1] for the definition of a complex isoparametric submanifold. In particular, if each component of $(\pi \circ \phi)^{-1}(M)$ is proper complex isoparametric (i.e., complex isoparametric and, for each unit normal vector v, the complexified shape operator $A^{\mathbf{c}}_v$ is diagonalizable with respect to a pseudo-orthonormal base), then we call M a *proper complex equifocal submanifold*. For a complex equifocal submanifold, the following fact holds:

For a curvature-adapted and complex equifocal submanifold M, it is proper complex equifocal submanifold if and only if it has no focal point of non-Euclidean type on the ideal boundary of the ambient symmetric space.

Here the curvature-adaptedness means that, for each unit normal vector v, the Jacobi operator $R(\cdot, v)v$ (R : the curvature tensor of G/K) preserves the tangent space invariantly and it commutes with the shape operator A_v. See [K6] for the notion of a focal point of non-Euclidean type on the ideal boundary.

Next we recall the notions of an anti-Kaehlerian symmetric space associated with a sym-
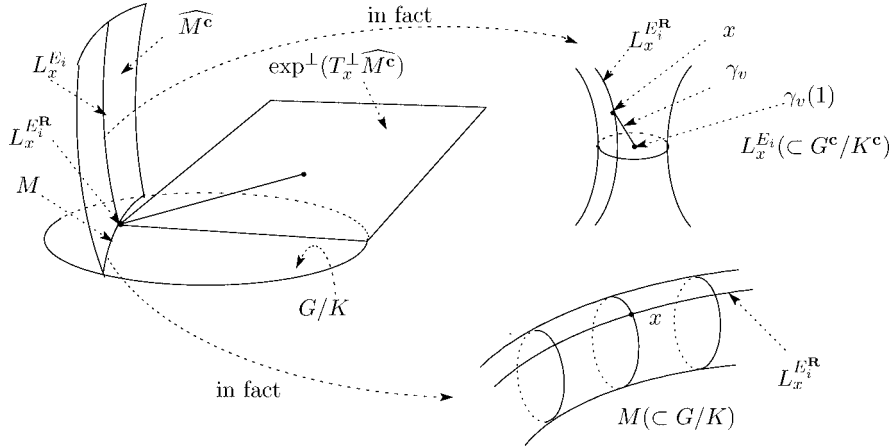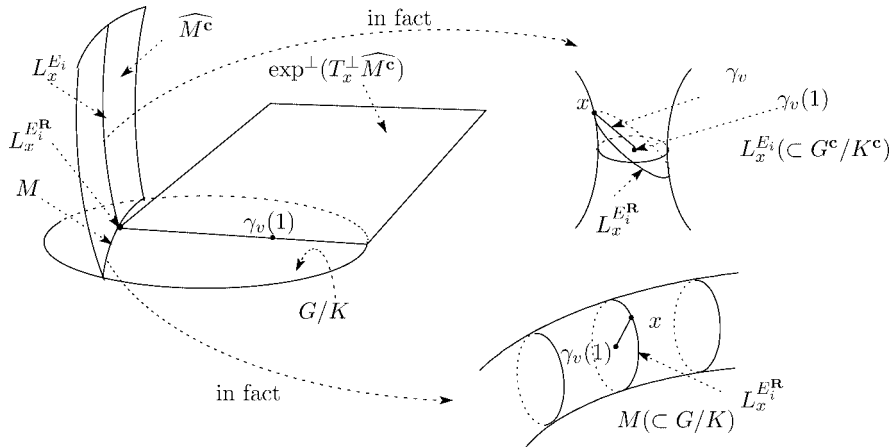
FIGURE 1.

FIGURE 2.

metric space of non-compact type which was introduced in [K2]. Let J be a parallel complex structure on an even dimensional pseudo-Riemannian manifold $(M, \langle \, , \, \rangle)$ of half index. If $\langle JX, JY \rangle = -\langle X, Y \rangle$ holds for every $X, Y \in TM$, then $(M, \langle \, , \, \rangle, J)$ is called an *anti-Kaehlerian manifold*.

Let $N = G/K$ be a symmetric space of non-compact type and $(\mathfrak{g}, \sigma)$ be its orthogonal symmetric Lie algebra. Let $\mathfrak{g} = \mathfrak{f} + \mathfrak{p}$ be the Cartan decomposition associated with a symmetric pair (G, K). Note that $\mathfrak{f}$ is the Lie algebra of K and $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$, where e is the identity element of G. Let $\langle \, , \, \rangle$ be the Ad(G)-

invariant non-degenerate inner product of $\mathfrak{g}$ inducing the Riemannian metric of N and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{a} + \sum\limits_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$, that is, $\mathfrak{p}_\alpha = \{X \in \mathfrak{p} \mid \mathrm{ad}(a)^2(X) = \alpha(a)^2 X$ for all $a \in \mathfrak{a}\}$. Let $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{f}^{\mathbf{c}}$, $\mathfrak{p}^{\mathbf{c}}$, $\mathfrak{a}^{\mathbf{c}}$ $\mathfrak{p}_\alpha^{\mathbf{c}}$ and $\langle \ , \ \rangle^{\mathbf{c}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{f}$, $\mathfrak{p}$, $\mathfrak{a}$, $\mathfrak{p}_\alpha$ and $\langle \ , \ \rangle$, respectively. If $\mathfrak{g}^{\mathbf{c}}$ and $\mathfrak{f}^{\mathbf{c}}$ are regarded as real Lie algebras, then $(\mathfrak{g}^{\mathbf{c}}, \mathfrak{f}^{\mathbf{c}})$ is a semi-simple symmetric pair, $\mathfrak{a}$ is a maximal split abelian subspace of $\mathfrak{p}^{\mathbf{c}}$ and $\mathfrak{p}^{\mathbf{c}} = \mathfrak{a}^{\mathbf{c}} + \sum\limits_{\alpha \in \triangle_+} \mathfrak{p}_\alpha^{\mathbf{c}}$ is the root space decomposition with respect to $\mathfrak{a}$. Here we note that $\mathfrak{a}^{\mathbf{c}}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{p}^{\mathbf{c}}$ and $\mathfrak{p}_\alpha^{\mathbf{c}} = \{X \in \mathfrak{p}^{\mathbf{c}} \mid (\mathrm{ad}(a)^{\mathbf{c}})^2(X) = \alpha(a)^2 X$ for all $a \in \mathfrak{a}\}$. See [R] and [OS] for general theory of a semi-simple symmetric pair. Let $G^{\mathbf{c}}$ (resp. $K^{\mathbf{c}}$) be the complexification of G (resp. K). The 2-multiple of the real part $\mathrm{Re}\langle \ , \ \rangle^{\mathbf{c}}$ of $\langle \ , \ \rangle^{\mathbf{c}}$ is the Killing form of $\mathfrak{g}^{\mathbf{c}}$ regarded as a real Lie algebra. The restriction $2\mathrm{Re}\langle \ , \ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}} \times \mathfrak{p}^{\mathbf{c}}}$ is an $\mathrm{Ad}(K^{\mathbf{c}})$-invariant non-degenerate inner product of $\mathfrak{p}^{\mathbf{c}}$ ($= T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$). Denote by $\langle \ , \ \rangle'$ the $G^{\mathbf{c}}$-invariant pseudo-Riemannian metric on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from $2\mathrm{Re}\langle \ , \ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}} \times \mathfrak{p}^{\mathbf{c}}}$. Define an almost complex structure J_0 of $\mathfrak{p}^{\mathbf{c}}$ by $J_0 X = \sqrt{-1}\, X$ ($X \in \mathfrak{p}^{\mathbf{c}}$). It is clear that J_0 is $\mathrm{Ad}(K^{\mathbf{c}})$-invariant. Denote by $\widetilde{J}$ the $G^{\mathbf{c}}$-invariant almost complex structure on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from J_0. It is shown that $(G^{\mathbf{c}}/K^{\mathbf{c}}, \langle \ , \ \rangle', \widetilde{J})$ is an anti-Kaehlerian manifold and a (semi-simple) pseudo-Riemannian symmetric space. We call this anti-Kaehlerian manifold an *anti-Kaehlerian symmetric space associated with G/K* and simply denote it by $G^{\mathbf{c}}/K^{\mathbf{c}}$. Next we shall recall the notion of an anti-Kaehlerian equifocal submanifold which was introduced in [K2]. Let f be an isometric immersion of an anti-Kaehlerian manifold $(M, \langle \ , \ \rangle, J)$ into $G^{\mathbf{c}}/K^{\mathbf{c}}$. If $\widetilde{J} \circ f_* = f_* \circ J$, then M is called an *anti-Kaehlerian submanifold* immersed by f. If, for each $x \in M$, $\exp^\perp(T_x^\perp M)$ is totally geodesic, then M is called a *a submanifold with section*. Denote by $\exp^\perp$ the normal exponential map of M. Let $v \in T_x^\perp M$. If $\exp^\perp(av_x + bJv_x)$ is a focal point of (M, x), then we call the complex number $a + b\sqrt{-1}$ a *complex focal radius along the geodesic γ_{v_x}*. Assume that the normal bundle of M is abelian and globally flat and that, for each unit normal vector field v, the number (which may be ∞) of distinct complex focal radii along the geodesic γ_{v_x} is independent of the choice of $x \in M$. Then we can define the complex radius functions as above. If, for a parallel unit normal vector field v, the number of distinct complex focal radii along γ_{v_x} is independent of the choice of $x \in M$, complex focal radius functions for v are constant on M and they have constant multiplicity, then M is called an *anti-Kaehlerian equifocal submanifold*. Let $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$. See Section 6 of [K2] for the definition of the parallel transport map. This map $\phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion. Let $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection. It is shown that M is anti-Kaehlerian equifocal if and only if each component of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M)$ is anti-Kaehlerian isoparametric. See Section 5 of [K2] for the definition of an anti-Kaehlerian isoparametric submanifold. In particular, if each component of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M)$ is proper anti-Kaehlerian isoparametric (i.e., anti-Kaehlerian isoparametric and, for each unit normal vector v, the shape operator A_v is diagonalizable with respect to an orthonormal base of the
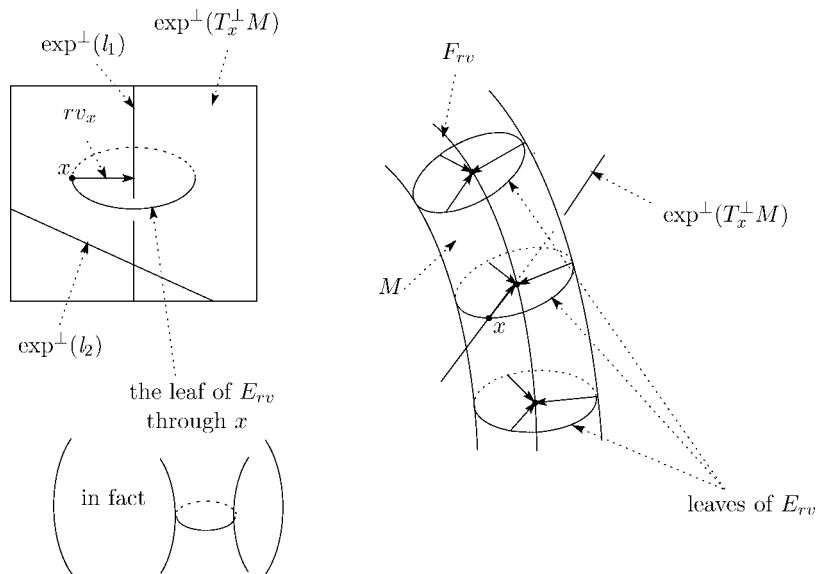
FIGURE 3.

tangent space regarded as a complex vector space), then we call M a *proper anti-Kaehlerian equifocal submanifold*. Assume that M is anti-Kaehlerian equifocal. Let r be a complex focal radius for a parallel unit normal vector field v. Then rv is called a *focal normal vector field* of M. Then a focal map $f_{rv} : M \to G^{\mathbf{c}}/K^{\mathbf{c}}$ is defined by $f_{rv}(x) = \exp^{\perp}(rv_x)$ ($x \in M$). Set $F_{rv} := f_{rv}(M)$. We call F_{rv} the *focal submanifold of M for rv*. Define a distribution E_{rv} on M by $(E_{rv})_x := \mathrm{Ker}(f_{rv})_{*x}$ ($x \in M$). We call E_{rv} the *focal distribution on M for rv*. It is clear that E_{rv} is integrable. It is shown that the focal set of M at x consists of the images by $\exp^{\perp}$ of infinitely many complex hyperplanes (which are called complex focal hyperplanes) in $T_x^{\perp}M$ (see [K2]). Denote by S the set of all complex focal hyperplanes of M at x. If $\sharp\{l \in S \mid rv_x \in l\} = 1$, then the leaves of E_{rv} are the images by the normal exponential map of complex spheres in normal spaces of F_{rv}, where $\sharp(\cdot)$ is the cardinal number of $(\cdot)$. Let r_1 (resp. r_2) be a complex focal radius for a parallel unit normal vector field v_1 (resp. v_2). If $\{l \in S \mid r_1(v_1)_x \in l\} = \{l \in S \mid r_2(v_2)_x \in l\}$, then we have $E_{r_1 v_1} = E_{r_2 v_2}$.

Next we recall the notion of the extrinsic complexification of a complete C^{ω}-submanifold in a symmetric space of non-compact type which was introduced in [K2]. First we recall the complexification of a complete C^{ω}-Riemannian manifold. Let M be a complete C^{ω}-Riemannian manifold. The notion of the adapted complex structure on a neighborhood U of the 0-section of the tangent bundle TM is defined as the complex structure (on U) such that, for each geodesic $\gamma : \mathbf{R} \to M$, the restriction of its differential $\gamma_* : T\mathbf{R} = \mathbf{C} \to TM$ to $\gamma_*^{-1}(U)$ is holomorphic. We take U as large as possible under the condition that $U \cap T_x M$ is a
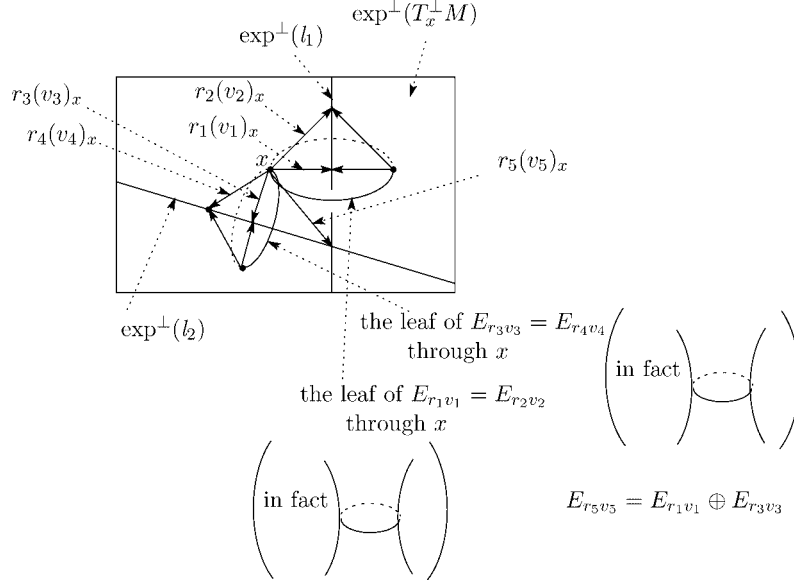
FIGURE 4.

star-shaped neighborhood of 0_x for each $x \in M$, where 0_x is the zero vector of $T_x M$. If M is of non-negative curvature, then we have $U = TM$. Also, if all sectional curvatures of M are larger than or equal to c ($c < 0$), then U contains the ball bundle $T^r M := \{X \in TM \mid \|X\| < r\}$ of radius $r := \frac{\pi}{2\sqrt{-c}}$. In detail, see [Sz1~4]. Denote by J_A the adapted complex structure on U. The complex manifold (U, J_A) is interpreted as the complexification of N. We denote (U, J_A) by $M^{\mathbf{c}}$ and call it the complexification of M, where we note that $M^{\mathbf{c}}$ is given no Riemannian metric. In particular, in the case of $M = \mathbf{R}^m$ (the Euclidean space), we have $(U, J_A) = \mathbf{C}^m$. Also, in the case where M is a symmetric space G/K of non-compact type, there exists the holomorphic diffeomorphism δ of (U, J_A) onto an open subset of $G^{\mathbf{c}}/K^{\mathbf{c}}$. Let M be an immersed (complete) C^ω-submanifold in G/K. Denote by f its immersion. Let $M^{\mathbf{c}}$ be the complexification of M (defined as above). We shall define the complexification $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of f, where we shrink $M^{\mathbf{c}}$ to a neighborhood of the 0-section of TM if necessary. For its purpose, we first define the complexification of a C^ω-curve $\alpha : \mathbf{R} \to G/K$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with G/K and $W : \mathbf{R} \to \mathfrak{p}$ be the curve in $\mathfrak{p}$ with $(\exp W(t))K = \alpha(t)$ ($t \in \mathbf{R}$), where we note that W is uniquely determined because G/K is of non-compact type. Since α is of class C^ω, so is also W. Let $W^{\mathbf{c}} : D \to \mathfrak{p}^{\mathbf{c}}$ (D : a neighborhood of $\mathbf{R}$ in $\mathbf{C}$) be the holomorphic extension of W. We define the complexification $\alpha^{\mathbf{c}} : D \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of α by $\alpha^{\mathbf{c}}(z) = (\exp W^{\mathbf{c}}(z))K^{\mathbf{c}}$. It is shown that this complexification of a C^ω-curve in G/K is a holomorphic curve in $G^{\mathbf{c}}/K^{\mathbf{c}}$. By using this complexification of a C^ω-curve in G/K, we define the complexification $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of f by $f^{\mathbf{c}}(X) :=$

$(f \circ \gamma_X^M)^{\mathbf{c}}(\sqrt{-1})$ $(X \in M^{\mathbf{c}} (\subset TM))$, where γ_X^M is the geodesic in M with $\dot{\gamma}_X^M(0) = X$. Here we shrink $M^{\mathbf{c}}$ to a neighborhood of the 0-section of TM if necessary in order to assure that $\sqrt{-1}$ belongs to the domain of $(f \circ \gamma_X^M)^{\mathbf{c}}$ for each $X \in M^{\mathbf{c}}$. It is shown that the map $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ is holomorphic and that the restriction of $f^{\mathbf{c}}$ to a neighborhood U' of the 0-section of TM is an immersion, where we take U' as large as possible. Denote by $M^{\mathbf{c}}$ this neighborhood U' from now on. Give $M^{\mathbf{c}}$ the Riemannian metric induced from that of $G^{\mathbf{c}}/K^{\mathbf{c}}$ by $f^{\mathbf{c}}$. Then $M^{\mathbf{c}}$ is an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$ immersed by $f^{\mathbf{c}}$. We call this anti-Kaehlerian submanifold $M^{\mathbf{c}}$ immersed by $f^{\mathbf{c}}$ the *extrinsic complexification* of the submanifold M. We consider the case where M is (extrinsically) homogeneous. Concretely we consider the case where $M = H(g_0 K)$ and f is the inclusion map of M into G/K, where H is a closed subgroup of G. Let ι be a natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$, that is, $\iota(gK) = gK^{\mathbf{c}}$ $(X \in \mathfrak{g})$. It is shown that ι is totally geodesic. Let $\mathfrak{g}_H^{\mathbf{c}}$ be the complexification of the Lie algebra of H and set $H^{\mathbf{c}} := \exp \mathfrak{g}_H^{\mathbf{c}}$. For a homogeneous submanifold $M = H(g_0 K)$, the image $f^{\mathbf{c}}(M^{\mathbf{c}})$ is an open subset of the orbit $H^{\mathbf{c}}(g_0 K^{\mathbf{c}})$. Hence this orbit is the complete extension of $M^{\mathbf{c}}$. It is shown that M is complex equifocal if and only if $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal (see Theorem 5 of [K2]). Also, it is shown that M is proper complex equifocal if and only if $M^{\mathbf{c}}$ is proper anti-Kaehlerian equifocal.

3. aks-representations

In this section, we shall first introduce the notions of an anti-Kaehlerian symmetric pair and an anti-Kaehlerian symmetric Lie algebra, and investigate the correspondence relations of those notions with an anti-Kaehlerian symmetric space. Let $(M, J, \langle \ , \ \rangle)$ be an anti-Kaehlerian manifold (i.e., $J^2 = -\mathrm{id}, \nabla J = 0$ (∇ : the Levi-Civita connection of $\langle \ , \ \rangle$)) and $\langle JX, JY \rangle = -\langle X, Y \rangle$ $(X, Y \in TM))$. We denote by the same symbol id the identity transformations of various sets. If there exists an involutive holomorphic isometry s_p of M having p as an isolated fixed point for each $p \in M$, then we call $(M, J, \langle \ , \ \rangle)$ an *anti-Kaehlerian symmetric space*. Also, if there exists a local involutive holomorphic isometry defined on a neighborhood of p having p as an isolated fixed point for each $p \in M$, then we call $(M, J, \langle \ , \ \rangle)$ a *locally anti-Kaehlerian symmetric space*. In this section, we introduce the notions of an anti-Kaehlerian symmetric pair and an anti-Kaehlerian symmetric Lie algebra in relation with an anti-Kaehlerian symmetric spaces. Let G be a connected complex Lie group and K be a closed complex subgroup of G. If there exists an involutive (complex) automorphism ρ of G such that $G_\rho^0 \subset K \subset G_\rho$ (G_ρ : the group of all fixed points of ρ, G_ρ^0 : the identity component of G_ρ) then we call the pair (G, K) an *anti-Kaehlerian symmetric pair*. If $\mathfrak{g}$ be a complex Lie algebra and τ be a complex involution of $\mathfrak{g}$, then we call such a pair $(\mathfrak{g}, \tau)$ an *anti-Kaehlerian symmetric Lie algebra*. Let $\mathfrak{f} := \mathrm{Ker}(\tau - \mathrm{id})$ and $\mathfrak{p} := \mathrm{Ker}(\tau + \mathrm{id})$. Denote by Ad_G and $\mathrm{ad}_{\mathfrak{g}}$ the adjoint representations of G and $\mathfrak{g}$, respectively. Also, denote by j the complex structure of $\mathfrak{g}$. Let $\mathfrak{p}_{\mathbf{R}}$ be the totally real subspace of $\mathfrak{p}$ such that $\langle \ , \ \rangle|_{\mathfrak{p}_{\mathbf{R}} \times j \mathfrak{p}_{\mathbf{R}}} = 0$ and that $\langle \ , \ \rangle|_{\mathfrak{p}_{\mathbf{R}} \times \mathfrak{p}_{\mathbf{R}}}$ is positive definite. Here we note that such a totally real subspace is determined uniquely. Set $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}}(\mathfrak{f}) := \{\mathrm{ad}_{\mathfrak{g}}(X)|_{\mathfrak{p}} \mid X \in \mathfrak{f}\}$, $\mathrm{Ad}_G|_{\mathfrak{p}}(K) := \{\mathrm{Ad}_G(k)|_{\mathfrak{p}} \mid k \in K\}$,

$\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}) := \{\mathrm{pr}_{\mathfrak{p}_{\mathbf{R}}} \circ \mathrm{ad}_{\mathfrak{g}}(X)|_{\mathfrak{p}_{\mathbf{R}}} \mid X \in \mathfrak{f}\}$ and $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K) := \exp_{\mathrm{GL}(\mathfrak{p}_{\mathbf{R}})}(\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))$, where $\exp_{\mathrm{GL}(\mathfrak{p}_{\mathbf{R}})}$ is the exponential map of $\mathrm{GL}(\mathfrak{p}_{\mathbf{R}})$. Let $SO_{AK}(\mathfrak{p})$ be the identity component of the group $\{A \in \mathrm{GL}(\mathfrak{p}) \mid A^* \langle\ ,\ \rangle = \langle\ ,\ \rangle,\ A \circ j = j \circ A\}$ and set $\mathfrak{so}_{AK}(\mathfrak{p}) := \{A \in \mathfrak{gl}(\mathfrak{p}) \mid A \circ j = j \circ A,\ \langle AX, Y \rangle = -\langle X, AY \rangle\ (\forall\, X, Y \in \mathfrak{p})\}$, which is the Lie algebra of $SO_{AK}(\mathfrak{p})$. Then we have the following fact.

LEMMA 3.1. *The complexification* $\mathfrak{so}(\mathfrak{p}_{\mathbf{R}})^{\mathfrak{c}}$ *of* $\mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$ *coincides with* $\mathfrak{so}_{AK}(\mathfrak{p})$ *and hence* $SO(\mathfrak{p}_{\mathbf{R}})$ *is a half-dimensional totally real compact subgroup of* $SO_{AK}(\mathfrak{p})$. *Also, the complexification* $(\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))^{\mathfrak{c}}$ *of* $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$ *coincides with* $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}}(\mathfrak{f})$ *and* $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$ *is contained in* $\mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$. *Hence* $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$ *is a half-dimensional totally real compact subgroup of* $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$ *contained in* $SO(\mathfrak{p}_{\mathbf{R}})$.

PROOF.   For $A \in \mathfrak{gl}(\mathfrak{p}_{\mathbf{R}})$, denote by $\widetilde{A}$ the element of $\mathfrak{gl}(\mathfrak{p}, j) := \{B \in \mathfrak{gl}(\mathfrak{p}) \mid B \circ j = j \circ B\}$ whose restriction to $\mathfrak{p}_{\mathbf{R}}$ is equal to A. Let $C \in \mathfrak{so}_{AK}(\mathfrak{p})$. Set $A := \mathrm{pr}_{\mathfrak{p}_{\mathbf{R}}} \circ C|_{\mathfrak{p}_{\mathbf{R}}}$ and $B := -j \circ \mathrm{pr}_{j\mathfrak{p}_{\mathbf{R}}} \circ C|_{\mathfrak{p}_{\mathbf{R}}}$. Then we have $C = \widetilde{A} + j\widetilde{B}$. Take $X, Y \in \mathfrak{p}_{\mathbf{R}}$. Then it follows from $\langle \mathfrak{p}_{\mathbf{R}}, j\mathfrak{p}_{\mathbf{R}} \rangle = 0$ that $\langle CX, jY \rangle = -\langle BX, Y \rangle$ and $\langle X, C(jY) \rangle = -\langle BY, X \rangle$. Hence it follows from $\langle CX, jY \rangle = -\langle X, C(jY) \rangle$ that $\langle BX, Y \rangle = -\langle X, BY \rangle$. Thus we have $B \in \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$. Also we have $\langle CX, Y \rangle = \langle AX, Y \rangle$ and $\langle X, CY \rangle = -\langle X, AY \rangle$. Hence we have $\langle AX, Y \rangle = -\langle X, AY \rangle$. Thus we have $A \in \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$. Therefore we have $C \in \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})^{\mathfrak{c}}$. Thus we have $\mathfrak{so}_{AK}(\mathfrak{p}) \subset \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})^{\mathfrak{c}}$. Since $\mathfrak{so}_{AK}(\mathfrak{p})$ and $\mathfrak{so}(\mathfrak{p}_{\mathbf{R}})^{\mathfrak{c}}$ are of the same dimension, we have $\mathfrak{so}_{AK}(\mathfrak{p}) = \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})^{\mathfrak{c}}$. Therefore the first-half statement of this lemma is shown. Let $C \in \mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f})$. Set $A := \mathrm{pr}_{\mathfrak{p}_{\mathbf{R}}} \circ C|_{\mathfrak{p}_{\mathbf{R}}}$ and $B := -j \circ \mathrm{pr}_{j\mathfrak{p}_{\mathbf{R}}} \circ C|_{\mathfrak{p}_{\mathbf{R}}}$. From the definition of $\mathrm{ad}_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$, we have $A \in \mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$. Also, it follows from $-j \circ C \in \mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f})$ that $-(\mathrm{pr}_{\mathfrak{p}_{\mathbf{R}}} \circ j \circ C)|_{\mathfrak{p}_{\mathbf{R}}} \in \mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$. Clearly we have $-(\mathrm{pr}_{\mathfrak{p}_{\mathbf{R}}} \circ j \circ C)|_{\mathfrak{p}_{\mathbf{R}}} = B$. Thus we have $B \in \mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$. Therefore we have $C(= \widetilde{A} + j \circ \widetilde{B}) \in (\mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))^{\mathfrak{c}}$. Thus $\mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f}) \subset (\mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))^{\mathfrak{c}}$ is obtained. From $\dim_{\mathbf{R}} \mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f}) = \dim_{\mathbf{R}}(\mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))^{\mathfrak{c}}$, it follows that $\mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f}) = (\mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))^{\mathfrak{c}}$. Also, since $C \in \mathrm{ad}|_{\mathfrak{p}}(\mathfrak{f}) \subset \mathfrak{so}_{AK}(\mathfrak{p})$, we can show $A \in \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$ as above. Therefore we obtain $\mathrm{ad}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}) \subset \mathfrak{so}(\mathfrak{p}_{\mathbf{R}})$. Hence $\mathrm{Ad}|_{\mathfrak{p}_{\mathbf{R}}}(K) \subset SO(\mathfrak{p}_{\mathbf{R}})$ is obtained. Furthermore, since $\mathrm{Ad}|_{\mathfrak{p}_{\mathbf{R}}}(K)$ is closed in $SO(\mathfrak{p}_{\mathbf{R}})$, it is compact. Thus the second-half statement of this lemma follows.   q.e.d.

Now we show that an anti-Kaehlerian symmetric pair arises from an anti-Kaehlerian symmetric space.

PROPOSITION 3.2. *Let* $(M, J, \langle\ ,\ \rangle)$ *be an anti-Kaehlerian symmetric space,* G *be the identity component of the isometry group of* $(M, J, \langle\ ,\ \rangle)$ *and* K *be the isotropy group of* G *at some* $p_0 \in M$. *Then the pair* (G, K) *is an anti-Kaehlerian symmetric pair.*

PROOF.   Identify M with G/K under the correspondence $g(p_0) \leftrightarrow gK$ $(g \in G)$. Define a map $\rho : G \to G$ by $\rho(g) = s_{p_0} \circ g \circ s_{p_0}$ $(g \in G)$, which is an involutive automorphism of G. Easily we can show that $G_{\rho}^0 \subset K \subset G_{\rho}$ (see the proof of (ii) of Theorem 3.3 of Chapter IV in [H]). Let $\mathfrak{f} := \mathrm{Ker}(\rho_{*e} - \mathrm{id})$ and $\mathfrak{p} := \mathrm{Ker}(\rho_{*e} + \mathrm{id})$, where e is the identity element of G. The space $\mathfrak{p}$ is identified with $T_{p_0}M$. Define the $\sqrt{-1}$-multiple in $\mathfrak{g}$ by $\sqrt{-1}X = J_{p_0}X$

($X \in \mathfrak{p} = T_{p_0}M$) and $[\sqrt{-1}Y, Z] = [Y, J_{p_0}Z]$ ($Y \in \mathfrak{f}$, $Z \in \mathfrak{p}$), where $[\ ,\ ]$ is the Lie bracket product of $\mathfrak{g}$. Note that this $\sqrt{-1}$-multiple in $\mathfrak{g}$ is well-defined because $\mathfrak{f}$ acts on $\mathfrak{p}$ effectively. Since $\mathrm{ad}(X) \circ J_{p_0} = J_{p_0} \circ \mathrm{ad}(X)$ on $\mathfrak{p}$ ($X \in \mathfrak{f}$), $[Y, Z] = -R_{p_0}(Y, Z)$ ($Y, Z \in \mathfrak{p}$) and $R_{p_0}(J_{p_0}Y, Z) = J_{p_0}R_{p_0}(Y, Z)$ ($Y, Z \in \mathfrak{p}$) by anti-Kaehlerity of M, we see that $(\mathfrak{g}, [\ ,\ ])$ is a complex Lie algebra under this $\sqrt{-1}$-multiple. Also, it is easy to show that $\mathfrak{f}$ is a complex Lie subalgebra and ρ_{*e} is the complex involution. Hence G, K and ρ are regarded as a complex Lie group, a complex Lie subgroup of G and an involutive complex automorphism of G, respectively.   q.e.d.

By using Lemma 3.1, we show that an anti-Kaehlerian symmetric space arises from an anti-Kaehlerian symmetric pair.

PROPOSITION 3.3. *Let (G, K) be an anti-Kaehlerian symmetric pair. Then there exists an anti-Kaehlerian structure $(J, \langle\ ,\ \rangle)$ of G/K such that $(G/K, J, \langle\ ,\ \rangle)$ is an anti-Kaehlerian symmetric space.*

PROOF. Since (G, K) is an anti-Kaehlerian symmetric pair, there exists an involutive (complex) automorphism ρ of G with $G_\rho^0 \subset K \subset G_\rho$. Let $\mathfrak{g} := \mathrm{Lie}\,G$, $\mathfrak{f} := \mathrm{Lie}\,K$ and $\mathfrak{p} := \mathrm{Ker}(\rho_{*e} + \mathrm{id})$. Then we can show $\mathrm{Ad}_G(K)(\mathfrak{p}) \subset \mathfrak{p}$ (see the first part of the proof of Proposition 3.4 of Chapter IV in [H]). Define an almost complex structure j of $\mathfrak{p}$ by $j(X) = \sqrt{-1}X$ ($X \in \mathfrak{p}$). It is clear that j is $\mathrm{Ad}_G(K)$-invariant. Denote by J the G-invariant almost complex structure on G/K arising from j. Let $\mathrm{GL}((\mathfrak{p}, j)) := \{A \in \mathrm{GL}(\mathfrak{p}) \mid A \circ j = j \circ A\}$, where $\mathrm{GL}(\mathfrak{p})$ is the group of all (real) linear isomorphisms of $\mathfrak{p}$. Take a half-dimensional subspace $\mathfrak{p}_{\mathbf{R}}$ of $\mathfrak{p}$ with $\mathfrak{p}_{\mathbf{R}} \oplus j\mathfrak{p}_{\mathbf{R}} = \mathfrak{p}$. The group $\mathrm{GL}(\mathfrak{p}_{\mathbf{R}})$ of all linear isomorphisms of $\mathfrak{p}_{\mathbf{R}}$ is regarded as a half-dimensional totally real subgroup of $\mathrm{GL}((\mathfrak{p}, j))$ by identifying each $A \in \mathrm{GL}(\mathfrak{p}_{\mathbf{R}})$ with $\widetilde{A} \in \mathrm{GL}((\mathfrak{p}, j))$ defined by $\widetilde{A}(X + jY) = AX + jAY$ ($X, Y \in \mathfrak{p}_{\mathbf{R}}$). Let $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f})$ be as in the proof of Proposition 3.2 and $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K) := \exp_{\mathrm{GL}(\mathfrak{p}_{\mathbf{R}})}(\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{p}_{\mathbf{R}}}(\mathfrak{f}))$. It is clear that the group $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$ is regarded as a half-dimensional totally real subgroup of $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$. By taking an anti-Kaehlerian inner product β of $(\mathfrak{p}, j)$ such that $\beta|_{\mathfrak{p}_{\mathbf{R}} \times j\mathfrak{p}_{\mathbf{R}}} = 0$ and that $\beta|_{\mathfrak{p}_{\mathbf{R}} \times \mathfrak{p}_{\mathbf{R}}}$ is positive definite and using Lemma 3.1, $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$ is a half-dimensional totally real compact subgroup of $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$. Define a real bilinear form β_0 on $\mathfrak{p}$ by

$$\beta_0(X, Y) = \int_{a \in \mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)} \beta(aX, aY)\omega \quad (X, Y \in \mathfrak{p}),$$

where ω is the Haar measure of $\mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$ and each $a \in \mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$ is extended to the linear transformation of $\mathfrak{p}$ in the natural manner. We shall show that β_0 is an anti-Kaehlerian inner product of $(\mathfrak{p}, j)$. Let $X \in \mathfrak{p}_{\mathbf{R}}$. Since $\beta(aX, aX) \geq 0$ for any $a \in \mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$, we have $\beta_0(X, X) \geq 0$. If $\beta_0(X, X) = 0$, then we have $\beta(aX, aX) = 0$ for any $a \in \mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$. In particular, we have $\beta(X, X) = 0$, that is, $X = 0$. Thus $\beta_0|_{\mathfrak{p}_{\mathbf{R}} \times \mathfrak{p}_{\mathbf{R}}}$ is positive definite. Let Y be another vector of $\mathfrak{p}_{\mathbf{R}}$. Since $\beta(aX, ajY) = 0$ ($a \in \mathrm{Ad}_G|_{\mathfrak{p}_{\mathbf{R}}}(K)$), we have $\beta_0(X, jY) = 0$. Thus it follows from the arbitrariness of X and Y that $\beta_0|_{\mathfrak{p}_{\mathbf{R}} \times j\mathfrak{p}_{\mathbf{R}}} = 0$. On the other hand, it is clear that $\beta_0(jZ, jW) = -\beta_0(Z, W)$ ($Z, W \in \mathfrak{p}$). These facts imply that β_0 is an anti-

Kaehlerian inner product of $(\mathfrak{p}, j)$. Next we shall show that β_0 is $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$-invariant. It is clear that β_0 is $\mathrm{Ad}_G|_{\mathfrak{p_R}}(K)$-invariant. Fix $X, Y \in \mathfrak{p}$. Define a complex-valued function f on $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$ by $f(a) = \beta_0(aX, aY) - \sqrt{-1}\beta_0(aX, ajY)$ $(a \in \mathrm{Ad}_G|_{\mathfrak{p}}(K))$. Since $f \equiv \beta_0(X, Y) - \sqrt{-1}\beta_0(X, jY)$ on $\mathrm{Ad}_G|_{\mathfrak{p_R}}(K)$, f is holomorphic and $\mathrm{Ad}_G|_{\mathfrak{p_R}}(K)$ is a half-dimensional totally real subgroup of $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$, we see that $f \equiv \beta_0(X, Y) - \sqrt{-1}\beta_0(X, jY)$ on $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$, which implies that β_0 is $\mathrm{Ad}_G|_{\mathfrak{p}}(K)$-invariant. Denote by $\langle\ ,\ \rangle$ the G-invariant pseudo-Riemannian metric on G/K arising from β. It is clear that $(G/K, J, \langle\ ,\ \rangle)$ is an anti-Kaehlerian manifold. Next we shall show that $(G/K, J, \langle\ \ \rangle)$ is an anti-Kaehlerian symmetric space. Let $\pi : G \to G/K$ be the natural projection. Define a map $s_o : G/K \to G/K$ by $s_o(\pi(g)) = \pi(\rho(g))$ $(g \in G)$. It is clear that s_o is well-defined and that $s_o^2 = \mathrm{id}$. Also, it is shown that s_o is an isometry of $(G/K, \langle\ ,\ \rangle)$ (see the proof of Proposition 3.4 of Chapter IV in [H]). Furthermore, it is shown that s_o is holomorphic. Also, we have $s_{o*\pi(e)} \circ \pi_{*e} = \pi_{*e} \circ \rho_{*e} = -\pi_{*e}$ on $\mathfrak{p}$, that is, $s_{o*\pi(e)} = -\mathrm{id}$, which implies that $\pi(e)$ is an isolated fixed point of s_o. For each $g \in G$, define a map $s_{\pi(g)} : G/K \to G/K$ by $s_{\pi(g)} = g \circ s_o \circ g^{-1}$. Easily we can show that $s_{\pi(g)}$ is an involutive holomorphic isometry of $(G/K, J, \langle\ ,\ \rangle)$ having $\pi(g)$ as an isolated fixed point. Thus $(G/K, J, \langle\ ,\ \rangle)$ is an anti-Kaehlerian symmetric space. q.e.d.

Let $(\mathfrak{g}, \tau)$ be an anti-Kaehlerian symmetric Lie algebra and $\mathfrak{f} := \mathrm{Ker}(\tau - \mathrm{id})$. Let G be a connected complex Lie group with $\mathrm{Lie}\,G = \mathfrak{g}$ and K be a complex Lie subgroup of G with $\mathrm{Lie}\,K = \mathfrak{f}$. We call such a pair (G, K) a *pair associated with* $(\mathfrak{g}, \tau)$.

PROPOSITION 3.4.   *Let $(\mathfrak{g}, \tau)$ be an anti-Kaehlerian symmetric Lie algebra, (G, K) be a pair associated with $(\mathfrak{g}, \tau)$ such that K is connected and $(\widetilde{G}, \widetilde{K})$ be a pair associated with $(\mathfrak{g}, \tau)$ such that $\widetilde{G}$ is simply connected and that $\widetilde{K}$ is connected. Then the following statements* (i) *and* (ii) *hold*:

(i)   *$(\widetilde{G}, \widetilde{K})$ is an anti-Kaehlerian symmetric pair.*

(ii)   *Assume that K is closed. Let $(J, \langle\ ,\ \rangle)$ be a G-invariant anti-Kaehlerian structure on G/K defined as in the proof of Proposition* 3.3. *Then $(G/K, J, \langle\ ,\ \rangle)$ is a locally anti-Kaehlerian symmetric space and the universal anti-Kaehlerian covering of $(G/K, J, \langle\ ,\ \rangle)$ is isometric to an anti-Kaehlerian symmetric space $\widetilde{G}/\widetilde{K}$ equipped with a suitable anti-Kaehlerian structure defined as in the proof of Proposition* 3.3.

PROOF.   First we shall show the statement (i). Since $\widetilde{G}$ is simply connected, there uniquely exists an involutive (complex) automorphism ρ of $\widetilde{G}$ with $\rho_{*e} = \tau$. By a standard method, we can show that $\widetilde{K}$ is equal to the identity component $\widetilde{G}_\rho^0$ of the group of all fixed points of ρ because $\widetilde{K}$ is connected. Thus $(\widetilde{G}, \widetilde{K})$ is an anti-Kaehlerian symmetric pair.

Next we shall show the statement (ii). The groups $\mathrm{Ad}_G(K)$ and $\mathrm{Ad}_{\widetilde{G}}(\widetilde{K})$ coincide with each other because they are connected complex Lie subgroups of the adjoint group $\mathrm{int}\,\mathfrak{g}$ and have the same Lie algebra. Let $(J, \langle\ ,\ \rangle)$ (resp. $(\widetilde{J}, \langle\ ,\ \rangle^{\sim})$) be a G (resp. $\widetilde{G}$)-invariant anti-Kaehlerian structure on G/K (resp. $\widetilde{G}/\widetilde{K}$) as in the proof of Proposition 3.3. Let ψ be the

homomorphism of $\widetilde{G}$ onto G with $\psi_{*e} = \mathrm{id}$. It is clear that $\widetilde{K}$ is the identity component of $\psi^{-1}(K)$. Hence a map $\bar{\psi} : \widetilde{G}/\widetilde{K} \to G/K$ is well-defined by $\bar{\psi}(\widetilde{g}\widetilde{K}) = \psi(\widetilde{g})K$ $(\widetilde{g} \in \widetilde{G})$. It is shown that this map $\bar{\psi}$ is a covering map (see Lemma 13.4 of Chapter I in [H]). It is easy to show that $\bar{\psi}$ is an anti-Kaehlerian covering map of $(\widetilde{G}/\widetilde{K}, \widetilde{J}, \langle\ ,\ \rangle^{\sim})$ onto $(G/K, J, \langle\ ,\ \rangle)$. Hence $(G/K, J, \langle\ ,\ \rangle)$ is a locally anti-Kaehlerian symmetric space. Since $\widetilde{G}/\widetilde{K}$ is simply connected (see the proof of Proposition 3.6 of Chapter IV in [H]), $(\widetilde{G}/\widetilde{K}, \widetilde{J}, \langle\ ,\ \rangle^{\sim})$ is the universal anti-Kaehlerian covering of $(G/K, J, \langle\ ,\ \rangle)$.   q.e.d.

Let $(M, J, \langle\ ,\ \rangle)$ be an irreducible anti-Kaehlerian symmetric space, G be the identity component of the isometry group of $(M, J, \langle\ ,\ \rangle)$ and K be the isotropy group of G at some point $p_0 \in M$, where the irreducibility implies that M is not decomposed into the non-trivial product of two anti-Kaehlerian symmetric spaces. Assume that $(M, J, \langle\ ,\ \rangle)$ does not have a pseudo-Euclidean part in its de Rham decomposition. Note that an anti-Kaehlerian symmetric space without pseudo-Euclidean part is not necessarily semi-simple (see [CP],[W1]). Also, let $(\mathfrak{g}, \tau)$ be the anti-Kaehlerian symmetric Lie algebra associated with the anti-Kaehlerian symmetric pair (G, K) and $\mathfrak{p} := \mathrm{Ker}(\tau + \mathrm{id})$. The space $\mathrm{Ker}(\tau - \mathrm{id})$ is equal to the Lie algebra $\mathfrak{k}$ of K and $\mathfrak{p}$ is identified with $T_{p_0}M (= T_{eK}(G/K))$. We call the linear isotropy representation $\mathrm{Ad}_G|_{\mathfrak{p}} : K \to \mathrm{GL}(\mathfrak{p})$ an *aks-representation*, where $\mathfrak{p}$ is regarded as an anti-Kaehlerian space under the identification $\mathfrak{p} = T_{p_0}M$. Let $\mathfrak{a}_s$ be a maximal split abelian subspace of $\mathfrak{p}$ (see [R] or [OS] for the definition of a maximal split abelian subspace) and $\mathfrak{p} = \mathfrak{p}_0 + \sum\limits_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}_s$ (i.e., the simultaneous eigenspace decomposition of $\mathrm{ad}(a)^2$'s $(a \in \mathfrak{a}_s)$), where the space $\mathfrak{p}_\alpha$ is defined by $\mathfrak{p}_\alpha := \{X \in \mathfrak{p}\,|\,\mathrm{ad}(a)^2(X) = \alpha(a)^2 X$ for all $a \in \mathfrak{a}_s\}$ and $\triangle_+$ is the positive root system with respect to $\mathfrak{a}_s$ under some lexicographic ordering of $\mathfrak{a}_s^*$. Set $\mathfrak{a} := \mathfrak{p}_0 (\supset \mathfrak{a}_s)$, $j := J_{p_0}$ and $\langle\ ,\ \rangle_0 := \langle\ ,\ \rangle_{p_0}$. It is shown that $\langle\ ,\ \rangle_0|_{\mathfrak{a}_s \times \mathfrak{a}_s}$ is positive (or negative) definite, $\mathfrak{a} = \mathfrak{a}_s \oplus j\mathfrak{a}_s$ and $\langle\ ,\ \rangle_0|_{\mathfrak{a}_s \times j\mathfrak{a}_s} = 0$. Note that $\mathfrak{p}_\alpha = \{X \in \mathfrak{p}\,|\,\mathrm{ad}(a)^2(X) = \alpha^{\mathbf{c}}(a)^2 X$ for all $a \in \mathfrak{a}\}$ for each $\alpha \in \triangle_+$, where $\alpha^{\mathbf{c}}$ is the complexification of $\alpha : \mathfrak{a}_s \to \mathbf{R}$, $\mathfrak{a}$ is regarded as the complexification $\mathfrak{a}_s^{\mathbf{c}}$ of $\mathfrak{a}_s$ and $\alpha^{\mathbf{c}}(a)^2 X$ means $\mathrm{Re}(\alpha^{\mathbf{c}}(a)^2)X + \mathrm{Im}(\alpha^{\mathbf{c}}(a)^2)jX$. Let $l_\alpha := (\alpha^{\mathbf{c}})^{-1}(0)$ $(\alpha \in \triangle)$ and $D := \mathfrak{a}\setminus\bigcup\limits_{\alpha \in \triangle_+} l_\alpha$. Take $u \in D$ and let M be the orbit through u of the K-action by the linear isotropy representation $(\mathrm{Ad}_G|_{\mathfrak{p}})|_K$. Since $u \in D$, M is a principal orbit. Denote by A the shape tensor of M. Take $v \in T_u^\perp M (= \mathfrak{a})$. Then we have $T_uM = \sum\limits_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ and $A_v|_{\mathfrak{p}_\alpha} = -\frac{\alpha^{\mathbf{c}}(v)}{\alpha^{\mathbf{c}}(u)}\mathrm{id}$ $(\alpha \in \triangle_+)$. It is easy to show that the K-action by $(\mathrm{Ad}_G|_{\mathfrak{p}})|_K$ is an anti-Kaehlerian polar action having $\mathfrak{a}$ as a section, where an anti-Kaehlerian polar action means the finite dimensional version of an anti-Kaehlerian polar action on an infinite dimensional anti-Kaehlerian space defined in [K2]. Furthermore, from $A_v|_{\mathfrak{p}_\alpha} = -\frac{\alpha^{\mathbf{c}}(v)}{\alpha^{\mathbf{c}}(u)}\mathrm{id}_{\mathfrak{p}_\alpha}$ and the arbitrariness of v and u, we see that each principal orbit of the K-action is proper anti-Kaehlerian isoparametric in the sense of [K4].

In the 2-dimensional anti-Kaehlerian space $V = (\mathbf{R}^2, J_0, \langle\,,\,\rangle_0)$, there uniquely exists a 1-dimensional totally real subspace W of V such that $\langle W, J_0 W\rangle_0 = 0$ and that $\langle\,,\,\rangle_0|_{W \times W}$ is positive definite. Let $w \in W \cup J_0 W$. The quotient manifold $V/\mathbf{Z}w$ is a flat anti-Kaehlerian manifold whose universal anti-Kaehlerian covering is V. We call such an anti-Kaehlerian manifold an *anti-Kaehlerian cylinder*. Let $(G/K, J, \langle\,,\,\rangle)$ be a semi-simple anti-Kaehlerian symmetric space and $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} = T_{eK}(G/K)$. It is easy to show that $\exp \mathfrak{a}$ is a flat totally geodesic submanifold in G/K and that it is holomorphic and isometric to the product of some anti-Kaehlerian cylinders. We call $\exp \mathfrak{a}$ a *maximal anti-Kaehlerian cylindrical product*. Here we note that, if $(M, J, \langle\,,\,\rangle)$ is not semi-simple, then $\exp \mathfrak{a}$ is holomorphic and isometric to the product of some anti-Kaehlerian cylinders and an anti-Kaehlerian space.

At the end of this section, we shall recall the notion of the anti-Kaehlerian symmetric space associated with a Riemannian symmetric space of non-compact type which was introduced in [K2]. Let G/K be a Riemannian symmetric space of non-compact type and ρ be the Cartan involution, where G is assumed to be a connected semi-simple Lie group admitting a faithful real representation and K can be assumed to be a maximal compact subgroup of G. Let $\mathfrak{g} := \operatorname{Lie} G$, $\mathfrak{k} := \operatorname{Lie} K$ and $\mathfrak{p} := \operatorname{Ker}(\rho_{*e} + \operatorname{id})$, where $\mathfrak{p}$ is identified with $T_{eK}(G/K)$. Also, let $\mathfrak{g}^{\mathbf{c}}$ (resp. $\rho_{*e}^{\mathbf{c}}$) be the complexification of $\mathfrak{g}$ (resp. ρ_{*e}). Since G admits a faithful real representation, we can define the complexification $G^{\mathbf{c}}$ (resp. $K^{\mathbf{c}}$) of G (resp. K) and the compact dual $G^*(\subset G^{\mathbf{c}})$ of G. It is shown that $(G^{\mathbf{c}}, K^{\mathbf{c}})$ is an anti-Kaehlerian symmetric pair. Let β be the $\operatorname{Ad}_G(K)$-invariant (positive definite) inner product of $\mathfrak{p}$ arising the Riemannian metric of G/K. Let $\langle\,,\,\rangle$ be the pseudo-Riemannian metric of $G^{\mathbf{c}}/K^{\mathbf{c}}$ arising from $\operatorname{Re} \beta^{\mathbf{c}}$ ($\mathfrak{p}^{\mathbf{c}} \times \mathfrak{p}^{\mathbf{c}} \to \mathbf{R}$) and J be the natural almost complex structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$, where $\mathfrak{p}^{\mathbf{c}}$ is identified with $T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$. Then $(G^{\mathbf{c}}/K^{\mathbf{c}}, J, \langle\,,\,\rangle)$ is an anti-Kaehlerian symmetric space. We call this anti-Kaehlerian symmetric space the *anti-Kaehlerian symmetric space associated with G/K*, where we note that $G^{\mathbf{c}}/K^{\mathbf{c}}$ is a semi-simple anti-Kaehlerian symmetric space.

REMARK 3.1. If β is the Killing-Cartan form of $\mathfrak{g}$, $2\operatorname{Re} \beta^{\mathbf{c}}$ is that of $\mathfrak{g}^{\mathbf{c}}$ regarded as a real Lie algebra.

4. Anti-Kaehlerian holonomy systems

Let (V, R, G) be a triple consisting of a Euclidean space V, a curvature-like tensor R ($\in V^* \otimes V^* \otimes V^* \otimes V$) and a compact connected Lie subgroup G of the linear isometry group $O(V)$ of V. J. Simons [Si] called (V, R, G) a *holonomy system* if $R(v_1, v_2) \in \operatorname{Lie} G$ for all $v_1, v_2 \in V$. In this section, we introduce the notion of an anti-Kaehlerian holonomy system and show some facts concerning such a system. Let $(V, J, \langle\,,\,\rangle)$ be a (finite dimensional) anti-Kaehlerian space and R ($\in V^* \otimes V^* \otimes V^* \otimes V$) be a curvature-like tensor. Let $SO_{AK}(V)$ be the identity component of the group $\{A \in GL(V) \mid A^*\langle\,,\,\rangle = \langle\,,\,\rangle, [A, J] = 0\}$ and G be a connected complex Lie subgroup of $SO_{AK}(V)$. We call the triple $((V, J, \langle\,,\,\rangle), R, G)$ an *anti-Kaehlerian holonomy system* if the following two conditions hold:

(AH-i) $J \circ R(v_1, v_2) = R(Jv_1, v_2) = R(v_1, v_2) \circ J$ for all $v_1, v_2 \in V$,

(AH-ii) $R(v_1, v_2) \in \text{Lie } G$ for all $v_1, v_2 \in V$.

Furthermore, if the following condition (S) holds, then we say that the triple is *symmetric*:

(S) $R(gv_1, gv_2)gv_3 = gR(v_1, v_2)v_3$ for all $v_i \in V$ ($i = 1, 2, 3$) and all $g \in G$.

Also, if G is weakly irreducible, then we say that the triple is *weakly irreducible*, where the weak irreducibility of G implies that there exists no G-invariant non-degenerate subspace W of V with $W \neq \{0\}$ and $W \neq V$. Here we give examples of an anti-Kaehlerian holonomy system.

EXAMPLE 1.   Let $(M, J, \langle \ , \ \rangle)$ be an anti-Kaehlerian manifold. Let ∇ be the Levi-Civita connection of $\langle \ , \ \rangle$, R be the curvature tensor of ∇ and Φ_x be the restricted holonomy group of ∇ at x ($\in M$). Then the triple $((T_x M, J_x, \langle \ , \ \rangle_x), R_x, \Phi_x)$ is an anti-Kaehlerian holonomy system. In particular, if $(M, J, \langle \ , \ \rangle)$ is locally symmetric (resp. irreducible), then this anti-Kaehlerian holonomy system is symmetric (resp. weakly irreducible).

EXAMPLE 2.   Let $(M, J, \langle \ , \ \rangle)$ be a complex n-dimensional anti-Kaehlerian submanifold in an anti-Kaehlerian manifold $(\widetilde{M}, \widetilde{J}, \langle \ , \ \rangle^{\sim})$, $T^{\perp}M$ be the normal bundle, A be the shape tensor, $\nabla^{\perp}$ be the normal connection, $R^{\perp}$ be the curvature tensor of $\nabla^{\perp}$ and $\Phi_x^{\perp}$ be the restricted holonomy group of $\nabla^{\perp}$ at x ($\in M$). Define $\bar{R}_x^{\perp} \in T_x^{\perp}M^* \otimes T_x^{\perp}M^* \otimes T_x^{\perp}M^* \otimes T_x^{\perp}M$ by

$$\bar{R}_x^{\perp}(v_1, v_2)v_3 := \sum_{i=1}^{2n} \langle e_i, e_i \rangle R_x^{\perp}(A_{v_1}e_i, A_{v_2}e_i)v_3 \,,$$

where $(e_1, \ldots, e_{2n})$ is an orthonormal base of $T_x M$. Then the triple $((T_x^{\perp}M, \widetilde{J}_x|_{T_x^{\perp}M}, \langle \ , \ \rangle_x^{\sim}|_{T_x^{\perp}M \times T_x^{\perp}M}), \bar{R}_x^{\perp}, \Phi_x^{\perp})$ is an anti-Kaehlerian holonomy system.

We have the following fact for a weakly irreducible symmetric anti-Kaehlerian holonomy system.

LEMMA 4.1.   *Let $S = ((V, J, \langle \ , \ \rangle), R, G)$ be a weakly irreducible symmetric anti-Kaehlerian holonomy system with $R \neq 0$. Then the G-action on V is equivalent to an aks-representation.*

PROOF.   Let $\mathfrak{g}^R$ be the Lie algebra generated by the set $\{R(v_1, v_2) \,|\, v_1, v_2 \in V\}$ ($\subset \mathfrak{so}_{AK}(V) := \text{Lie}(SO_{AK}(V))$) and $G^R := \exp \mathfrak{g}^R$, where $\exp$ is the exponential map of $SO_{AK}(V)$. Set $\mathfrak{L} := \mathfrak{g}^R \oplus V$. Define the $\sqrt{-1}$-multiples of elements of $\mathfrak{L}$ by $\sqrt{-1}v := Jv$ ($v \in V$) and $\sqrt{-1}R(v_1, v_2) := J \circ R(v_1, v_2)$ ($v_1, v_2 \in V$). Also, define $[\ , \ ]$ ($: \mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$) by $[A_1, A_2] := A_1 \circ A_2 - A_2 \circ A_1$ ($A_1, A_2 \in \mathfrak{g}^R$), $[v_1, v_2] := R(v_1, v_2)$ ($v_1, v_2 \in V$) and $[A, v] := A(v)$ ($A \in \mathfrak{g}^R$, $v \in V$). Then it follows from the symmetry of S that $(\mathfrak{L}, [\ , \ ])$ is a complex Lie algebra. Define a (complex) involution ρ of $(\mathfrak{L}, [\ , \ ])$ by $\rho|_{\mathfrak{g}^R} = \text{id}$ and $\rho|_V = -\text{id}$. Take a totally real subspace W of V such that $\langle \ , \ \rangle|_{W \times JW} = 0$ and that $\langle \ , \ \rangle|_{W \times W}$ is positive definite. Let $(\mathfrak{g}^R)_W := \{\text{pr}_W \circ R(v_1, v_2)|_W \,|\, v_1, v_2 \in V\}$. By imitating the proof of Lemma 3.1, we can show that $(\mathfrak{g}^R)_W$ is a Lie subalgebra of $\mathfrak{so}(W)$ and

$((\mathfrak{g}^R)_W)^{\mathbf{c}} = \mathfrak{g}^R$. Thus $((\mathfrak{L}, [\ ,\ ]), \rho)$ is an anti-Kaehlerian symmetric Lie algebra. Let $(\widetilde{L}, \widetilde{G})$ be a pair associated with $((\mathfrak{L}, [\ ,\ ]), \rho)$ such that $\widetilde{L}$ is simply connected and that $\widetilde{G}$ is connected. According to Proposition 3.4, $(\widetilde{L}, \widetilde{G})$ is an anti-Kaehlerian symmetric pair. Hence, it follows from Proposition 3.3 that there exists an anti-Kaehlerian structure $(J, \langle\ ,\ \rangle)$ such that $(\widetilde{L}/\widetilde{G}, J, \langle\ ,\ \rangle)$ is an anti-Kaehlerian symmetric space. On the other hand, we can show that the G-action on V is equivalent to both the restricted holonomy group action G^R of $\widetilde{L}/\widetilde{G}$ at $e\widetilde{G}$ and the linear isotropy group action $\mathrm{Ad}_{\widetilde{L}}|_{T_{e\widetilde{G}}(\widetilde{L}/\widetilde{G})}(\widetilde{G})$ (see P359$\sim$360 of [W1]). Since the G-action is weakly irreducible by the assumption, $\widetilde{L}/\widetilde{G}$ is irreducible. Hence, $\mathrm{Ad}_{\widetilde{L}}|_{T_{e\widetilde{G}}(\widetilde{L}/\widetilde{G})}(\widetilde{G})$-action is an aks-representation. Therefore, we obtain the statement of this lemma.   q.e.d.

Now we shall define the notion of the complexification of a holonomy system. Let $S = ((V, \langle\ ,\ \rangle), R, G)$ be a holonomy system. Then the triple $S^{\mathbf{c}} := ((V^{\mathbf{c}}, \mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}), R^{\mathbf{c}}, G^{\mathbf{c}})$ gives an anti-Kaehlerian holonomy system, where $V^{\mathbf{c}}$, $\langle\ ,\ \rangle^{\mathbf{c}}$, $R^{\mathbf{c}}$ and $G^{\mathbf{c}}$ are the complexifications of V, $\langle\ ,\ \rangle$, R and G, respectively. We call this system $S^{\mathbf{c}}$ the *complexification of S*. Next we shall define the notion of a totally real holonomy subsystem of an anti-Kaehlerian holonomy system. Let $S = ((V, J, \langle\ ,\ \rangle), R, G)$ be an anti-Kaehlerian holonomy system. Take a totally real subspace W of V such that $\langle\ ,\ \rangle|_{W \times JW} = 0$ and that $\langle\ ,\ \rangle|_{W \times W}$ is positive definite. Set $R_W := \mathrm{pr}_W \circ R|_{W \times W \times W}$. Let $\mathfrak{g}_W$ be the Lie subalgebra of $\mathfrak{so}(W)$ spanned by $\{\mathrm{pr}_W \circ A|_W \mid A \in \mathfrak{g}\}$ and $G_W := \exp_{SO(W)}(\mathfrak{g}_W)$. It is shown that G_W is compact and connected. Hence the triple $S_W := ((W, \langle\ ,\ \rangle|_{W \times W}), R_W, G_W)$ is a holonomy system. If $G_W^{\mathbf{c}} = G$, then we have $S_W^{\mathbf{c}} = S$. Then we call S_W a *totally real holonomy subsystem of S*. Note that, if S is symmetric and $R \neq 0$, then $G_W^{\mathbf{c}} = G$ automatically holds. In fact, the Lie algebra $\mathfrak{g}$ of G is then generated by $\{R(v_1, v_2) \mid v_1, v_2 \in V\}$ and the Lie algebra $\mathfrak{g}_W$ of G_W includes $\{R_W(w_1, w_2) \mid w_1, w_2 \in W\}$. Hence we have $\mathfrak{g} \subset \mathfrak{g}_W^{\mathbf{c}}$, that is, $G \subset G_W^{\mathbf{c}}$. On the other hand, it is clear that $G_W^{\mathbf{c}} \subset G$. Thus we have $G_W^{\mathbf{c}} = G$.

Now we show the following fact for a weakly irreducible anti-Kaehlerian holonomy system.

LEMMA 4.2. *Let $S = ((V, J, \langle\ ,\ \rangle), R, G)$ be a weakly irreducible anti-Kaehlerian holonomy system. Assume that there exists a totally real holonomy subsystem of S having non-zero scalar curvature. Then the G-action on V is equivalent to an aks-representation.*

PROOF. Let $S' := ((W, \langle\ ,\ \rangle|_{W \times W}), R_W, G_W)$ be a totally real holonomy subsystem of S having non-zero scalar curvature, which is irreducible because S is weakly irreducible. According to the proof of Theorem 5 of [Si], we can construct a non-zero curvature-like tensor R' (: $W \times W \times W \to W$) such that $((W, \langle\ ,\ \rangle|_{W \times W}), R', G_W)$ is a symmetric holonomy system. Define $\psi : G \times V^3 \to V$ by $\psi(g, v_1, v_2, v_3) = gR'^{\mathbf{c}}(g^{-1}v_1, g^{-1}v_2)g^{-1}v_3 - R'^{\mathbf{c}}(v_1, v_2)v_3$ $((g, v_1, v_2, v_3) \in G \times V^3)$, where $R'^{\mathbf{c}}$ is the complexification of R'. Since ψ is holomorphic and $\psi = 0$ over a totally real submanifold $G_W \times W^3$ of $G \times V^3$, we have $\psi \equiv 0$ by the theorem of identity. Then the triple $((V, J, \langle\ ,\ \rangle), R'^{\mathbf{c}}, G)$ is a weakly irreducible symmetric anti-Kaehlerian holonomy system. Hence we obtain the statement of this lemma by Lemma 4.1.   q.e.d.

5.   Partial tubes with flat and abelian normal bundle

For a submanifold in a Riemannian symmetric space of non-positive (or non-negative) curvature, M. Brück [B] defined a certain kind of partial tube with abelian normal bundle including the normal holonomy tube, where the submanifold is assumed to admit the ε-tube for a sufficiently small positive number ε. In this section, we shall define the similar partial tube for an anti-Kaehlerian submanifold in a non-flat anti-Kaehlerian symmetric space of non-positive (or non-negative) curvature. Let M be an anti-Kaehlerian submanifold in such an anti-Kaehlerian symmetric space $N = G/K$. Let $\varepsilon_\gamma := \inf\{|r| \mid r : \text{focal radius of } M \text{ along } \gamma\}$, where γ is a unit speed normal geodesic of M. Denote by ε_+^M (resp. ε_-^M) $\inf\{\varepsilon_\gamma \mid \gamma :$ unit speed spacelike (resp. timelike) normal geodesic$\}$. Assume that $\varepsilon_+^M > 0$ (resp. $\varepsilon_-^M > 0$). Denote the metric, the curvature tensor and the complex structure of N by $\langle \ , \ \rangle$, $\widetilde{R}$ and $\widetilde{J}$, respectively. Fix $x_0 \in M$. Let $\mathfrak{C}_{x_0} := \{c : [0,1] \to M : \text{a piecewise smooth path with } c(0) = x_0\}$, $\Phi_{x_0}^0$ be the restricted normal holonomy group of M at x_0 and $\mathfrak{L}_{x_0}$ be the Lie subalgebra of $\mathfrak{so}_{AK}(T_{x_0}^\perp M)$ generated by $\{P_c^{-1} \circ \mathrm{pr}_{T_{c(1)}^\perp M} \circ \widetilde{R}_{c(1)}(P_c v_1, P_c v_2) \circ P_c \mid v_1, v_2 \in T_{x_0}^\perp M, \ c \in \mathfrak{C}_{x_0}\}$, where $\mathfrak{so}_{AK}(T_{x_0}^\perp M) := \{A \in \mathfrak{gl}(T_{x_0}^\perp M) \mid \langle A v_1, v_2 \rangle_{x_0} + \langle v_1, A v_2 \rangle_{x_0} = 0 \ (\forall v_1, v_2 \in T_{x_0}^\perp M), \ [A, \widetilde{J}_{x_0}|_{T_{x_0}^\perp M}] = 0\}$, P_c is the parallel transport along c with respect to the normal connection $\nabla^\perp$ of M and $\mathrm{pr}_{T_{c(1)}^\perp M}$ is the orthogonal projection onto $T_{c(1)}^\perp M$. Also, let $\widehat{\mathfrak{L}}_{x_0}$ be the Lie algebra generated by $\mathfrak{L}_{x_0}$ and $\mathrm{Lie}\,\Phi_{x_0}^0$. Let $L_{x_0} := \exp \mathfrak{L}_{x_0}$ and $\widehat{L}_{x_0} := \exp \widehat{\mathfrak{L}}_{x_0}$, where $\exp$ is the exponential map of $GL(T_{x_0}^\perp M)$. Note that L_{x_0} and $\widehat{L}_{x_0}$ are Lie subgroups of $SO_{AK}(T_{x_0}^\perp M) := \{A \in GL(T_{x_0}^\perp M) \mid \langle A v_1, A v_2 \rangle_{x_0} = \langle v_1, v_2 \rangle_{x_0} \ (\forall v_1, v_2 \in T_{x_0}^\perp M), \ [A, \widetilde{J}_{x_0}|_{T_{x_0}^\perp M}] = 0\}$. Set $\widetilde{R}_c := P_c^{-1} \circ \mathrm{pr}_{T_{c(1)}^\perp M} \circ \widetilde{R}_{c(1)}(P_c(\cdot), P_c(\cdot)) \circ P_c$ for each $c \in \mathfrak{C}_{x_0}$. For each $c \in \mathfrak{C}_{x_0}$, it is clear that $S_c := (T_{x_0}^\perp M, \widetilde{R}_c, L_{x_0})$ is an anti-Kaehlerian holonomy system. Fix $c_0 \in \mathfrak{C}_{x_0}$ and a totally real subspace W of $T_{x_0}^\perp M$ such that $\langle \ , \ \rangle_{x_0}|_{W \times W}$ is positive definite. Let $\mathfrak{L}_{x_0}^W$ be the Lie subalgebra of $\mathfrak{so}(W)$ generated by $\{\mathrm{pr}_W \circ \widetilde{R}_c(v_1, v_2)|_W \mid v_1, v_2 \in V, \ c \in \mathfrak{C}_{x_0}\}$ and set $L_{x_0}^W := \exp \mathfrak{L}_{x_0}^W$, where $\exp$ is the exponential map of $GL(W)$. The group $L_{x_0}^W$ is compact because it is a closed subgroup of the compact group $SO(W)$. Hence $S_{c_0}|_W := ((W, \langle \ , \ \rangle_{x_0}|_{W \times W}), \mathrm{pr}_W \circ \widetilde{R}_{c_0}|_{W \times W \times W}, L_{x_0}^W)$ is a holonomy system. Clearly we have $(\mathfrak{L}_{x_0}^W)^{\mathbf{c}} = \mathfrak{L}_{x_0}$, that is, $(L_{x_0}^W)^{\mathbf{c}} = L_{x_0}$. It is shown that $S_{c_0}|_W$ is a totally real holonomy subsystem of S_{c_0}. Let $W = W_0 \oplus W_1 \oplus \cdots \oplus W_k$ be the decomposition of W such that W_i $(i = 0, 1, \ldots, k)$ are $L_{x_0}^W$-invariant, $L_{x_0}^{W_0} = \{\mathrm{id}_{W_0}\}$ and that $L_{x_0}^{W_i}$ $(i = 1, \ldots, k)$ are irreducible (non-trivial), where $L_{x_0}^{W_i} := \{g|_{W_i} \mid g \in L_{x_0}^W\}$ $(i = 0, 1, \ldots, k)$. Let $V_i := W_i \oplus J W_i (= W_i^{\mathbf{c}})$ $(i = 0, 1, \ldots, k)$. Note that the Lie algebra of $L_{x_0}^{W_i}$ is equal to $\{\mathrm{pr}_{W_i} \circ \widetilde{R}_c(v_1, v_2)|_{W_i} \mid v_1, v_2 \in V, \ c \in \mathfrak{C}_{x_0}\}$. Let $\mathfrak{L}_{x_0}^{V_i}$ $(i = 0, 1, \ldots, k)$ be the Lie subalgebra of $\mathfrak{so}_{AK}(V_i)$ generated by $\{\mathrm{pr}_{V_i} \circ \widetilde{R}_c(v_1, v_2)|_{V_i} \mid v_1, v_2 \in V, \ c \in \mathfrak{C}_{x_0}\}$ and $L_{x_0}^{V_i} := \exp \mathfrak{L}_{x_0}^{V_i}$,

where exp is the exponential map of $\mathrm{GL}(V_i)$. Clearly we have $T_{x_0}^\perp M = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ and $L_{x_0}^{V_i} = (L_{x_0}^{W_i})^{\mathbf{c}}$ $(i = 0, 1, \ldots, k)$. Also, it is easy to show that V_i $(i = 0, 1, \ldots, k)$ are L_{x_0}-invariant, $L_{x_0}^{V_0} = \{\mathrm{id}_{V_0}\}$ and that $L_{x_0}^{V_i}$ $(i = 1, \ldots, k)$ are weakly irreducible (non-trivial).

We have the following fact.

LEMMA 5.1. *The action of $L_{x_0}^{V_i}$ on V_i is equivalent to an aks-representation.*

PROOF. It is easy to show that $S_i := (V_i, \mathrm{pr}_{V_i} \circ \widetilde{R}_{c_0}|_{V_i \times V_i \times V_i}, L_{x_0}^{V_i})$ is a weakly irreducible anti-Kaehlerian holonomy system and that $(W_i, \mathrm{pr}_{W_i} \circ \widetilde{R}_{c_0}|_{W_i \times W_i \times W_i}, L_{x_0}^{W_i})$ is an irreducible totally real holonomy subsystem of S_i. Since N is of non-positive (or non-negative) curvatures, we see that the scalar curvature of $\mathrm{pr}_{W_i} \circ \widetilde{R}_{c_0}|_{W_i \times W_i \times W_i}$ does not vanish. Hence, it follows from Lemma 4.2 that the $L_{x_0}^{V_i}$-action is equivalent to an aks-representation. q.e.d.

As in Lemma 3.3 of [B], we have the following statements.

LEMMA 5.2. (i) V_i $(i = 0, 1, \ldots, k)$ *are $\Phi_{x_0}^0$-invariant.*

(ii) $\Phi_{x_0}^0|_{V_i} \subset L_{x_0}^{V_i}$ $(i = 1, \ldots, k)$, *where $\Phi_{x_0}^0|_{V_i} := \{g|_{V_i} \mid g \in \Phi_{x_0}^0\}$.*

(iii) *Let $W_0 = W_{0,0} \oplus W_{0,1} \oplus \cdots \oplus W_{0,l}$ be the decomposition of W_0 such that $W_{0,j}$ $(j = 0, 1, \ldots, l)$ are $\Phi_{x_0}^0|_{W_0}$-invariant, $\Phi_{x_0}^0|_{W_{0,0}} = \{\mathrm{id}_{W_{0,0}}\}$ and that $\Phi_{x_0}^0|_{W_{0,j}}$ $(j = 1, \ldots, l)$ are irreducible, where $\Phi_{x_0}^0|_{W_{0,j}} := \{g|_{W_{0,j}} \mid g \in \Phi_{x_0}^0\}$ $(j = 0, 1, \ldots, l)$. Set $V_{0,j} := W_{0,j}^{\mathbf{c}}$ $(j = 1, \ldots, l)$. Then the $\Phi_{x_0}^0|_{V_{0,j}}$-action on $V_{0,j}$ is equivalent to an aks-representation $(j = 1, \ldots, l)$.*

PROOF. From the definition of L_{x_0}, it follows that $\Phi_{x_0}^0$ is contained in the normalizer of L_{x_0} in $\mathrm{SO}_{\mathrm{AK}}(T_{x_0}^\perp M)$. Hence V_i $(i = 0, 1, \ldots, k)$ are $\Phi_{x_0}^0$-invariant. The group $\Phi_{x_0}^0|_{V_i}$ is contained in the normalizer $N(L_{x_0}^{V_i})$ of $L_{x_0}^{V_i}$ $(i \geq 1)$. On the other hand, according to Theorem 5 of [Si], the normalizer of $L_{x_0}^{W_i}$ coincides with oneself. From this fact, $N(L_{x_0}^{V_i}) = L_{x_0}^{V_i}$ follows. Hence we have $\Phi_{x_0}^0|_{V_i} \subset L_{x_0}^{V_i}$ (≥ 1). We define $\bar{R}_{x_0}^\perp \in T_{x_0}^\perp M^* \otimes T_{x_0}^\perp M^* \otimes T_{x_0}^\perp M^* \otimes T_{x_0}^\perp M$ by $\bar{R}_{x_0}^\perp(v_1, v_2)v_3 := \sum_{i=1}^{2n} \langle e_i, e_i \rangle R_{x_0}^\perp(A_{v_1}e_i, A_{v_2}e_i)v_3$, where $(e_1, \ldots, e_{2n})$ is an orthonormal base of $T_{x_0}M$. Let $(\bar{R}_{x_0}^\perp)_{W_0} := \mathrm{pr}_{W_0} \circ \bar{R}_{x_0}^\perp|_{W_0 \times W_0 \times W_0}$ and $(\Phi_{x_0}^0)_{W_0}$ be the image by the exponential map of the Lie subalgebra of $\mathfrak{so}(W_0)$ generated by $\{\mathrm{pr}_{W_0} \circ P_c^{-1} \circ R_{c(1)}^\perp(P_c X, P_c Y) \circ P_c|_{W_0} \mid X, Y \in T_{x_0}M, \ c \in \mathfrak{C}_{x_0}\}$. The triple $(W_0, (\bar{R}_{x_0}^\perp)_{W_0}, (\Phi_{x_0}^0)_{W_0})$ is a holonomy system. Since $\widetilde{R}(w_1, w_2) = 0$ for all $w_1, w_2 \in W_0$, we have

(5.1) $\qquad \langle R_{x_0}^\perp(X, Y)w_1, w_2 \rangle = \langle [A_{w_2}, A_{w_1}]X, Y \rangle$ $(X, Y \in T_{x_0}M, \ w_1, w_2 \in W_0)$

by the Ricci equation. By using this relation, we have

(5.2) $\qquad \langle (\bar{R}_{x_0}^\perp)_{W_0}(w_1, w_2)w_3, w_4 \rangle = \frac{1}{2}\mathrm{Tr}([A_{w_1}, A_{w_2}] \circ [A_{w_3}, A_{w_4}])$ $(w_1, \ldots, w_4 \in W_0)$.

By imitating the proof of Theorem 3.1 of [O] (in terms of (5.1) and (5.2)), we can show that the triples $S_{W_{0,j}} := (W_{0,j}, (\mathrm{pr}_{W_{0,j}} \circ \bar{R}^\perp_{x_0})|_{W_{0,j} \times W_{0,j} \times W_{0,j}}, (\Phi^0_{x_0})_{W_0}|_{W_{0,j}})$ $(j = 1, \ldots, l)$ are holonomy systems having non-zero scalar curvature, where we use the fact that N is of non-positive (or non-negative) curvature. Also, it is clear that $S_{V_{0,j}} := (V_{0,j}, (\mathrm{pr}_{V_{0,j}} \circ \bar{R}^\perp_{x_0})|_{V_{0,j} \times V_{0,j} \times V_{0,j}}, \Phi^0_{x_0}|_{V_{0,j}})$ $(j = 1, \ldots, l)$ are weakly irreducible anti-Kaehlerian holonomy systems having $S_{W_{0,j}}$ as a totally real holonomy subsystem. Hence it follows from Lemma 4.2 that the $\Phi^0_{x_0}|_{V_{0,j}}$-action $(j = 1, \ldots, l)$ is equivalent to an aks-representation.   q.e.d.

From these lemmas, we have the following fact directly.

THEOREM 5.3.   *There exists a decomposition* $T^\perp_{x_0} M = V_0 \oplus V_1 \oplus \cdots \oplus V_l \oplus V'_1 \oplus \cdots \oplus V'_k$ *of* $T^\perp_{x_0} M$ *such that* V_i $(i = 0, 1, \ldots, l)$ *and* V'_i $(i = 1, \ldots, k)$ *are* $\widehat{L}_{x_0}$-*invariant*, $\widehat{L}_{x_0}|_{V_0} = \{\mathrm{id}_{V_0}\}$, *the* $\widehat{L}_{x_0}|_{V_i}$-*actions* $(i = 1, \ldots, l)$ *and the* $\widehat{L}_{x_0}|_{V'_i}$-*actions* $(i = 1, \ldots, k)$ *are equivalent to aks-representations*, $\widehat{L}_{x_0}|_{V_1 \oplus \cdots \oplus V_l} = \Phi^0_{x_0}|_{V_1 \oplus \cdots \oplus V_l}$ *and that* $\widehat{L}_{x_0}|_{V'_1 \oplus \cdots \oplus V'_k} = L_{x_0}|_{V'_1 \oplus \cdots \oplus V'_k}$.

For $v_0 \in T^\perp_{x_0} M$, define a subbundle $B_{v_0}(M)$ of $T^\perp M$ by

$$B_{v_0}(M) := \{P_c(g v_0) \mid g \in \widehat{L}_{x_0}, \ c \in \mathfrak{C}_{x_0}\}$$

and $\widetilde{B}_{v_0}(M) := \exp^\perp(B_{v_0}(M))$, where $\exp^\perp$ is the normal exponential map of M. For each spacelike (resp. timelike) vector v_0 with $\|v_0\| < \varepsilon^M_+$ (resp. ε^M_-), $\widetilde{B}_{v_0}(M)$ is an immersed submanifold, that is, a partial tube over M whose fibre over x_0 is $\exp^\perp(\widehat{L}_{x_0} v_0)$. This partial tube $\widetilde{B}_{v_0}(M)$ is a notion similar to a partial tube defined for a submanifold in a Riemannian symmetric space of non-positive (or non-negative) curvature by M. Brück [B]. Denote by $\mathrm{Hol}_{v_0}(M)$ the normal holonomy tube over M through v_0. Clearly we have $\mathrm{Hol}_{v_0}(M) \subset \widetilde{B}_{v_0}(M)$. Also, we have the following facts.

THEOREM 5.4.   *Assume that* $\widehat{L}_{x_0} v_0$ *is a principal orbit of the* $\widehat{L}_{x_0}$-*action. Then the following statements* (i)$\sim$(iii) *hold.*
   (i)   *The normal connection of* $\widetilde{B}_{v_0}(M)$ *is flat,*
   (ii) $\widetilde{B}_{v_0}(M)$ *has abelian normal bundle,*
   (iii)   *Assume that* M *is simply connected. The* $\widehat{L}_{x_0}$-*action and the normal parallel transport map of* M *preserve the focal structure of* M *if and only if* $\widetilde{B}_{v_0}(M)$ *is anti-Kaehlerian equifocal. Then* M *is a focal submanifold of* $\widetilde{B}_{v_0}(M)$.

PROOF.   These statements are shown by imitating the discussions in Sections 4.2 $\sim$ 4.4 of [B].   q.e.d.

6.   Anti-Kaehlerian submanifolds with abelian normal bundle

Let $N = G/K$ be a semi-simple anti-Kaehlerian symmetric space. Denote by $\langle \, , \, \rangle$ (resp. J) the metric (resp. the complex structure) of N. Let E be a vector bundle along a

smooth curve $c : [0, 1] \to N$ (i.e., E : a subbundle of c^*TN) such that each fibre E_t ($t \in [0, 1]$) is an anti-Kaehlerian and abelian subspace of $T_{c(t)}N$ and that each $\exp_N(E_t)$ ($t \in [0, 1]$) is properly embedded into N. Since N is semi-simple, $\exp_N(E_t)$ is an anti-Kaehlerian cylindrical product. There exists a totally real subspace $E_t^{\mathbf{R}}$ of E_t such that $\exp_N(E_t^{\mathbf{R}})$ is a torus (with a flat pseudo-Riemannian metric). Denote by G the full holomorphic isometry group of N from now on. Also, denote by K_t the isotropy group of G at $c(t)$ and denote by $(K_t)_{v_0}$ the isotropy group of the linear isotropy action $K_t \times T_{c(t)}N \to T_{c(t)}N$ at $v_0 \in T_{c(t)}N$. Then we have the following fact.

LEMMA 6.1. *The set* $E^l := \bigcup_{t \in [0,1]} \{v_0 \in E_t \,|\, \dim(K_t)_{v_0} \le l\}$ *is open in E for each* $l \in \mathbf{N}$.

PROOF. The statement of this lemma is shown by imitating the discussion in Page 81 of [PT]. q.e.d.

Set $l_0 := \min\{l \,|\, E^l \neq \emptyset\}$. Fix $t_0 \in [0, 1]$ and $v_0 \in E_{t_0} \cap E^{l_0}$. By using some J-orthonormal frame field $(\widetilde{v}_1, J\widetilde{v}_1, \ldots, \widetilde{v}_r, J\widetilde{v}_r)$ of E, we define maps $\psi_{t_0 t} : E_{t_0} \to E_t$ ($t \in [0, 1]$) by $\psi_{t_0 t}((\widetilde{v}_i)_{t_0}) = (\widetilde{v}_i)_t$ and $\psi_{t_0 t}(J(\widetilde{v}_i)_{t_0}) = J(\widetilde{v}_i)_t$ ($i = 1, \ldots, r$). Let $v_t := \psi_{t_0 t}(v_0)$. Let I_0 be the maximal sub-interval of $[0, 1]$ containing t_0 such that $v_t \in E^{l_0}$ for all $t \in I_0$, which is open because E^{l_0} is open in E. Take a smooth curve $\hat{c} : I_0 \to G$ satisfying $\hat{c}(t_0) = e$ (e : the identity element of G) and $\hat{c}(t)(c(t_0)) = c(t)$ for all $t \in I_0$. Let $\widehat{E}_t := \hat{c}(t)_*^{-1}(E_t)$ and $h(t) := \hat{c}(t)_*^{-1}(v_t)$ ($t \in I_0$). Take a tubular neighborhood T of the principal orbit $K_{t_0}v_0$ in $T_{c(t_0)}N$. Let I_1 be the maximal sub-interval of I_0 containing t_0 satisfying $h(I_1) \subset T$ and define $\gamma : I_1 \to K_{t_0}v_0$ by $h(t) \in S_{\gamma(t)}$ ($t \in I_1$), where $S_{\gamma(t)}$ is the slice of $K_{t_0}v_0$ through $\gamma(t)$. Let $o : I_1 \to K_{t_0}$ be a smooth curve such that $o(t_0) = e$ and $o(t)(v_0) = \gamma(t)$ for all $t \in I_1$. Then we can prove the following fact by imitating the proof of Lemma 5.2 of [B].

LEMMA 6.2. *The set* $\bigcup_{t \in I_1} o(t)^{-1}(\widehat{E}_t)$ *is contained in a maximal abelian anti-Kaehlerian subspace of* $T_{c(t_0)}N$.

PROOF. Let $w \in S_{\gamma(t)} \cap \widehat{E}_t$. From $w \in S_{\gamma(t)}$, we have $(K_{t_0})_w \subset (K_{t_0})_{\gamma(t)}$ (see Page 81 of [PT]). This together with $\dim(K_{t_0})_{v_0} = l_0$ deduces that $\dim(K_{t_0})_w = \dim(K_{t_0})_{\gamma(t)}$, which implies that $(K_{t_0})_w = (K_{t_0})_{\gamma(t)}$. Let $\mathfrak{a}_t := T_{\gamma(t)}^{\perp}(K_{t_0}v_0)$ ($t \in I_1$), which is the maximal abelian anti-Kaehlerian subspace of $T_{c(t_0)}N$ containing $\gamma(t)$. Since $K_{t_0}w$ is parallel to $K_{t_0}v_0$ and $w \in S_{\gamma_t}$, we have $T_w^{\perp} K_{t_0}w = T_{\gamma(t)}^{\perp} K_{t_0}v_0$. Similarly we have $T_{h(t)}^{\perp} K_{t_0}h(t) = T_{\gamma(t)}^{\perp} K_{t_0}v_0 = \mathfrak{a}_t$, where we use $h(t) \in E^{l_0}$. Hence, since $\mathfrak{a}_t$ is the maximal abelian anti-Kaehlerian subspace containing $h(t)$, $h(t) \in \widehat{E}_t$ and $\widehat{E}_t$ is abelian, we have $\widehat{E}_t \subset \mathfrak{a}_t$, that is, $o(t)^{-1}\widehat{E}_t \subset \mathfrak{a}_0$. Thus the statement of this lemma follows. q.e.d.

Furthermore we can show the following fact by imitating the proof of Lemma 5.3 of [B].

LEMMA 6.3. *The space $o(t)^{-1}(\widehat{E}_t)$ is independent of the choice of $t \in I_1$.*

PROOF. According to Lemma 6.2, $\bigcup_{t \in I_1} o(t)^{-1}(\widehat{E}_t)$ is contained in some maximal abelian anti-Kaehlerian subspace $\mathfrak{a}_0$ of $T_{c(t_0)}N$. Since N is semi-simple, $\exp \mathfrak{a}_0$ is an anti-Kaehlerian cylindrical product. There exists a totally real subspace $\mathfrak{a}_0^{\mathbf{R}}$ of $\mathfrak{a}_0$ such that $\exp \mathfrak{a}_0^{\mathbf{R}}$ is a torus. Denote $\exp \mathfrak{a}_0^{\mathbf{R}}$ by T^k ($k = \frac{1}{2}\mathrm{rank}\,N$). Since $\exp E_t$ is an anti-Kaehlerian cylindrical product by the assumption, so is also $\exp(o(t)^{-1}(\widehat{E}_t))$. Hence $\exp(o(t)^{-1}(\widehat{E}_t) \cap \mathfrak{a}_0^{\mathbf{R}})$ is a torus, which we denote by T_t^r ($r = \frac{1}{2}\dim E_t$). Let $\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$ be the lattice of T^k. Since T_t^r is a sub-torus of T^k, the lattice of T_t^r is expressed as $\{\mathbf{a}_i := \sum_{j=1}^{k} a_{ij}(t)\mathbf{e}_j \mid i = 1, \dots, r\}$ ($a_{ij}(t) \in \mathbf{Z}$). Furthermore, since T_t^r varies continuously with respect to t, the a_{ij}'s are continuous. Hence the a_{ij}'s are constant, so T_t^r is independent of the choice of t. This implies that $o(t)^{-1}(\widehat{E}_t)$ is independent of the choice of t.   q.e.d.

From this lemma, we have the following fact.

LEMMA 6.4. *There exists a smooth curve $w : I_1 \to G$ with $w(t)_* E_{t_0} = E_t$ ($t \in I_1$).*

PROOF. Define a smooth curve $w : I_1 \to G$ by $w(t) := \hat{c}(t) \circ o(t)$ ($t \in I_1$). This curve w is the desired curve.   q.e.d.

Furthermore, we can show the following fact from this lemma.

LEMMA 6.5. *There exists a smooth curve $w : [0, 1] \to G$ with $w(t)_* E_0 = E_t$ ($t \in [0, 1]$).*

PROOF. Let $G_{2r}^{AK}(N) := \bigcup_{x \in N} \{\Pi \mid \Pi : 2r\text{-dimensional anti-Kaehlerian subspace}$ of $T_x N\}$, which is a submanifold of the Grassmann bundle of N consisting of $2r$-dimensional subspaces. The group G acts on $G_{2r}^{AK}(N)$ naturally. Let I_2 be the maximal interval such that $t_0 \in I_2$ and $\bigcup_{t \in I_2} E_t \subset G(E_{t_0})$. From Lemma 6.4, it follows that I_2 is open. On the other hand, since $t \to E_t$ ($t \in [0, 1]$) is a continuous curve in $G_{2r}^{AK}(N)$, I_2 is closed. Therefore we have $I_2 = [0, 1]$, which implies that the above interval I_1 is equal to $[0, 1]$.   q.e.d.

Also we shall need the following lemma.

LEMMA 6.6. *Fix $t_0 \in [0, 1]$. Let $g : (-\varepsilon, \varepsilon) \to G$ be a smooth curve such that $g(0) = e$ and that $\frac{d}{dt}|_{t=0}g(t)c(t_0)$ is orthogonal to E_{t_0}, and X be the vector field along $\exp E_{t_0}$ defined by $X_x := \frac{d}{dt}|_{t=0}g(t)x$ ($x \in \exp E_{t_0}$). Then X is a normal vector field of $\exp E_{t_0}$.*

PROOF. Denote by $X_{\mathbf{R}}^T$ the $T(\exp E_{t_0}^{\mathbf{R}})$- component of $X|_{\exp E_{t_0}^{\mathbf{R}}}$. Let $\gamma : \mathbf{R} \to \exp E_{t_0}^{\mathbf{R}}$ be a geodesic in $\exp E_{t_0}^{\mathbf{R}}$ (and hence N). Define a map $\delta : (-\varepsilon, \varepsilon) \times \mathbf{R} \to N$ by $\delta(t, s) = g(t)\gamma(s)$. Since δ is a geodesic variation, the variational vector field $\frac{\partial\delta}{\partial t}|_{t=0} (= X \circ \gamma)$ is a Jacobi field along γ. Hence $X_{\mathbf{R}}^T \circ \gamma$ is also a Jacobi field. By using this fact, we have

$$\frac{d^2}{ds^2}\langle X_{\mathbf{R}}^T \circ \gamma, \dot\gamma \rangle = \langle \widetilde\nabla_{\dot\gamma}\widetilde\nabla_{\dot\gamma}(X_{\mathbf{R}}^T \circ \gamma), \dot\gamma \rangle = -\langle \widetilde R(X_{\mathbf{R}}^T \circ \gamma, \dot\gamma)\dot\gamma, \dot\gamma \rangle = 0\,.$$

Hence we can express as $\langle X_{\mathbf{R}}^T \circ \gamma, \dot\gamma \rangle(s) = as + b$ $(a, b \in \mathbf{R})$. Since $\gamma(\mathbf{R})$ is contained in the compact set $\exp E_{t_0}^{\mathbf{R}}$, we have $\sup \|\langle X_{\mathbf{R}}^T \circ \gamma, \dot\gamma \rangle\| < \infty$. Therefore, we see that $\langle X_{\mathbf{R}}^T \circ \gamma, \dot\gamma \rangle$ is constant. Hence we have $\langle \widetilde\nabla_{\dot\gamma}(X_{\mathbf{R}}^T \circ \gamma), \dot\gamma \rangle = 0$. Since this relation holds for any geodesic γ in $\exp E_{t_0}^{\mathbf{R}}$, $X_{\mathbf{R}}^T$ is a Killing vector field on a flat torus $\exp E_{t_0}^{\mathbf{R}}$. This fact together with $(X_{\mathbf{R}}^T)_{c(t_0)} = 0$ implies that $X_{\mathbf{R}}^T \equiv 0$. Denote by X^T the $T(\exp E_{t_0})$-component of X. We have only to show $X^T \equiv 0$. Since X^T is real holomorphic (i.e., $X^T - \sqrt{-1}JX^T$: holomorphic) and $X_{\mathbf{R}}^T = 0$ on the totally real submanifold $\exp E_{t_0}^{\mathbf{R}}$ of $\exp E_{t_0}$, we see that $X^T = 0$ along $\exp E_{t_0}^{\mathbf{R}}$. It follows from the theorem of identity that $X^T = 0$ on the whole of $\exp E_{t_0}$. This completes the proof. q.e.d.

Let M be an anti-Kaehlerian submanifold with abelian normal bundle in N. Assume that $\exp_N(T_x^\perp M)$ is properly embedded for each $x \in M$. By using Lemma 6.6, we can show the following fact.

LEMMA 6.7. *Let x be a point of M and $g : \mathbf{R} \to G$ be a C^∞-curve such that $g(0) = e$, $g(t)x \in M$ $(t \in \mathbf{R})$ and that $g(t)_* T_x^\perp M = T_{g(t)x}^\perp M$ $(t \in \mathbf{R})$. Let $c(t) := g(t)x$ $(t \in \mathbf{R})$. Then $g(t)_* : T_x^\perp M \to T_{c(t)}^\perp M$ is the parallel transport along $c|_{[0,t]}$ with respect to the normal connection $\nabla^\perp$ of M.*

PROOF. Take an arbitrary $v \in T_x^\perp M$. Let γ_v be the geodesic in $\exp^\perp(T_x^\perp M)$ with $\dot\gamma_v(0) = v$ and define a map $\delta : \mathbf{R}^2 \to N$ by $\delta(t, s) := g(t)(\gamma_v(s))$. Since $\delta_*(\frac{\partial}{\partial t})$ is a normal vector field of $\exp(T_{c(t)}^\perp M)$ by Lemma 6.6 and $\exp(T_{c(t)}^\perp M)$ is totally geodesic, we have

$$\widetilde\nabla_{\dot c} g(t)_* v = \widetilde\nabla_{\frac{\partial}{\partial t}|_{s=0}} \delta_*\left(\frac{\partial}{\partial s}\right) = \widetilde\nabla_{\frac{\partial}{\partial s}|_{s=0}} \delta_*\left(\frac{\partial}{\partial t}\right)$$

$$= \nabla^{\perp_t}_{\frac{\partial}{\partial s}|_{s=0}} \delta_*\left(\frac{\partial}{\partial t}\right) \in T_{c(t)}^\perp \exp(T_{c(t)}^\perp M) = T_{c(t)} M\,,$$

where $\nabla^{\perp_t}$ is the normal connection of $\exp(T_{c(t)}^\perp M)$. Hence we have $\nabla^\perp_{\dot c} g(t)_* v = 0$. From the arbitrariness of v, this implies that $g(t)_* : T_x^\perp M \to T_{c(t)}^\perp M$ is the parallel transport along $c|_{[0,t]}$ with respect to $\nabla^\perp$. q.e.d.

By using Lemmas 6.5 and 6.7, we can show the following fact.

THEOREM 6.8.   *Let M be as above. The normal connection of M is flat.*

PROOF.   Let $c : I \to M$ be a loop at $x(\in M)$ such that the homotopy class $[c]$ of c is the identity element of the fundamental group $\pi_1(M, x)$. From the assumption, it follows that $t \to T^\perp_{c(t)} M$ satisfies the same conditions as the above $t \to E_t$. Hence it follows from Lemma 6.5 that there exists a smooth curve $w : I \to G$ with $w(t)_*(T^\perp_x M) = T^\perp_{c(t)} M$ ($t \in I$). Furthermore, it follows from Lemma 6.7 that $w(1)_* : T^\perp_x M \to T^\perp_x M$ is the parallel transport along c with respect to $\nabla^\perp$. The element $w(1)$ of G is an isometry of the anti-Kaehlerian cylindrical product $\exp_N(T^\perp_x M)$ having x as a fixed point. Furthermore, since $[c]$ is the identity element of $\pi_1(M, x)$, $w(1)$ preserves the orientation. Hence, since the full orientation-preserving isometry group of an anti-Kaehlerian cylindrical product is a free action, $w(1)$ is the identity transformation of $\exp_N(T^\perp_x M)$ and hence $w(1)_*$ (i.e., the parallel transport along c with respect to $\nabla^\perp$) is the identity transformation of $T^\perp_x M$. From the arbitrariness of c, it follows that the restricted normal holonomy group of M at x is trivial, that is, the normal connection of M is flat.   q.e.d.

7.   Proofs of Theorems A, B and C

Let M and F be as in Theorem A. Fix $x_0 \in F$ and $v_o \in T^\perp_{x_0} F$ with $\exp^\perp(v_o) \in M$. Without loss of generality, we may assume that $0 < \langle v_o, v_o \rangle < (\varepsilon^+_F)^2$ or $0 > \langle v_o, v_o \rangle > -(\varepsilon^-_F)^2$, where $\varepsilon^\pm_F$ is as in Section 5. Let L_{x_0}, $\widehat{L}_{x_0}$, $B_{v_o}(F)$ and $\widetilde{B}_{v_o}(F)$ be the quantities as in Section 5 defined for F. Let $\pi_F : M \to F$ be the focal map onto F and $M^0_{x_0}$ be the component containing v_o of $(\exp^{\perp_F} |_{T^\perp_{x_0} F})^{-1}(\pi_F^{-1}(x_0))$, where $\exp^{\perp_F}$ is the normal exponential map of F. Then we can show the following fact.

LEMMA 7.1.   *The intersection $\widehat{L}_{x_0} v_o \cap M^0_{x_0}$ is open in $\widehat{L}_{x_0} v_o$.*

PROOF.   By imitating the proof of (11) in Page 91 of [B], we can show the statement of this lemma.   q.e.d.

By using Theorem 5.3, Lemmas 6.5, 6.7 and 7.1, we prove Theorem A.

PROOF OF THEOREM A.   It suffices to show that $M^0_{x_0}$ is an open portion of $\widehat{L}_{x_0} v_o$. In fact, M is then an open portion of $\widetilde{B}_{v_o}(F)$ and each fibre of $\widetilde{B}_{v_o}(F)$ are the image by the normal exponential map of a principal orbit of a pseudo-orthogonal representation on the normal space of F which is equivalent to the direct sum representation of an aks-representation and a trivial representation by Theorem 5.3. Let $c : [0, 1] \to M^0_{x_0}$ be a smooth curve with $c(0) = v_o$ and v_1 be an element of $T^\perp_{\exp^{\perp_F}(v_o)} M$ with $\exp^{\perp_M}(v_1) = x_0$. Let $\widetilde{v}_1$ be the $\nabla^\perp$-parallel vector field along $\widetilde{c} := \exp^{\perp_F} \circ c$ with $\widetilde{v}_1(0) = v_1$, where $\nabla^\perp$ is the normal connection of M. Define a vector bundle E along $\widetilde{c}$ by $E_t := T^\perp_{\widetilde{c}(t)} M$ ($t \in [0, 1]$). For simplicity, set $N := G/K$. Since

E_t is an anti-Kaehlerian and abelian subspace of $T_{\widetilde{c}(t)}N$ and $\exp_N(E_t)$ is properly embedded by the assumption, it follows from Lemma 6.5 that there exists a smooth curve $w : [0, 1] \to G$ with $w(t)(\exp^{\perp_F}(v_o)) = \widetilde{c}(t)$ and $w(t)_* E_0 = E_t$ ($t \in [0, 1]$). Furthermore, it follows from Lemma 6.7 that $w(t)_* : E_0 \to E_t$ is the parallel transport along $\widetilde{c}|_{[0,t]}$ with respect to $\nabla^{\perp}$. Hence we have $w(t)_* v_1 = \widetilde{v}_1(t)$. From this fact and $\exp^{\perp_M}(\widetilde{v}_1(t)) = x_0$ ($t \in [0, 1]$), we have

$$w(t)(x_0) = w(t)(\exp^{\perp_M}(v_1)) = \exp^{\perp_M}(w(t)_* v_1) = x_0 \,,$$

that is, $w(t) \in K_{x_0}$, where K_{x_0} is the isotropy group of G at x_0. Also, we have

$$\exp_N(c(t)) = \exp^{\perp_F}(c(t)) = w(t)(\exp^{\perp_F}(v_o)) = \exp_N(w(t)_*(v_o))$$

and hence $c(t) = w(t)_*(v_o) \in K_{x_0} v_o$. From the arbitrariness of c, it follows that

$$(7.1) \qquad\qquad\qquad M^0_{x_0} \subset K_{x_0} v_o \,.$$

Let $\mathfrak{H}$ be the Lie subalgebra of $\mathfrak{so}_{AK}(T^{\perp}_{x_0}F)$ generated by the set $\{\mathrm{pr}_{T^{\perp}_{x_0}F} \circ \widetilde{R}(v_1, v_2)|_{T^{\perp}_{x_0}F} \mid v_1, v_2 \in T^{\perp}_{x_0}F\}$ and set $H := \exp_{\mathrm{SO}_{AK}(T^{\perp}_{x_0}F)} \mathfrak{H}$, where $\exp_{\mathrm{SO}_{AK}(T^{\perp}_{x_0}F)}$ is the exponential map of $\mathrm{SO}_{AK}(T^{\perp}_{x_0}F)$. Clearly we have $H \subset \widehat{L}_{x_0}$. Let $v \in T^{\perp}_{v_0}Hv_0 \cap T^{\perp}_{x_0}F$. Then we have $\langle \widetilde{R}(v_0, v)v_0, v \rangle = 0$ because $\widetilde{R}(v_0, v)v_0 \in T_{v_0}Hv_0$. This implies that $\mathrm{Span}\{v_0, v\}$ is an abelian subspace of $T^{\perp}_{x_0}F$. Hence we see that $\mathrm{Span}\{v_0, v\} \subset T^{\perp}_{v_0}(K_{x_0}v_0)$, that is, $v \in T^{\perp}_{v_0}(K_{x_0}v_0)$. From the arbitrariness of v, we have $T^{\perp}_{v_0}Hv_0 \cap T^{\perp}_{x_0}F \subset T^{\perp}_{v_0}(K_{x_0}v_0)$ and hence $T_{v_0}(K_{x_0}v_0) \cap T^{\perp}_{x_0}F \subset T_{v_0}Hv_0$. On the other hand, it follows from Lemma 7.1 and (7.1) that

$$T_{v_0}Hv_0 \subset T_{v_0}(\widehat{L}_{x_0}v_0) \subset T_{v_0}M^0_{x_0} \subset T_{v_0}(K_{x_0}v_0) \cap T^{\perp}_{x_0}F \,.$$

Therefore, we obtain $T_{v_0}(\widehat{L}_{x_0}v_0) = T_{v_0}M^0_{x_0}$. Similarly, we obtain $T_v(\widehat{L}_{x_0}v_0) = T_v M^0_{x_0}$ for other $v \in M^0_{x_0}$. Hence we see that $M^0_{x_0}$ is an open portion of $\widehat{L}_{x_0}v_0$. This completes the proof. q.e.d.

We need the following lemma to prove Theorem B.

LEMMA 7.2. *Let $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection, $\phi^{\mathbf{c}} : H^0([0, 1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$ and H_u be the horizontal space of the submersion $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ at u ($\in H^0([0, 1], \mathfrak{g}^{\mathbf{c}})$). Then the restriction $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})|_{H_u}$ of $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ to H_u can be regarded as the exponential map of $G^{\mathbf{c}}/K^{\mathbf{c}}$ at $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u)$ under the identification of H_u with $T_{(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u)}(G^{\mathbf{c}}/K^{\mathbf{c}})$.*

PROOF. Let γ ($: \mathbf{R} \to G^{\mathbf{c}}/K^{\mathbf{c}}$) be a geodesic in $G^{\mathbf{c}}/K^{\mathbf{c}}$ and γ^L_u be the horizontal lift of γ to $u \in (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(\gamma(0))$. Since $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion, γ^L_u is a geodesic in $H^0([0, 1], \mathfrak{g}^{\mathbf{c}})$. Since $H^0([0, 1], \mathfrak{g}^{\mathbf{c}})$ is a flat space, we have $\gamma^L_u(t) = u + t\dot{\gamma}^L_u(0)$ ($\in H_u$), where $t \in \mathbf{R}$. From this fact, the statement of this lemma follows. q.e.d.

PROOF OF THEOREM B.   Let $M \hookrightarrow G/K$ be as in the statement of Theorem B and $M^{\mathbf{c}} \hookrightarrow G^{\mathbf{c}}/K^{\mathbf{c}}$ be the (extrinsic) complexification of M, where we note that $G^{\mathbf{c}}/K^{\mathbf{c}}$ is a semi-simple anti-Kaehlerian symmetric space of non-positive curvature. Define a distribution E_0 on $M^{\mathbf{c}}$ by $(E_0)_x := \bigcap_{v \in T_x^\perp M^{\mathbf{c}}} \left(\mathrm{Ker}\, A_v^{\mathbf{c}} \cap \mathrm{Ker}\, R^{\mathbf{c}}(\cdot, v)v \right)$ $(x \in M^{\mathbf{c}})$, where $A^{\mathbf{c}}$ is the shape tensor of $M^{\mathbf{c}}$ and $R^{\mathbf{c}}$ is the curvature tensor of $G^{\mathbf{c}}/K^{\mathbf{c}}$. Then $M^{\mathbf{c}}$ is an open portion of a product submanifold $M^{\mathbf{c}'} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}\, (\subset\, G^{\mathbf{c}'}/K^{\mathbf{c}'} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}} = G^{\mathbf{c}}/K^{\mathbf{c}})$, where the decomposition $G^{\mathbf{c}'}/K^{\mathbf{c}'} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ is an anti-Kaehlerian product such that the distribution $T(G_0^{\mathbf{c}}/K_0^{\mathbf{c}})$ on $M^{\mathbf{c}'} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ is the extension of E_0 and $M^{\mathbf{c}'}$ is an anti-Kaehlerian equifocal submanifold in $G^{\mathbf{c}'}/K^{\mathbf{c}'}$. Denote from now on $M^{\mathbf{c}'} \times G_0^{\mathbf{c}}/K_0^{\mathbf{c}}$ by $M^{\mathbf{c}}$ and $T(G_0^{\mathbf{c}}/K_0^{\mathbf{c}})$ by E_0. Fix $x \in M^{\mathbf{c}}$. Since $M^{\mathbf{c}}$ is proper anti-Kaehlerian equifocal, the focal set F of $M^{\mathbf{c}}$ at x consists of infinitely many complex hyperplanes $\{l_\lambda\}_{\lambda \in \Lambda}$ in $T_x^\perp(M^{\mathbf{c}})$. Take a focal normal vector field v such that $v_x \in l_{\lambda_0}$ for some $\lambda_0 \in I$ and that $v_x \notin l_\lambda$ $(\lambda \in I \setminus \{\lambda_0\})$. Denote by E the focal distribution for v. Now we shall show that each leaf of E is the image by the normal exponential map of an open portion of a complex sphere of a normal space of the focal submanifold $F := f_v(M^{\mathbf{c}})$, where f_v is the focal map for v. Let L be a leaf of E. Denote by $\widetilde{E}$ the focal distribution on $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ corresponding to E. Set $\widetilde{F} := (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(F)$, which is a focal submanifold corresponding to $\widetilde{E}$. It is clear that L is the image of some leaf $\widetilde{L}$ of $\widetilde{E}$ by $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$. According to Theorem 2 of [K2], $\widetilde{L}$ is an open portion of a complex sphere in the normal space $T_{u_0}^\perp \widetilde{F}$ of $\widetilde{F}$ at some $u_0 \in \widetilde{F}$. According to Lemma 7.2, $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})|_{T_{u_0}^\perp \widetilde{F}}$ is regarded as the normal exponential map $\exp^\perp_{(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u_0)}$ of F at $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u_0)$ under the identification of $T_{u_0}^\perp \widetilde{F}\, (\subset T_{u_0} H^0([0, 1], \mathfrak{g}^{\mathbf{c}}) = H^0([0, 1], \mathfrak{g}^{\mathbf{c}}))$ with $T_{(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u_0)}^\perp F$. Therefore, we see that L is the image of an open portion of a complex sphere in $T_{(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u_0)}^\perp F$ by $\exp^\perp_{(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u_0)}$. Let $\mathfrak{E} := \{E_i\}_{i \in I}$ be the family of all focal distributions on $M^{\mathbf{c}}$ whose leaves are the images by the normal exponential map of open portions of complex spheres of the normal spaces of focal submanifolds. Then it follows from the above fact that $E_0 \oplus \sum_{i \in I} E_i = TM^{\mathbf{c}}$. Also, it is clear that I is finite. Let $\mathfrak{E} = \{E_1, \ldots, E_k\}$. Take a focal normal vector field v_1 with $\mathrm{Ker}\, f_{v_1 *} = E_1$ and that $F_1 := f_{v_1}(M^{\mathbf{c}})$. Take $w_1 \in T^\perp F_1$ with $\exp^{\perp F_1}(w_1) \in M^{\mathbf{c}}$, where $\exp^{\perp F_1}$ is the normal exponential map of F_1. According to the proof of Theorem A, the partial tube $\widetilde{B}_{w_1}(F_1)$ includes $M^{\mathbf{c}}$ as an open portion. It is clear that $\widetilde{B}_{w_1}(F_1)$ is proper anti-Kaehlerian equifocal. Let $\{\widetilde{E}_1, \ldots, \widetilde{E}_k\}$ be the family of all focal distributions of $\widetilde{B}_{w_1}(F_1)$ with $\widetilde{E}_i|_{M^{\mathbf{c}}} = E_i$ $(i = 1, \ldots, k)$. Take a focal normal vector field v_2 of $\widetilde{B}_{w_1}(F_1)$ with $\mathrm{Ker}\, f_{v_2 *} = \widetilde{E}_2$ and set $F_2 := f_{v_2}(\widetilde{B}_{w_1}(F_1))$. Take $w_2 \in T^\perp F_2$ with $\exp^{\perp F_2}(w_2) \in \widetilde{B}_{w_1}(F_1)$, where $\exp^{\perp F_2}$ is the normal exponential map of F_2. According to the proof of Theorem A, the partial tube $\widetilde{B}_{w_2}(F_2)$ includes $\widetilde{B}_{w_1}(F_1)$ as an open portion. It is clear that $\widetilde{B}_{w_2}(F_2)$ is proper anti-Kaehlerian equifocal. By repeating $(k - 2)$-times the same process, we obtain the complete extension $\widehat{M^{\mathbf{c}}}$ of $M^{\mathbf{c}}$. From this construction of $\widehat{M^{\mathbf{c}}}$ and Theorem A, the statements

(i) and (ii) of Theorem B follow.   q.e.d.

Next we prove Theorem C.

PROOF OF THEOREM C.   Let $\{E_0, E_1, \ldots, E_k\}$ be as in the statement (ii) of Theorem B. Fix $x = gK \in M$. Since M is curvature-adapted, each $(E_i)_x$ $(i = 1, \ldots, k)$ is expressed as $(E_i)_x = \bigoplus_{(\lambda, \mu) \in S} (\mathrm{Ker}(A_v - \lambda\,\mathrm{id}) \cap \mathrm{Ker}(R(\cdot, v)v - \mu\,\mathrm{id}))^{\mathbf{c}}$ for some unit normal vector v of M at x, where A is the shape tensor of M and R is the curvature tensor of G/K, S is a subset of $(\mathrm{Spec}\,A_v \times \mathrm{Spec}\,R(\cdot, v)v) \setminus \{(0, 0)\}$. Hence we have $(E_i)_x \cap T_x M = \bigoplus_{(\lambda, \mu) \in S} (\mathrm{Ker}(A_v - \lambda\,\mathrm{id}) \cap \mathrm{Ker}(R(\cdot, v)v - \mu\,\mathrm{id}))$. Also, we have $(E_0)_x \cap T_x M = \bigcap_{v \in T_x^\perp M} (\mathrm{Ker}\,A_v \cap \mathrm{Ker}\,R(\cdot, v)v)$. From these relations, the statement of Theorem C follows.

                                                                                     q.e.d.

8.   Examples

Let M be a principal orbit of a Hermann type action $H \times G/K \to G/K$ and θ be the Cartan involution of G with $(\mathrm{Fix}\,\theta)_0 \subset K \subset \mathrm{Fix}\,\theta$ and σ be an involution of G with $(\mathrm{Fix}\,\sigma)_0 \subset H \subset \mathrm{Fix}\,\sigma$. Without loss of generality, we may assume that $\sigma \circ \theta = \theta \circ \sigma$. It can be shown that M is proper complex equifocal and curvature-adapted (see [K3]). Denote by A the shape tensor of M. Then $H(eK)$ is a totally geodesic orbit (which is a singular orbit except for one case) of the H-action and M is regarded as a partial tube over $H(eK)$. Let $L := \mathrm{Fix}(\sigma \circ \theta)$. The submanifold $\exp^\perp(T_{eK}^\perp(H(eK)))$ is totally geodesic and it is isometric to the symmetric space $L/H \cap K$, where $\exp^\perp$ is the normal exponential map of $H(eK)$. Let $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{h}$ be the Lie algebras of G, K and H. Denote by the same symbols the involutions of $\mathfrak{g}$ associated with θ and σ. Set $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$ $(\subset \mathfrak{g})$ and $\mathfrak{q} := \mathrm{Ker}(\sigma + \mathrm{id})$ $(\subset \mathfrak{g})$. Take $x := \exp^\perp(\xi) = \exp_G(\xi)K \in M \cap \exp^\perp(T_{eK}^\perp(H(eK)))$, where $\xi \in \mathfrak{p} = \mathrm{Ker}(\theta + \mathrm{id})$ $(\subset \mathfrak{g})$. For simplicity, set $g := \exp_G(\xi)$. Let Σ be the section of M through x, which pass through eK. Let $\mathfrak{b} := T_{eK}\Sigma$, $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p} := T_{eK}(G/K)$ containing $\mathfrak{b}$, $\triangle$ be the root system with respect to $\mathfrak{a}$ and $\mathfrak{p} = \mathfrak{a} + \sum_{\alpha \in \triangle_+} \mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{a}$. Set $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{q}(= T_{eK}^\perp(H(eK)))$. The orthogonal complement $\mathfrak{p}'^\perp$ of $\mathfrak{p}'$ in $\mathfrak{p}$ is equal to $\mathfrak{p} \cap \mathfrak{h}$. Set $\overline{\triangle} = \{\alpha|_{\mathfrak{b}} \mid \alpha \in \triangle \text{ s.t. } \alpha|_{\mathfrak{b}} \neq 0\}$, which is a root system by Theorem B of [K6]. Let $\overline{\triangle}_+$ be a positive root system of $\overline{\triangle}$ with respect to some lexicographic ordering, $\mathfrak{p}_\beta := \sum_{\alpha \in \triangle_+ \text{ s.t. } \alpha|_{\mathfrak{b}} = \pm\beta} \mathfrak{p}_\alpha$ for $\beta \in \overline{\triangle}_+$, $\overline{\triangle}_+^H := \{\beta \in \overline{\triangle}_+ \mid \mathfrak{p}'^\perp \cap \mathfrak{p}_\beta \neq \{0\}\}$ and $\overline{\triangle}_+^V := \{\beta \in \overline{\triangle}_+ \mid \mathfrak{p}' \cap \mathfrak{p}_\beta \neq \{0\}\}$. Since both $\mathfrak{p}'$ and $\mathfrak{p}'^\perp$ are Lie triple systems of $\mathfrak{p}$ and $\mathfrak{b}$ is contained in $\mathfrak{p}'$, we have $\mathfrak{p}'^\perp = \mathfrak{z}_{\mathfrak{p}'^\perp}(\mathfrak{b}) + \sum_{\beta \in \overline{\triangle}_+^H} (\mathfrak{p}'^\perp \cap \mathfrak{p}_\beta)$ and $\mathfrak{p}' = \mathfrak{b} + \sum_{\beta \in \overline{\triangle}_+^V} (\mathfrak{p}' \cap \mathfrak{p}_\beta)$. Note that

$\overline{\triangle}^V := \overline{\triangle}^V_+ \cup (-\overline{\triangle}^V_+)$ is the root system of the symmetric space $L/H \cap K$. Take $\eta \in T^\perp_x M$. For each $X \in \mathfrak{p}'^\perp \cap \mathfrak{p}_\beta$ ($\beta \in \overline{\triangle}^H_+$), we have $A_\eta \widetilde{X}_\xi = -\beta(\bar{\eta}) \tanh \beta(\xi) \widetilde{X}_\xi$ (see the proof of Theorem B of [K3]), where $\widetilde{X}_\xi$ is the horizontal lift of X to ξ (see Section 3 of [K3] for this definition) and $\bar{\eta}$ is the element of $\mathfrak{b}$ with $\exp^\perp_{*\xi}(\bar{\eta}) = \eta$ (where $\bar{\eta}$ is regarded as an element of $T_\xi \mathfrak{p}'$ under the natural identification of $\mathfrak{p}'$ with $T_\xi \mathfrak{p}'$). Also, for each $Y \in T_x(M \cap \exp^\perp(\mathfrak{p}')) \cap g_* \mathfrak{p}_\beta$ ($\beta \in \overline{\triangle}^V_+$), we have $A_\eta Y = -\frac{\beta(\bar{\eta})}{\tanh \beta(\xi)} Y$ (see the proof of Theorem B of [K3]). By using these relations, for the focal set F of $\widehat{M}^{\mathbf{c}}$ at x, we have

(8.1)
$$
g^{-1}_* F = \left(\bigcup_{\beta \in \overline{\triangle}^V_+} \bigcup_{j \in \mathbf{Z}} \left(-\xi + (\beta^{\mathbf{c}})^{-1}(j\pi\sqrt{-1}) \right) \right)
$$
$$
\cup \left(\bigcup_{\beta \in \overline{\triangle}^H_+} \bigcup_{j \in \mathbf{Z}} \left(-\xi + (\beta^{\mathbf{c}})^{-1}\left(\left(j + \frac{1}{2} \right)\pi\sqrt{-1} \right) \right) \right),
$$

where $\beta^{\mathbf{c}}$ is the complexification of β. Let $FD^{cs} := \{E_i \mid i = 1, \ldots, k\}$ be the family of all focal distributions of $\widehat{M}^{\mathbf{c}}$ whose leaves are the images by the normal exponential map of complex spheres in the normal spaces of focal submanifolds and $FD^{cs}_x := \{(E_i)_x \mid i = 1, \ldots, k\}$. For each $\beta \in \overline{\triangle}$, we set

$$
E^V_{\beta,x} := g_*(\mathfrak{p}_\beta \cap \mathfrak{p}')^{\mathbf{c}} \quad (\beta \in \overline{\triangle}^V_+) \quad \text{and} \quad E^H_{\beta,x} := g_*(\mathfrak{p}_\beta \cap \mathfrak{p}'^\perp)^{\mathbf{c}} \quad (\beta \in \overline{\triangle}^H_+).
$$

Then we have

(8.2)
$$
\mathfrak{z}_{\mathfrak{p}'^\perp}(\mathfrak{b}) \oplus \left(\bigoplus_{\beta \in \overline{\triangle}^V_+} E^V_{\beta,x} \right) \oplus \left(\bigoplus_{\beta \in \overline{\triangle}^H_+} E^H_{\beta,x} \right) = T_x \widehat{M}^{\mathbf{c}}.
$$

Also, for each subspace E of $T_x \widehat{M}^{\mathbf{c}}$, we set $FN(E) := \{v \in T^\perp_x \widehat{M}^{\mathbf{c}} \mid \mathrm{Ker}(f_{\widetilde{v}})_{*x} = E\}$, where $\widetilde{v}$ is the parallel normal vector field of $\widehat{M}^{\mathbf{c}}$ with $\widetilde{v}_x = v$ and $f_{\widetilde{v}}$ is the focal map for $\widetilde{v}$. For $\beta \in \overline{\triangle}^V_+$ with $2\beta, \frac{1}{2}\beta \notin \overline{\triangle}_+$, $E^V_{\beta,x}$ is a member of FD^{cs}_x and, for $\beta' \in \overline{\triangle}^H_+$ with $2\beta', \frac{1}{2}\beta' \notin \overline{\triangle}_+$, $E^H_{\beta',x}$ is a member of FD^{cs}_x. In fact, $E^V_{\beta,x}$ (resp. $E^H_{\beta',x}$) is the focal distribution for a focal normal vector field v with $v_x \in (-\xi + (\beta^{\mathbf{c}})^{-1}(0)) \setminus (g^{-1}_* F \setminus (-\xi + (\beta^{\mathbf{c}})^{-1}(0)))$ (resp. $v_x \in (-\xi + (\beta'^{\mathbf{c}})^{-1}(\frac{\pi}{2}\sqrt{-1})) \setminus (g^{-1}_* F \setminus (-\xi + (\beta'^{\mathbf{c}})^{-1}(\frac{\pi}{2}\sqrt{-1})))$). Hence, according to Theorem 2 in [K2], we have $E^V_{\beta,x}, E^H_{\beta',x} \in FD^{cs}_x$. However, for $\beta \in \overline{\triangle}^V_+$ with $2\beta \in \overline{\triangle}_+$ or $\frac{1}{2}\beta \in \overline{\triangle}_+$, $E^V_{\beta,x}$ is not necessarily a member of FD^{cs}_x but there exists $E \in FD^{cs}_x$ with $E \supset E^V_{\beta,x}$. For example, if $\beta \in \overline{\triangle}^V_+$, $\frac{1}{2}\beta \in \overline{\triangle}^H_+ \cap \overline{\triangle}^V_+$ and $2\beta \notin \overline{\triangle}_+$, then we have $E^V_{\beta,x} \notin FD^{cs}_x$ but $E^V_{\beta,x} \oplus E^H_{\frac{1}{2}\beta,x} \in FD^{cs}_x$ and $E^V_{\beta,x} \oplus E^V_{\frac{1}{2}\beta,x} \in FD^{cs}_x$. In fact, $E^V_{\beta,x} \oplus E^H_{\frac{1}{2}\beta,x}$ (resp. $E^V_{\beta,x} \oplus E^V_{\frac{1}{2}\beta,x}$) is the focal distribution for a focal normal vector field v with $v_x \in (-\xi + (\beta^{\mathbf{c}})^{-1}(\pi\sqrt{-1})) \setminus$

$(g_*^{-1}F \setminus (-\xi + (\beta^{\mathbf{c}})^{-1}(\pi\sqrt{-1})))$ (resp. $v_x \in (-\xi + (\beta^{\mathbf{c}})^{-1}(0)) \setminus (g_*^{-1}F \setminus (-\xi + (\beta^{\mathbf{c}})^{-1}(0))))$ but there exists no focal normal vector field having $E_{\beta,x}^V$ as a focal distribution. Similarly, for $\beta' \in \overline{\triangle}_+^H$ with $2\beta' \in \overline{\triangle}_+$ or $\frac{1}{2}\beta' \in \overline{\triangle}_+$, $E_{\beta,x}^H$ is not necessarily a member of FD_x^{cs} but there exists $E' \in FD_x^{cs}$ with $E' \supset E_{\beta',x}^H$. Thus, if $\overline{\triangle}$ (which is the root system) is reduced, then we have $T\widehat{M^{\mathbf{c}}} = \bigoplus_{i=0}^k E_i$ (orthogonal direct sum), where E_0 is defined by $(E_0)_x :=$

$\bigcap_{v \in T_x^\perp \widehat{M^{\mathbf{c}}}} (\mathrm{Ker}\,A_v^{\mathbf{c}} \cap \mathrm{Ker}\,R^{\mathbf{c}}(\cdot,v)v)$ $(x \in \widehat{M^{\mathbf{c}}})$ and $\{E_1, \ldots, E_k\} = FD_x^{cs}$. However, if $\overline{\triangle}$ is not

reduced, then we have $T\widehat{M^{\mathbf{c}}} = \sum_{i=0}^k E_i$ but the right-hand side is not necessarily an orthogonal direct sum. Assume that $\overline{\triangle}$ is reduced. For each $i \in \{1, \ldots, k\}$, we have $(E_i)_x = E_{\beta,x}^V$ or $E_{\beta,x}^H$ for some $\beta \in \overline{\triangle}$. It is easy to show that the leaves of $E_i^{\mathbf{R}} := E_i|_M \cap TM$ are diffeomorphic to a sphere (resp. an affine space) in the case of $(E_i)_x = E_{\beta,x}^V$ (resp. $E_{\beta,x}^H$).

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