Some Results on Additive Number Theory IV

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§ 1. The main theorem.

Let \( \omega(n) \) denote the number of distinct prime factors of a positive integer \( n \).

**Theorem.** Let \( \alpha < \beta \). Let \( A(N; \alpha, \beta) \) denote, for sufficiently large positive integer \( N \), the number of representations of \( N \) as the sum of the form \( N = p + n \), where \( p \) is prime, and \( n \) is a positive integer such that

\[
\log \log N + \alpha \sqrt{\log \log N} < \omega(n) < \log \log N + \beta \sqrt{\log \log N},
\]

then, as \( N \to \infty \), we have

\[
A(N; \alpha, \beta) \sim \frac{N}{\log N} \cdot \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/8} dx.
\]

We shall give a proof of this theorem in section 2. Our proof runs in the same lines as in my paper [6], but it uses also Bombieri's mean value theorem and Brun-Titchmarsh's inequality. It is to be noticed that somewhat analogous theorem was proved in Halberstam [3] using Siegel-Walfisz's theorem. It might perhaps be possible to prove our theorem in a similar style as in [3], but I hope that it would be of interest to prove the theorem in our way.

As was shown in Gallagher [2], Bombieri's theorem can be deduced rather simply from Siegel-Walfisz's theorem, and is far more conveniently applicable in our situation. For Bombieri's theorem cf. Bombieri [1], Gallagher [2], Halberstam-Richert [4], p. 111, Mitsui [5], Chap. 8.

We shall shorten the paper by omitting the similar parts of the proof as in Tanaka [6].

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§ 2. Proof of the main theorem.

Let $a$ and $b$ be non-negative integers. Then

\[
\sum_{c=0}^{b}(-1)^{c}\binom{a}{c}=\begin{cases}1, & \text{when } a=0, \\ \geq 0, & \text{when } a>0 \text{ and } b \text{ is even}, \\ \leq 0, & \text{when } a>0 \text{ and } b \text{ is odd}.
\end{cases}
\]

This is the same as Lemma 2 in [6].

Now we define some functions and sets which will be used in the sequel. The positive integer $N$ will be assumed to be sufficiently large as occasion demands.

We define the set $Q_N$ consisting of primes as

\[Q_N = \{p: p \nmid N, e^{(\log \log N)^{2/3}} < p < N^{(\log \log N)^{-2}}\}\]

and put

\[y(N) = \sum_{p \in Q_N} \frac{1}{p} .\]

Then we have

**Lemma 1.** $y(N) = \log \log N + O(\log \log \log N)$.

**Proof.** We can easily see that $\omega(N) = O(\log N)$, and hence

\[
\sum_{p \nmid N} \frac{1}{p} \leq \sum_{\omega(N) \leq} \frac{1}{p} = O(\log \log \log N).
\]

The lemma can be obtained similarly as Lemma 4 in [6].

We denote by $\omega_N(n)$ the number of distinct prime factors of a positive integer $n$, which belong to the set $Q_N$:

\[\omega_N(n) = \sum_{p \mid n, p \in Q_N} 1 .\]

For any positive integer $t$, we define the set $M_N(t)$ consisting of positive integers as

\[M_N(t) = \{m: m \text{ is squarefree}, m \text{ has } t \text{ prime factors}, m \text{ is composed only of primes } \in Q_N\} .\]

We put for convenience $M_N(0) = \{1\}$.

For any positive integer $t$, we denote by $F(N; t)$ the number of
representations of $N$ as the sum of the form $N = p + n$, where $p$ is prime, and $n$ is a positive integer such that $\omega_N(n) = t$.

For any positive integer $m$ such that $m \in M_N(t)$ with some positive integer $t$, we denote by $G(N; m)$ the number of representations of $N$ as the sum of the form $N = p + n$, where $p$ is prime, and $n$ is a positive integer such that

\[
\prod_{p \mid m, p \in Q_N} p = m.
\]

We obviously have

\[
F(N; t) = \sum_{m \in M_N(t)} G(N; m).
\]

For any positive integers $t$ and $T$, we put

\[
\mathcal{H}^{(0)}(N; t, T) = \sum_{m \in M_N(t)} \mathcal{H}^{(0)}(N; m, T),
\]

\[
\mathcal{H}^{(0)}(N; m, T) = \sum_{\tau = 0}^{2T} (-1)^\tau \mathcal{L}(N; m, \tau),
\]

\[
\mathcal{H}^{(1)}(N; t, T) = \sum_{m \in M_N(t)} \mathcal{H}^{(1)}(N; m, T),
\]

\[
\mathcal{H}^{(1)}(N; m, T) = \sum_{\tau = 0}^{2T+1} (-1)^\tau \mathcal{L}(N; m, \tau),
\]

\[
\mathcal{L}(N; m, \tau) = \sum_{p \in \mathcal{P}^N(\tau)} \sum_{m \mid n} 1.
\]

**Lemma 2.** $\mathcal{H}^{(1)}(N; t, T) \leq F(N; t) \leq \mathcal{H}^{(0)}(N; t, T)$.

**Proof.** We can write

\[
\mathcal{L}(N; m, \tau) = \sum_{p + m = N} \left( \frac{\omega_N(n) - t}{\tau} \right),
\]

so that

\[
\mathcal{H}^{(0)}(N; m, T) = \sum_{p + m = N} \sum_{\tau = 0}^{2T} (-1)^\tau \left( \frac{\omega_N(n) - t}{\tau} \right),
\]

\[
\mathcal{H}^{(1)}(N; m, T) = \sum_{p + m = N} \sum_{\tau = 0}^{2T+1} (-1)^\tau \left( \frac{\omega_N(n) - t}{\tau} \right).
\]

Now, since $m \in M_N(t)$ and $m \mid n$, (2) is equivalent to the equality $\omega_N(n) = t$. Hence, by (1), we have

\[
\mathcal{H}^{(1)}(N; m, T) \leq G(N; m) \leq \mathcal{H}^{(0)}(N; m, T).
\]
The lemma follows from this and the definitions of $F(N; t)$, $H^{(0)}(N; t, T)$ and $H^{(1)}(N; t, T)$.

We further put

$$
H^{(0)}(N; t, T) = \sum_{m \in M_{N}(t)} K^{(0)}(N; m, T),
$$

$$
K^{(0)}(N; m, T) = \sum_{\tau=0}^{2T} (-1)^{\tau} L(N; m, \tau),
$$

$$
H^{(1)}(N; t, T) = \sum_{m \in M_{N}(t)} K^{(1)}(N; m, T),
$$

$$
K^{(1)}(N; m, T) = \sum_{\tau=0}^{2T+1} (-1)^{\tau} L(N; m, \tau),
$$

$$
L(N; m, \tau) = \sum_{\mu \in M_{N}(\tau)} \frac{1}{\varphi(m\mu)},
$$

where $\varphi(m\mu)$ is Euler's function of $m\mu$.

**Lemma 3.** Let $T=[5y(N)]$. Then, as $N \to \infty$, we have

$$
H^{(0)}(N; t, T) = \frac{[y(N)]^{t}e^{-y(N)}}{t!}\{1+o(1)\},
$$

$$
H^{(1)}(N; t, T) = \frac{[y(N)]^{t}e^{-y(N)}}{t!}\{1+o(1)\}
$$

uniformly in $t$ with $t<2y(N)$.

**Proof.** The formulas in the lemma can be proved quite similarly as Lemma 6 in [6], if we replace the $L(N; m, \tau)$'s contained in the definitions of $H^{(0)}(N; t, T)$ and $H^{(1)}(N; t, T)$ by

$$
L^{*}(N; m, \tau) = \sum_{\mu \in M_{N}(\tau)} \frac{1}{m\mu}.
$$

Hence it will suffice for the proof of the lemma to show that

$$
L^{*}(N; m, \tau) = L(N; m, \tau)[1+o(1)]
$$

uniformly in the relevant $L(N; m, \tau)$'s.

Now, for each summand of $L(N; m, \tau)$, the pair of positive integers $m$ and $\mu$ is such that $(m, \mu)=1$, $m \in M_{N}(t)$, $t<2y(N)$, $\mu \in M_{N}(\tau)$, $\tau \leq 10y(N)+1$, so that, by the definitions of the sets $Q_{N}(t)$, $M_{N}(t)$, and Lemma 1, $m\mu$ is squarefree, $\omega(m\mu)<c \log \log N$, $c>0$, and each of the prime factors of $m\mu$ is greater than $e^{(\log \log N)^{3}}$. Hence
\[
\frac{1}{m\mu} < \frac{1}{\varphi(m\mu)} \leq \frac{1}{m\mu} \prod_{p|m\mu} \left(1 - \frac{1}{p}\right)^{-1} \leq \frac{1}{m\mu} \prod_{p|n\mu} \left(1 + \frac{2}{p}\right) < \frac{1}{m\mu} \left(1 + 2e^{-(\log\log N)^{2}}\right)^{c\log\log N} = \frac{1+o(1)}{m\mu},
\]
or
\[
\frac{1}{m\mu} = \frac{1+o(1)}{\varphi(m\mu)},
\]
from which we see that (3) holds with the required uniformity.

**Lemma 4.** Let $T$ be an increasing function of $N$ such that $T = O(\log \log N)$. Then, as $N \to \infty$, we have

\[
\mathcal{H}^{(0)}(N; t, T) - H^0(N; t, T) \ll N = o\left(\frac{N\{y(N)\}^{t}e^{-y(N)}}{t!\log N}\right),
\]

\[
\mathcal{H}^{(1)}(N; t, T) - H^0(N; t, T) \ll N = o\left(\frac{N\{y(N)\}^{t}e^{-y(N)}}{t!\log N}\right)
\]

uniformly in $t$ with $t < 2y(N)$, where $\text{li} \, N$ is the logarithmic integral of $N$.

**Proof.** The definition of $\mathcal{L}(N; m, \tau)$ can be rewritten as

\[
\mathcal{L}(N; m, \tau) = \sum_{\mu eH_{N}(\tau)} \pi(N; m\mu, N)
\]

where $\pi(N; m\mu, N)$ is the number of primes $p$ such that $p < N$ and $p \equiv m\mu$ (mod $N$). Hence, by the definitions of $\mathcal{H}^{(0)}(N; t, T)$ and $H^0(N; t, T)$, we can write

\[
|\mathcal{H}^{(0)}(N; t, T) - H^0(N; t, T) \ll N| \\
\leq \sum_{m \in M_{N}(t)} \sum_{\tau=0}^{2T} \sum_{\mu eM_{N}(\tau)} \left| \pi(N; m\mu, N) - \frac{\text{li} \, N}{\varphi(m\mu)} \right|
\]

Put here $m\mu = \nu$, then the same value of $\nu$ occurs at most $d(\nu)$ times, where $d(\nu)$ is the number of divisors of $\nu$; by our assumptions, $\nu$ is squarefree and $\nu \in M_{N}(\tau)$, $\tau < c \log \log N$, so that $\omega(\nu) < c \log \log N$ and $d(\nu) < e^{d\log\log N} = \log^{c} \log^{e} N$, where $c$ is a suitable positive constant; by the definition of the set $Q_{N}$, each prime factor of $\nu$ is less than $N^{c(\log\log N)^{-2}}$, and so $\nu < N^{c(\log\log N)^{-1}}$. Hence we have

\[
|\mathcal{H}^{(0)}(N; t, T) - H^0(N; t, T) \text{li} \, N| < \log^{e} N \sum_{(\nu, N)=1} \left| \pi(N; \nu, N) - \frac{\text{li} \, N}{\varphi(\nu)} \right|
\]
Now it follows from Bombieri's theorem that

$$\mathcal{H}^{(0)}(N; t, T) - H^0(N; t, T) \sim N = O(N \log^{-\alpha} N)$$

with arbitrary positive constant $\alpha$. (For our purpose somewhat weaker result than Bombieri's would suffice.) Again, since we assume $t < 2y(N)$,

$$\frac{\{y(N)\}^t}{t!} > \left(\frac{t}{2}\right)^t \cdot \frac{1}{t^t} = 2^{-t} > e^{-2y(N)}.$$

Hence we have

$$\mathcal{H}^{(0)}(N; t, T) - H^0(N; t, T) \sim N = O\left(\frac{N\{y(N)\}^t e^{\epsilon y(N)}}{t! \log^\alpha N}\right).$$

Similar result can be obtained for $\mathcal{H}^{(1)}(N; t, T)$, and, since $y(N) \sim \log \log N$ by Lemma 1, the lemma follows when we take $\alpha$ sufficiently large.

**LEMMA 5.** Let $T = [5y(N)]$. Then, as $N \to \infty$,

$$\mathcal{H}^{(0)}(N; t, T) = \frac{N\{y(N)\}^t e^{-\epsilon y(N)}}{t! \log N} \{1 + o(1)\},$$

$$\mathcal{H}^{(1)}(N; t, T) = \frac{N\{y(N)\}^t e^{-\epsilon y(N)}}{t! \log N} \{1 + o(1)\}$$

uniformly in $t$ with $t < 2y(N)$.

**PROOF.** The lemma follows from Lemmas 3 and 4.

**LEMMA 6.** As $N \to \infty$,

$$F(N; t) = \frac{N\{y(N)\}^t e^{-\epsilon y(N)}}{t! \log N} \{1 + o(1)\}$$

uniformly in $t$ with $t < 2y(N)$.

**PROOF.** The lemma follows from Lemmas 2 and 5.

**LEMMA 7.** Let $\alpha < \beta$. Let $t$ be a positive integer such that $t = y(N) + u\sqrt{y(N)}$ with $\alpha < u < \beta$. Then, as $N \to \infty$,

$$F(N; t) = \frac{N}{\sqrt{2\pi y(N) \log N}} e^{-u^2 \beta \{1 + o(1)\}}$$

uniformly in $t$ with above-mentioned restrictions.
PROOF. This lemma corresponds to Lemma 13 in [6], and can be proved similarly. The Stirling formula plays an important role in the proof.

**Lemma 8.** Let \( \alpha < \beta \), and let \( A^{**}(N; \alpha, \beta) \) denote the number of representations of \( N \) as the sum of the form \( N = p + n \), where \( p \) is prime, and \( n \) is a positive integer such that

\[
y(N) + \alpha \sqrt{y(N)} < \omega_N(n) < y(N) + \beta \sqrt{y(N)}.
\]

Then, as \( N \to \infty \), we have

\[
A^{**}(N; \alpha, \beta) \sim \frac{N}{\log N} \cdot \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} dx.
\]

**Proof.** This lemma corresponds to Lemma 14 in [6], and can be proved similarly.

**Lemma 9.** Let \( \alpha < \beta \), and let \( A^{*}(N; \alpha, \beta) \) denote the number of representations of \( N \) as the sum of the form \( N = p + n \), where \( p \) is prime, and \( n \) is a positive integer such that

\[
y(N) + \alpha \sqrt{y(N)} < \omega(n) < y(N) + \beta \sqrt{y(N)}.
\]

Then, as \( N \to \infty \), we have

\[
A^{*}(N; \alpha, \beta) \sim \frac{N}{\log N} \cdot \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} dx.
\]

**Proof.** We shall estimate the sum

\[
S(N) = \sum_{p < N} (\omega(N-p) - \omega_N(N-p))
\]

in utilizing Brun-Titchmarsh's inequality. For this inequality, cf. Halberstam-Richert [4], p. 110, Mitsui [5], p. 154. Now, noting the fact that a positive integer has at most one prime factor greater than the square root of itself, we argue as

\[
S(N) = \sum_{p < N} \sum_{q \in Q_N} 1 = \sum_{p < N} \sum_{q \in Q_N} \sum_{(q | (N-p))} 1 + O \left( \sum_{p < N} 1 \right)
\]

\[
= \sum_{p < N} \sum_{q \in Q_N} 1 + O \left( \frac{N}{\log N} \right) = \sum_{q \in Q_N} \pi(N; q, N) + O \left( \frac{N}{\log N} \right)
\]

where \( q \) runs through the primes satisfying the specified conditions. On
applying Brun-Titchmarsh's inequality to the last sum, we have

$$\sum_{\substack{q \leq 
 \in Q_N \sum_{q \leq \sqrt{N}} \frac{1}{q} \not\in Q_N}} \pi(N; q, N) = O\left( \sum_{\substack{q \leq \sqrt{N} \not\in Q_N}} \frac{N}{q \log(N/q)} \right) = O\left( \frac{N}{\log N} \sum_{\substack{q \leq \sqrt{N} \not\in Q_N}} \frac{1}{q} \right).$$

Again, similarly as in the proof of Lemma 4 in [6], we obtain

$$\sum_{\substack{q \leq \sqrt{N} \not\in Q_N}} \frac{1}{q} = O(\log \log \log N).$$

Thus it has been proved that

$$S(N) = O\left( \frac{N}{\log N} \log \log \log N \right).$$

Now we can prove the lemma similarly as in the proof of Lemma 15 in [6], using this result in the form

$$\sum_{n+p=N} \left\{ \omega(n) - \omega_N(n) \right\} = O\left( \frac{N}{\log N} \sqrt{y(N)} \right).$$

It follows from this that, for any given $\epsilon > 0$, we can take $N_1 = N_1(\epsilon)$ so large that, when $N > N_1$, the number of representations of $N$ as the sum of the form $N = p + n$ such that the inequality $\omega(n) - \omega_N(n) > \epsilon \sqrt{y(N)}$ holds, is less than $\epsilon N / \log N$. Hence, for $N > N_1$,

$$A^{**}(N; \alpha, \beta - \epsilon) - \epsilon \frac{N}{\log N} < A^*(N; \alpha, \beta) < A^{**}(N; \alpha - \epsilon, \beta) + \epsilon \frac{N}{\log N}.$$
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References


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