The Class Group of the Rees Algebras over Polynomial Rings

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Introduction

Let $A$ be a commutative ring with unit element and let $A[X]$ denote a polynomial ring over $A$ with an indeterminate $X$. For an ideal $a$ of $A$ we put $\mathcal{R}(A, a)=A[[aX; a \in a], X^{-1}]$, the $A$-subalgebra of $A[X, X^{-1}]$ generated by $\{aX; a \in a\}$ and $X^{-1}$, and we call it the Rees algebra of $a$ over $A$.

$\mathcal{R}(A, a)$ is a graded subring of $A[X, X^{-1}]$, whose graduation is given by $\mathcal{R}_n(A, a)=a^nX^n$ for $n \geq 0$ and $\mathcal{R}_n(A, a)=A$ for $n<0$. Note that $\mathcal{R}(A, a)$ is canonically identified with the ring $\bigoplus_{a \in a} a^n$ where $a^n=A$ for $n<0$.

The aim of this paper is to prove the following theorem.

THEOREM. Let $k$ be a Krull domain and let $W_1, \cdots, W_l$ be indeterminates over $k$. Then, for every positive integer $n$, $\mathcal{R}(k[W_1, \cdots, W_l], (W_1, \cdots, W_l)^*)$ is a Krull domain and $C(\mathcal{R})=C(k) \oplus \mathbb{Z}/n\mathbb{Z}$. (Here $C(\cdot)$ denotes the divisor class group.)

By the theorem we have the following result immediately.

COROLLARY. If $k$ is a field, then $\mathcal{R}(k[W_1, \cdots, W_l], (W_1, \cdots, W_l)^*)$ is a Macaulay normal domain and $C(\mathcal{R})=\mathbb{Z}/n\mathbb{Z}$.

§ 1. Proof of Theorem.

Let $k, W_1, \cdots, W_l, n$ be as in the introduction and let $X^{-1}=U$. We denote $\mathcal{R}(k[W_1, \cdots, W_l], (W_1, \cdots, W_l)^*)$ by $T$. Let $A_n$ be the set of the indexes $(\alpha)=(\alpha_1, \cdots, \alpha_n)$ where $\alpha_i$'s are nonnegative integers with $\sum_{i=1}^{n} \alpha_i=n$ and let $R=k[W_1, \cdots, W_l, U]$. Then $T=k[W_1, \cdots, W_l, U, W^{(\alpha)}/U]_{\alpha \in A_n}$ and $T$ is a $k$-subalgebra of $R[X]$, where $W^{(\alpha)}$ denotes $W_1^{\alpha_1} \cdots W_l^{\alpha_l}$.

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Now we give a graduation to $R$ and $R[X]$ by putting $R_{0}=k$, degree $U=n$ and degree $W_{j}=1$ for every $1 \leq j \leq s$, then they become graded rings. Moreover as degree $W^{(a)}/U=0$ in $R[X]$, we have $T_{s}=T_{s}[W_{1}, \ldots, W_{s}, U]$, where $T_{s}=k[(W^{(a)}/U), U \in T_{s}$ and $W_{j} \in T_{1}$ for every $1 \leq j \leq s$, is also a graded subring of $R[X]$. We put $p=T_{+}=(W_{1}, \ldots, W_{s}, U)T$. Of course $p$ is a prime ideal of $T$ and we have

**Proposition 1.** (1) $p=\text{rad}(UT)$.

(2) $T_{s}$ is a discrete valuation ring and $v_{s}(U)=n$. (Here $v_{s}$ denotes the discrete valuation corresponding to $T_{s}$.)

**Proof.** (1) For any $q \in \text{spec}(T)$ such that $q \supset U$, we have $W_{j}=U \cdot W_{j}/U \in q$. Then $W_{j} \in q$ for every $1 \leq j \leq s$. Thus $q \supset p$. Therefore we have $p=\text{rad}(UT)$.

(2) As $p \cap T_{s}=(0)$, we have $p \cap k=(0)$. Thus we may assume that $k$ is a field. Since $W_{j}/U \in T_{s}p, U=W_{j}^{*}/U/W_{j}^{*}$ and $W_{j}=W_{j} \cdot W_{j}^{*}/U \cdot U/W_{j}^{*}$ are contained in $W_{j}T_{s}$ for every $2 \leq j \leq s$. Therefore $pT_{s}=(W_{j})T_{s}$. Thus $T_{s}$ is a discrete valuation ring.

Next we prove $v_{s}(U)=n$. As $W^{(a)}=U \cdot W^{(a)}/U$, we have $p^{*}T_{s} \subset (U)T_{s}$. On the other hand, as $U=W_{j}^{*} \cdot U/W_{j}^{*} \in p^{*}T_{s}$, we have $p^{*}T_{s} \supset (U)T_{s}$. Thus we have $v_{s}(U)=n$.

We need the following Proposition 2 that is a result of Valla [3]. Here we give a simple proof for it.

**Proposition 2.** Let $A$ be a Macaulay ring and let $\{a_{1}, \cdots, a_{r}\}$ be an $A$-regular sequence. Then, for any positive integer $n$, $\mathcal{R}(A, (a_{1}, \cdots, a_{r})^{n})$ is a Macaulay ring.

**Proof.** We put $a=(a_{1}, \cdots, a_{r})$. Let $\varphi$ be an $A$-algebra endomorphism of $A[X, X^{-1}]$ defined by $\varphi(X)=X^{a}$ and $\varphi'$ be the restriction of $\varphi$ to $\mathcal{R}(A, a^{*})$.

\[
\mathcal{R}(A, a^{*}) \hookrightarrow A[X, X^{-1}]
\]

\[
\downarrow \varphi' \quad \downarrow \varphi
\]

\[
\mathcal{R}(A, a) \hookrightarrow A[X, X^{-1}]
\]

Then $\varphi'$ is an injection and its image is the Veronesean ring $\mathcal{R}(A, a^{(a)})$. Therefore if $\mathcal{R}(A, a)$ is a Macaulay ring, $\mathcal{R}(A, a^{*})$ is a Macaulay ring since $\mathcal{R}(A, a^{*})$ is a direct summand of $\mathcal{R}(A, a)$ and $\mathcal{R}(A, a)$ is integral over $\mathcal{R}(A, a^{*})$. (cf. [2] Proposition 12) Thus we may assume $n=1$. As $\mathcal{R}(A, a)/U \mathcal{R}(A, a)=G_{a}(A)$, we have only to prove that $G_{a}(A)$ is a Macaulay ring. This follows immediately from the fact that $G_{a}(A)$ is a
polynomial ring over \(A/a\) since \(\{a_1, \cdots, a_r\}\) is a regular sequence.

We put \(B=R[X]\). Note that we have also \(B=T[X]\).

**Lemma.** \(T=T_\mathfrak{p} \cap B\).

**Proof.** First, we assume that \(k\) is a field. Then \(T\) is a Macauly ring by the above proposition. Thus \(T=\bigcap_{htq=1} T_q\). Let \(q \in \text{spec}(T)\) of \(htq=1\) and suppose \(q \neq \mathfrak{p}\). As \(\mathfrak{p}=\text{rad}(UT)\), we have \(q \not\supset U\). Thus we have \(T_q \cap T[X]=B\) and hence \(T \supset T_q \cap B\). The opposite inclusion is trivial.

Now suppose that \(k\) is not necessarily a field and let \(f \in T_\mathfrak{p} \cap B\). Then \(rf \in T\) for some \(r \in k\setminus(0)\) by virtue of the result in case \(k\) is a field. On the other hand, since \(f \in B=T[X]\), we can express \(U^nf=g \in T\) for some integer \(N>0\). Therefore \(U^nf=rg\) in \(T\) where \(a=rf\). Since \(T/rT \cong R(k/rk[W_1, \cdots, W_s])\) and \(U\) is a nonzero divisor on \(R(k/rk[W_1, \cdots, W_s])\), we have \(\{r, U\}\) is a \(T\)-regular sequence. Therefore we have \(a \in rT\). Hence we have \(f \in T\).

**Proof of Theorem.** If \(k\) is a Krull domain, then \(R\) is also a Krull domain. As \(U\) is a prime element of \(R, B\) is a Krull domain. Also by Proposition 1 \(T_\mathfrak{p}\) is a discrete valuation ring. From these results and the above lemma, \(T\) is a Krull domain.

Next we have an exact sequence

\[
0 \longrightarrow \mathfrak{g}(k) \longrightarrow C(T) \longrightarrow C(B) \longrightarrow 0.
\]

Since we have \(C(B)=C(R)=C(k)\) by Cor. 7.3 and Prop. 8.9 in [1], the natural map \(C(k)\rightarrow C(T)\) makes the sequence split. Hence we have \(C(T) = C(k) \oplus \mathfrak{g}(k)\).

Now we must prove that \(\mathfrak{g}(k)\) is of order \(n\) in \(C(T)\). Put \(m=\text{order}(\mathfrak{g}(k))\), \((0<m \leq n)\), and we have \(m\cdot \mathfrak{g}(k)=\mathfrak{g}(aT)\) for some nonzero \(a \in Q(T)\), where \(Q(\cdot)\) denotes the quotient field. Hence we have \(aT = A: (A: p^n) = \bigcap_{htq=1} p^n T_q = p^n(T)\). Thus we have \(p^n(T)=aT\) for some nonzero \(a \in T\). Now we claim that \(p^n(T)\) is a graded ideal and \(a\) is a homogeneous element in \(T\) with degree \(m\). Indeed, we put \(\mathfrak{T}=Q(T)[U, W_1, \cdots, W_s]\). Then we have \(p^n(T)=p^n(U) \cap T=[p^n(T) \cap \mathfrak{T}] \cap T=p^n\mathfrak{T} \cap T\). Hence \(p^n(T)\) is a graded ideal. And \(p^n\mathfrak{T}=W_1^n \mathfrak{T}=a \mathfrak{T}\). Hence \(a\) is a homogeneous element and \(a\) equals to \(W_1^n\) up to unit in \(\mathfrak{T}\). Thus we have degree \(a=\deg W_1^n=m\).

If \(m<n\), \(p^n(T) \supset p^n(T) \not\supset U\). Thus we have \(U=ab \in aT\). Since \(a \in T\) and degree \(a=m\), we can write \(a=\sum_{\lambda} c_{(\lambda)} U^1 W_1^{\lambda_1} \cdots W_s^{\lambda_s}\) where \(n\lambda_0 + \lambda_1 + \cdots + \lambda_s = m\). Hence \(\lambda_0 = 0\) as \(m<n\). Thus we have \(a \in (W_1, \cdots, W_s)\) and

\(T \subset R[X]\), we can express \(U=ab=d/U^i e/U^j\) where \(d \in (W_1, \cdots, W_s)\) and
Thus we have $U^{i+j+1} \in (W_1, \ldots, W_s)R$, which is a contradiction since $U, W_1, \ldots, W_s$ are indeterminates. The proof of the theorem is now complete.

**Proof of Corollary.** As $k$ is a field, $T = \mathcal{R}(k[W_1, \ldots, W_s], (W_1, \ldots, W_s)^*)$ is a Macaulay ring by Proposition 2. By the theorem, $T$ is a Krull domain and $C(T) = C(k) \oplus \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/n\mathbb{Z}$. Since $T$ is a Noetherian ring and completely integrally closed, $T$ is a normal domain.

**References**


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