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The Class Group of the Rees Algebras over Polynomial Rings

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Introduction

Let A be a commutative ring with unit element and let $A[X]$ denote a polynomial ring over A with an indeterminate X. For an ideal $\mathfrak{a}$ of A we put $\mathscr{R}(A, \mathfrak{a})=A[\{aX; a\in\mathfrak{a}\}, X^{-1}]$, the A-subalgebra of $A[X, X^{-1}]$ generated by $\{aX; a\in\mathfrak{a}\}$ and X^{-1}, and we call it the Rees algebra of $\mathfrak{a}$ over A.

$\mathscr{R}(A, \mathfrak{a})$ is a graded subring of $A[X, X^{-1}]$, whose graduation is given by $\mathscr{R}_n(A, \mathfrak{a})=\mathfrak{a}^n X^n$ for $n\geqq 0$ and $\mathscr{R}_n(A, \mathfrak{a})=A$ for $n<0$. Note that $\mathscr{R}(A, \mathfrak{a})$ is canonically identified with the ring $\bigoplus_{n\in Z}\mathfrak{a}^n$ where $\mathfrak{a}^n=A$ for $n<0$.

The aim of this paper is to prove the following theorem.

THEOREM. *Let k be a Krull domain and let $W_1, \cdots, W_s$ be indeterminates over k. Then, for every positive integer n, $\mathscr{R}(k[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$ is a Krull domain and $C(\mathscr{R})=C(k)\oplus Z/nZ$. (Here $C(\cdot)$ denotes the divisor class group.)*

By the theorem we have the following result immediately.

COROLLARY. *If k is a field, then $\mathscr{R}(k[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$ is a Macaulay normal domain and $C(\mathscr{R})=Z/nZ$.*

§ 1. Proof of Theorem.

Let $k, W_1, \cdots, W_s, n$ be as in the introduction and let $X^{-1}=U$. We denote $\mathscr{R}(k[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$ by T. Let Λ_n be the set of the indexes $(\alpha)=(\alpha_1, \cdots, \alpha_s)$ where α_i's are nonnegative integers with $\sum_{j=1}^{s}\alpha_j=n$ and let $R=k[W_1, \cdots, W_s, U]$. Then $T=k[W_1, \cdots, W_s, U, \{W^{(\alpha)}/U\}_{\alpha\in\Lambda_n}]$ and T is a k-subalgebra of $R[X]$, where $W^{(\alpha)}$ denotes $W_1^{\alpha_1}\cdots W_s^{\alpha_s}$.

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Now we give a graduation to R and $R[X]$ by putting $R_0=k$, degree $U=n$ and degree $W_j=1$ for every $1\leqq j\leqq s$, then they become graded rings. Moreover as degree $W^{(\alpha)}/U=0$ in $R[X]$, $T=T_0[W_1, \cdots, W_s, U]$, where $T_0=k[\{W^{(\alpha)}/U\}]$, $U\in T_n$ and $W_j\in T_1$ for every $1\leqq j\leqq s$, is also a graded subring of $R[X]$. We put $\mathfrak{p}=T_+=(W_1, \cdots, W_s, U)T$. Of course $\mathfrak{p}$ is a prime ideal of T and we have

PROPOSITION 1. (1) $\mathfrak{p}=\mathrm{rad}(UT)$.

(2) *$T_{\mathfrak{p}}$ is a discrete valuation ring and $v_{\mathfrak{p}}(U)=n$. (Here $v_{\mathfrak{p}}$ denotes the discrete valuation corresponding to $T_{\mathfrak{p}}$.)*

PROOF. (1) For any $\mathfrak{q}\in \mathrm{spec}(T)$ such that $\mathfrak{q}\ni U$, we have $W_j^n=U\cdot W_j^n/U\in\mathfrak{q}$. Then $W_j\in\mathfrak{q}$ for every $1\leqq j\leqq s$. Thus $\mathfrak{q}\supset\mathfrak{p}$. Therefore we have $\mathfrak{p}=\mathrm{rad}(UT)$.

(2) As $\mathfrak{p}\cap T_0=(0)$, we have $\mathfrak{p}\cap k=(0)$. Thus we may assume that k is a field. Since $W_1^n/U\in T\backslash\mathfrak{p}$, $U=W_1^n\cdot U/W_1^n$ and $W_j=W_1\cdot W_jW_1^{n-1}/U\cdot U/W_1^n$ are contained in $W_1T_{\mathfrak{p}}$ for every $2\leqq j\leqq s$. Therefore $\mathfrak{p}T_{\mathfrak{p}}=(W_1)T_{\mathfrak{p}}$. Thus $T_{\mathfrak{p}}$ is a discrete valuation ring.

Next we prove $v_{\mathfrak{p}}(U)=n$. As $W^{(\alpha)}=U\cdot W^{(\alpha)}/U$, we have $\mathfrak{p}^nT_{\mathfrak{p}}\subset(U)T_{\mathfrak{p}}$. On the other hand, as $U=W_1^n\cdot U/W_1^n\in\mathfrak{p}^nT_{\mathfrak{p}}$ we have $\mathfrak{p}^nT_{\mathfrak{p}}\supset(U)T_{\mathfrak{p}}$. Thus we have $v_{\mathfrak{p}}(U)=n$.

We need the following Proposition 2 that is a result of Valla [3]. Here we give a simple proof for it.

PROPOSITION 2. *Let A be a Macaulay ring and let $\{a_1, \cdots, a_r\}$ be an A-regular sequence. Then, for any positive integer n, $\mathscr{R}(A, (a_1, \cdots, a_r)^n)$ is a Macaulay ring.*

PROOF. We put $\mathfrak{a}=(a_1, \cdots, a_r)$. Let φ be an A-algebra endomorphism of $A[X, X^{-1}]$ defined by $\varphi(X)=X^n$ and φ' be the restriction of φ to $\mathscr{R}(A, \mathfrak{a}^n)$.

$$\begin{array}{ccc} \mathscr{R}(A, \mathfrak{a}^n) & \hookrightarrow & A[X, X^{-1}] \\ \downarrow{\scriptstyle \varphi'} & & \downarrow{\scriptstyle \varphi} \\ \mathscr{R}(A, \mathfrak{a}) & \hookrightarrow & A[X, X^{-1}] \end{array}$$

Then φ' is an injection and its image is the Veronesean ring $\mathscr{R}(A, \mathfrak{a})^{(n)}$. Therefore if $\mathscr{R}(A, \mathfrak{a})$ is a Macaulay ring, $\mathscr{R}(A, \mathfrak{a}^n)$ is a Macaulay ring since $\mathscr{R}(A, \mathfrak{a}^n)$ is a direct summand of $\mathscr{R}(A, \mathfrak{a})$ and $\mathscr{R}(A, \mathfrak{a})$ is integral over $\mathscr{R}(A, \mathfrak{a}^n)$. (cf. [2] Proposition 12) Thus we may assume $n=1$. As $\mathscr{R}(A, \mathfrak{a})/U\mathscr{R}(A, \mathfrak{a})=G^{\cdot}_{\mathfrak{a}}(A)$, we have only to prove that $G^{\cdot}_{\mathfrak{a}}(A)$ is a Macaulay ring. This follows immediately from the fact that $G^{\cdot}_{\mathfrak{a}}(A)$ is a polynomial ring over $A/\mathfrak{a}$ since $\{a_1, \cdots, a_r\}$ is a regular sequence.

We put $B=R[X]$. Note that we have also $B=T[X]$.

LEMMA. $T=T_{\mathfrak{p}}\cap B$.

PROOF. First, we assume that k is a field. Then T is a Macaulay ring by the above proposition. Thus $T=\bigcap_{\mathrm{ht}\mathfrak{q}=1} T_{\mathfrak{q}}$. Let $\mathfrak{q}\in \mathrm{spec}(T)$ of $\mathrm{ht}\,\mathfrak{q}=1$ and suppose $\mathfrak{q}\neq\mathfrak{p}$. As $\mathfrak{p}=\mathrm{rad}(UT)$, we have $\mathfrak{q}\not\ni U$. Thus we have $T_{\mathfrak{q}}\cap T[X]=B$ and hence $T\supset T_{\mathfrak{p}}\cap B$. The opposite inclusion is trivial.

Now suppose that k is not necessarily a field and let $f\in T_{\mathfrak{p}}\cap B$. Then $rf\in T$ for some $r\in k\backslash(0)$ by virtue of the result in case k is a field. On the other hand, since $f\in B=T[X]$, we can express $U^N f=g\in T$ for some integer $N>0$. Therefore $U^N a=rg$ in T where $a=rf$. Since $T/rT\cong \mathscr{R}(k/rk[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$ and U is a nonzero divisor on $\mathscr{R}(k/rk[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$, we have $\{r, U\}$ is a T-regular sequence. Therefore we have $a\in rT$. Hence we have $f\in T$.

PROOF OF THEOREM. If k is a Krull domain, then R is also a Krull domain. As U is a prime element of R, B is a Krull domain. Also by Proposition 1 $T_{\mathfrak{p}}$ is a discrete valuation ring. From these results and the above lemma, T is a Krull domain.

Next we have an exact sequence

$$0\longrightarrow Z_{\mathrm{cl}(\mathfrak{p})}\longrightarrow C(T)\longrightarrow C(B)\longrightarrow 0 .$$

Since we have $C(B)=C(R)=C(k)$ by Cor. 7.3 and Prop. 8.9 in [1], the natural map $C(k)\to C(T)$ makes the sequence split. Hence we have $C(T)=C(k)\oplus Z_{cl(\mathfrak{p})}$.

Now we must prove that $\mathrm{cl}(\mathfrak{p})$ is of order n in $C(T)$. Put $m=\mathrm{order}(\mathrm{cl}(\mathfrak{p}))$, $(0<m\leqq n)$, and we have $m\cdot\mathrm{cl}(\mathfrak{p})=\mathrm{cl}(aT)$ for some nonzero $a\in Q(T)$, where $Q(\cdot)$ denotes the quotient field. Hence we have $aT=A:(A:\mathfrak{p}^m)=\bigcap_{\mathrm{ht}_T\mathfrak{q}=1}\mathfrak{p}^m T_{\mathfrak{q}}=\mathfrak{p}^{(m)}$. Thus we have $\mathfrak{p}^{(m)}=aT$ for some nonzero $a\in T$. Now we claim that $\mathfrak{p}^{(m)}$ is a graded ideal and a is a homogeneous element in T with degree m. Indeed, we put $\tilde{T}=Q(T_0)[U, W_1, \cdots, W_s]=Q(T_0)[W_1]$. Then we have $\mathfrak{p}^{(m)}=\mathfrak{p}^m T_{\mathfrak{p}}\cap T=[\mathfrak{p}^m T_{\mathfrak{p}}\cap\tilde{T}]\cap T=\mathfrak{p}^m\tilde{T}\cap T$. Thus $\mathfrak{p}^{(m)}$ is a graded ideal. And $\mathfrak{p}^m\tilde{T}=W_1^m\tilde{T}=a\tilde{T}$. Hence a is a homogeneous element and a equals to W_1^m up to unit in $\tilde{T}$. Thus we have degree $a=$ degree $W_1^m=m$.

If $m<n$, $\mathfrak{p}^{(m)}\supset\mathfrak{p}^{(n)}\ni U$. Thus we have $U=ab\in aT$. Since $a\in T$ and degree $a=m$, we can write $a=\sum_\lambda c_{(\lambda)}U^{\lambda_0}W_1^{\lambda_1}\cdots W_s^{\lambda_s}$ where $n\lambda_0+\lambda_1+\cdots+\lambda_s=m$. Hence $\lambda_0=0$ as $m<n$. Thus we have $a\in(W_1, \cdots, W_s)T$. As $T\subset R[X]$, we can express $U=ab=d/U^i\cdot e/U^j$ where $d\in(W_1, \cdots, W_s)R$ and

$e \in R$. Thus we have $U^{i+j+1} \in (W_1, \cdots, W_s)R$, which is a contradiction since $U, W_1, \cdots, W_s$ are indeterminates. The proof of the theorem is now complete.

PROOF OF COROLLARY. As k is a field, $T = \mathscr{R}(k[W_1, \cdots, W_s], (W_1, \cdots, W_s)^n)$ is a Macaulay ring by Proposition 2. By the theorem, T is a Krull domain and $C(T) = C(k) \oplus Z/nZ = Z/nZ$. Since T is a Noetherian ring and completely integrally closed, T is a normal domain.

References

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TOKYO, 158