Some Remarks on Subvarieties of Hopf Manifolds

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Introduction

A holomorphic automorphism $g$ of a complex space $\mathfrak{X}$ is called a \textit{contraction} to a point $O \in \mathfrak{X}$ if $g$ satisfies the following three conditions:

(i) $g(O)=O$,

(ii) $\lim_{\nu \to +\infty} g^\nu(x)=O$ for any point $x \in \mathfrak{X}$,

(iii) for any small neighborhood $U$ of $O$ in $\mathfrak{X}$, there exists an integer $\nu_0$ such that $g^\nu(U) \subset U$ for all $\nu \geq \nu_0$,

where $g^\nu$ is the $\nu$-times composite of $g$. By [2]*, the complex space $\mathfrak{X}$ which admits a contracting automorphism is holomorphically isomorphic to an algebraic subset of $C^N$ for some $N$. We identify $\mathfrak{X}$ to the algebraic subset of $C^N$. Then there exists a contracting automorphism $\tilde{g}$ of $C^N$ to the origin $O$ such that $\tilde{g}|_x=g$ ([2], [3]). Obviously the action of $\tilde{g}$ on $C^N-O$ is free and properly discontinuous. Hence the quotient space $H=C^N-O/\langle \tilde{g} \rangle$ is a compact complex manifold which is called a \textit{primary Hopf manifold}. Sometimes we indicate by $H^N$ an $N$-dimensional primary Hopf manifold. The compact complex space $\mathfrak{X}-O/\langle g \rangle$ is clearly an analytic subset of a primary Hopf manifold. A compact complex manifold $X$ of dimension $n \ (n \geq 2)$ is called a \textit{Hopf manifold} if its universal covering is holomorphically isomorphic to $C^n-O$ (Kodaira [4]).

The purpose of this paper is to show several properties of subvarieties of Hopf manifolds.

§ 1. Hopf manifolds.

The following proposition shows that it is sufficient to consider only subvarieties of primary Hopf manifolds.

\textbf{Proposition 1.} Any Hopf manifold is a submanifold of a (higher dimensional) primary Hopf manifold.

Received June 30, 1978

* In [2], the condition (iii) is forgotten.
Let $X$ be any Hopf manifold. Then, by definition, there exists a group $G$ of holomorphic transformations of $C^n-\{0\}$ such that $X=C^n-\{0\}/G$ ($n=\dim X \geq 2$). It follows from a theorem of Hartogs that any element of $G$ can be extended to a holomorphic transformation of $C^n$. Hence we may assume that each element of $G$ is a holomorphic transformation of $C^n$ which fixes the origin $O \in C^n$. By the same argument as in [4] pp. 694-695, $G$ contains a contraction.

For each element $x \in G$, we denote by $dx(O)$ the jacobian matrix at the origin $O \in C^n$.

**Lemma 1.** An element $x \in G$ is a contraction if and only if $|\det(dx(O))|<1$.

**Proof.** If $x \in G$ is a contraction, then any eigenvalue $\alpha$ of $dx(O)$ satisfies $|\alpha|<1$ (see [3] for the detail). Hence $|\det(dx(O))|<1$. Conversely, let $x$ be an element of $G$ satisfying $|\det(dx(O))|<1$. Let $g$ be a contraction contained in $G$. Since $C^n-\{0\}/\langle g \rangle$ is compact, the index of the infinite cyclic subgroup $\{g\}$ generated by $g$ is finite in $G$. Now assume that $x$ is not a contraction. Then $x^n$ is not a contraction for any integers $n$. Hence $x^n \neq g^n$ for any pair of integers $n$ and $m$ except $n=m=0$. This implies that $\{x\} \cap \{g\} = \{1\}$. This contradicts the fact that $\{g\}$ is of the finite index in $G$.

Let $U$ be a subgroup of $G$ defined by

$$U = \{x \in G: |\det(dx(O))| = 1\}.$$  

Obviously $U$ is a normal subgroup of $G$.

**Lemma 2.** There exists an infinite cyclic subgroup $Z$ of $G$ such that $G$ is the semi-direct product of $Z$ and $U$; $G = Z \cdot U$.

**Proof.** Define a group homomorphism $l: G \to R$ by

$$l(x) = -\log |\det(dx(O))| \quad (x \in G).$$  

Let $g_i \in G$ be a contraction. Then the index $d$ of the infinite cyclic group $\langle l(g_i) \rangle$ generated by $l(g_i)$ in $l(G)$ is finite. Hence $d^{-1}l(g_i)$ is a minimum positive element of $l(G)$. Let $g$ be an element of $G$ such that $l(g) = d^{-1}l(g_i)$. We put $Z = \{g\}$. Then it is clear that $G = Z \cdot U$. Q.E.D.

**Lemma 3.** $U$ is a finite normal subgroup of $G$.

**Proof.** Clear by Lemma 2.

Now continue the proof of Proposition 1. It is easy to see that any
holomorphic transformation \( u \) of \( \mathbb{C}^n \) which fixes the origin is linear, if \( u \) is of the finite order. Hence \( U \) is a finite subgroup of \( \text{GL}(n, \mathbb{C}) \).

Hence, by H. Cartan [1], \( \mathfrak{x} = \mathbb{C}^n / U \) is a complex space with unique possible singularity at \( \overline{O} \), where \( \overline{O} \) is the corresponding point to the origin \( O \in \mathbb{C}^n \).

The generator \( g \) of \( Z \) induces a contracting automorphism \( \overline{g} \) of \( \mathbb{C}^n \) such that \( \overline{g}(\overline{O}) = \overline{O} \). Hence \( X = \mathfrak{x} - \{\overline{O}\} / \langle \overline{g} \rangle \) is a submanifold of a primary Hopf manifold as we have seen in the introduction.

Q.E.D.

§ 2. Line bundles defined by divisors.

Let \( M \) be an arbitrary compact complex manifold and \( N \) be a divisor of \( M \). The line bundle \([N] \) defined by \( N \) is an element of \( H^1(M, \mathbb{C}^*) \). There is a natural homomorphism \( i: H^1(M, \mathbb{C}^*) \rightarrow H^1(M, 0^*) \) induced by the natural injection \( \mathbb{C}^* \rightarrow 0^* \). If \([N]\) is in the image of \( i \), then \([N]\) is called a flat line bundle. In other words, \([N]\) is locally flat if and only if its transition functions can be written by constant functions.

Now let \( \tilde{g} \) be any contracting automorphism of \( \mathbb{C}^N \) which fixes the origin \( O \in \mathbb{C}^N \). Then, by L. Reich ([6], [7]), we can choose a system of coordinates of \( \mathbb{C}^N \) such that \( \tilde{g} \) can be written in the following form:

\[
\begin{align*}
 z'_1 &= \alpha_1 z_1 \\
 z'_2 &= \alpha_2 z_2 \\
 & \vdots \\
 z'_{r_1} &= \alpha_{r_1} z_{r_1} \\
 z'_{r_1+1} &= \alpha_{r_1+1} z_{r_1+1} + P_{r_1+1}(z_1, \cdots, z_{r_1}) \\
 & \vdots \\
 z'_{r_1+r_2} &= \alpha_{r_1+r_2} z_{r_1+r_2} + P_{r_1+r_2}(z_1, \cdots, z_{r_1}) \\
 & \vdots \\
 z'_N &= \alpha_N z_N + P_N(z_1, \cdots, z_{r_1+r_2+\cdots+r_{\epsilon-1}}),
\end{align*}
\]

where \( 1 > |\alpha_1| \geq \cdots \geq |\alpha_N| > 0 \), \( \mu \) is the number of Jordan blocks of the linear part, \( P_j \) \((r_1 + \cdots + r_\epsilon < j \leq r_1 + \cdots + r_{\epsilon+1}) \) are finite sums of monomials \( z_1^{m_1} \cdots z_{r_\epsilon}^{m_\epsilon} \), which satisfy

\[
\alpha_j = \alpha_1^{m_1} \cdots \alpha_N^{m_N} , \\
m_1 + \cdots + m_{r_\epsilon} \geq 2 \text{ (all } m_i > 0). \]

Let \( \tilde{\omega}: \mathbb{C}^N - \{O\} \rightarrow H = \mathbb{C}^N - \{O\} / \langle \tilde{g} \rangle \) be the covering projection. For any analytic subset \( X \) in \( H \), the set \( \tilde{\omega}^{-1}(X) \) is an analytic subset in \( \mathbb{C}^N - \{O\} \).
If \( \dim X \geq 1 \), then by a theorem of Remmert-Stein, \( \mathfrak{X} = \tilde{\omega}^{-1}(X) \cup \{O\} \) is an analytic subset of \( C^N \). In what follows, we indicate by the script letters the analytic subsets in \( C^N \) corresponding in the above manner to the analytic subsets of \( H \) written by the Roman letters. An analytic subset is called a variety if it is irreducible.

Assume that \( X \) is an analytic subvariety in \( H \) of \( \dim X \geq 2 \) and that \( D \) is an analytic subvariety of codimension 1 in \( X \). It is clear that \( \mathfrak{X} \) and \( \mathscr{D} \) are both \( \mathcal{g} \)-invariant in \( C^N \), i.e., \( g(\mathfrak{X}) = \mathfrak{X} \) and \( g(\mathscr{D}) = \mathscr{D} \).

**Lemma 4** ([2]). There exists a non-constant holomorphic function \( f \) on \( \mathfrak{X} \) such that \( g^*f = \alpha f \) for some constant \( \alpha \) \((0 < |\alpha| < 1)\) and that \( f|_{\mathfrak{T}} = 0 \).

**Remark 1.** In [2], the word “variety” is used as “analytic set”.

Let \( X \) be a non-singular manifold. Consider \( f \) of Lemma 4 as a multiplicative multi-valued holomorphic function on \( X \) (K. Kodaira [4] p. 701). The divisor \( D_1 = (f) \) is well-defined. The equation \( g^*f = \alpha f \) implies that the line bundle \( [D_1] \) is flat of which the transition functions are some powers of \( \alpha \). We summarize these facts as follows.

**Theorem 1.** Let \( X \) be a submanifold of \( H \) and \( D \) an effective divisor on \( X \). Assume that \( \dim X \geq 2 \). Then then exists an effective divisor \( E \) on \( X \) such that the line bundle \( [D+E] \) is flat of which the transition functions are some powers of a certain constant \( \alpha \in C^* \) \((0 < |\alpha| < 1)\).

**Remark 2.** The following example shows that there are cases such that the “additional” effective divisor \( E \) of Theorem 1 is indispensable.

Let \((x_0, x_1, x_2, x_3)\) be a standard system of coordinates of \( C^4 \). Fix a complex number \( \alpha \) such that \( 0 < |\alpha| < 1 \). Let \( \mathfrak{g} \) be a contracting holomorphic automorphism of \( C^4 \) defined by

\[
\mathfrak{g} : (x_0, x_1, x_2, x_3) \mapsto (\alpha x_0, \alpha x_1, \alpha x_2, \alpha x_3) .
\]

Define \( \mathfrak{g} \)-invariant subvarieties of \( C^4 \) by

\[
\mathfrak{X} : x_0 x_1 = x_2 x_3
\]

and

\[
\mathfrak{A} : x_3 = 0 .
\]

Denote the intersection \( \mathfrak{X} \cap \mathfrak{A} \) by \( \mathcal{S} \). Then \( \mathcal{S} = \{x_0 - x_3 = 0\} \cup \{x_1 - x_3 = 0\} \). We put

\[
\mathcal{S}_1 = \{x_0 - x_3 = 0\}
\]
Then $S = \mathcal{L} - \{O\}/\langle g \rangle$, $S_1 = \mathcal{L}_1 - \{O\}/\langle g \rangle$ and $S_2 = \mathcal{L}_2 - \{O\}/\langle g \rangle$ are subvarieties of a compact complex manifold $X = \mathfrak{X} - \{O\}/\langle g \rangle$. It is clear that $[S_1 + S_2] = [S]$ is flat. We shall prove that either $[S_1]$ or $[S_2]$ is not flat. Assume that both $[S_1]$ and $[S_2]$ are flat. Let $\mathcal{U} = \{U_i\}$ be a sufficiently fine finite open covering of $X$. We represent $[S_1]$ as an $l$-cocycle $\{c_{1\lambda\mu}\} \in Z^1(\mathfrak{X}, C^*)$.

Since $\dim H^0(X, O[S_1]) > 0$, there exists a non-zero section $\varphi_1$ which vanishes exactly on $S_1$. Let $\varphi_{1\lambda} = c_{1t^\ell} \varphi_{1\mu}$ on $U_{\lambda} \cap U_{\mu}$. As we can easily see, $\eta_1 = \frac{d\varphi_{1\lambda}}{\varphi_{1\lambda}} = \frac{d\varphi_{1\mu}}{\varphi_{1\mu}} = \cdots$ is a meromorphic $1$-form on $X$. Since $\mathfrak{X} - \{0\}$ is simply connected, $f_1(x) = \exp \int^x \eta_1$ is a holomorphic function on $\mathfrak{X} - \{0\}$ such that $g \sim * f_1 = \beta_1 f_1 (\beta_1 \in C^*, 0 < |\beta_1| < 1)$ which vanishes exactly on $\mathcal{L}_1 - \{0\}$ with multiplicity 1. Since $\mathfrak{X}$ is normal at $O$, $f_1$ uniquely extends to a holomorphic function on $\tilde{\omega}^{-1}(X_{f})$. Hence, $\tilde{\omega}^{-1}(X_{f})$ is a countable union of points. Hence $\dim X_f = 0$. This contradicts $\dim X > 1$. This implies that either...
$[S_1]$ or $[S_2]$ is not flat.

**Remark 3.** If $\dim X = 2$, then $[D]$ is always flat ([3]).

§ 3. Some properties of subvarieties.

By Lemma 5 in [2], we have easily

**Proposition 2.** Let $Y_1$ and $Y_2$ be subvarieties of a (primary) Hopf
manifold $H$ such that $Y_1 \subset Y_2$ and $0 < n_1 = \dim Y_1 < n_2 = \dim Y_2$. Then there
exists a sequence of subvarieties $W_0, W_1, \ldots, W_p$ $(p = n_2 - n_1)$ in $H$ with
following properties:

(i) $W_0 = Y_1$, $W_p = Y_2$,

(ii) $W_i \subset W_{i+1}$ $(i = 0, \ldots, p - 1)$, $\dim W_i + 1 = \dim W_{i+1}$.

**Proposition 3.** Let $H^N = C^N / \langle \beta \rangle$ be a primary Hopf manifold. Then

(a) any positive dimensional subvariety in $H^N$ contains a curve,

(b) any irreducible curve in $H^N$ is non-singular elliptic,

(c) for any elliptic curve $C$ in $H^N$, there exist an eigenvalue $\alpha$ of
$\tilde{g}$, a constant $\beta$ and certain positive integers $m, n$ with $\alpha^n = \beta^m$ such that
$C$ is isomorphic to $C^* / \langle \beta \rangle$.

**Proof.** (a) Let $Y$ be a $n$-dimensional subvariety in $H^N (n \geq 1)$. For any
integer $k$ $(1 \leq k \leq N)$, the $(N-k)$-dimensional subspace $C^N - k$ defined by $z_1 =
\cdots = z_k = 0$ is $g$-invariant. There exists an integer $k$ such that $\dim (C^N - k \cap
\mathcal{Y}) = 1$. Then $\tilde{\omega}(C^N - (k-1) \cap \mathcal{Y}) - \{0\}$ is a 1-dimensional analytic subset of $Y$.

(b) Let $C$ be any irreducible curve in $H^N$. Then $C$ is a 1-dimensional analytic subset of $C^N$. Let $C_0$ be one of the irreducible components of $C$. Then, for some positive integer $n_0$, $g^{n_0}$ acts on $C_0$ as a contracting automorphism of $C_0$. Let $\lambda : C^* \rightarrow C_0$ be the normalization of $C_0$. Then $g^{n_0}$ naturally induces a contracting automorphism of $C^*$.

(c) Consider the $\tilde{g}$-invariant subspaces $C^N - k$ defined in (a). For $k = 0$, $C^N - k$ is the total space. Fix the integer $k$ $(0 \leq k \leq N - 1)$ such that
$\mathcal{Y} \subset C^N - k$ and $\mathcal{Y} \not\subset C^N - k - 1$. If $\mathcal{Y} \cap C^N - k - 1$ contains a point $p$ other than $O$, then $\mathcal{Y} \cap C^N - k - 1$ contains an infinite sequence of points $\tilde{g}(p) \rightarrow O (n = 1, 2, \ldots)$. Hence one of the irreducible components of $\mathcal{Y}$ is contained in $C^N - k - 1$. Since $\tilde{g}$ is transitive over all the irreducible components of $\mathcal{Y}$, this implies that $\mathcal{Y} \subset C^N - k - 1$, contradiction. Therefore $\mathcal{Y} \cap C^N - k - 1 = \{0\}$. Hence $f =
$z_{k+1} \mid C^{N-k}$, the restriction of $z_{k+1}$ to $C^{N-k}$, vanishes nowhere on $\mathscr{G}-\{O\}$. Moreover $f$ satisfies the equation $g^{*}f=\alpha_{k+1}f$. Hence we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathscr{G}-\{O\} & \xrightarrow{g} & \mathscr{G}-\{O\} \\
\uparrow f & & \uparrow f \\
C^{*} & \xrightarrow{\alpha_{k+1}} & C^{*}.
\end{array}
$$

This induces a covering $\tilde{f}: C\to C^{*}/\langle\alpha_{k+1}\rangle$. Since both $C$ and $C^{*}/\langle\alpha_{k+1}\rangle$ are non-singular elliptic curves, $\tilde{f}$ has no branch points by the Hurwitz's formula. Hence we get the following commutative diagram:

$$
\begin{array}{ccc}
\mathscr{G}-\{O\} & \xrightarrow{g} & \mathscr{G}-\{O\} \\
\uparrow f & & \uparrow f \\
C^{*} & \xrightarrow{\alpha_{k+1}} & C^{*}.
\end{array}
$$

This induces a covering $\tilde{f}: C\to C^{*}/\langle\alpha_{k+1}\rangle$. Since both $C$ and $C^{*}/\langle\alpha_{k+1}\rangle$ are non-singular elliptic curves, $\tilde{f}$ has no branch points by the Hurwitz's formula. Hence there exist $\beta\in C^{*}$ and positive integers $m, n$ such that $C\cong C^{*}/\langle\beta\rangle$ and $\alpha_{k+1}^{m}=\beta^{n}$.

Q.E.D.

REMARK 4. By Propositions 2 and 3 (a), it follows that any $n$-dimensional subvariety of a Hopf manifold contains subvarieties of arbitrary dimensions less than $n$.

§ 4. Subvarieties of algebraic dimension 0.

In general, let $M$ be a compact complex analytic subvariety. Then the field $\mathscr{M}(M)$ of all meromorphic functions on $M$ has the finite transcendental degree $a(M)$ over $C$. We call $a(M)$ the algebraic dimension of $M$. It is well-known that $a(M)\leq \dim M$. The number $\dim M-a(M)$ is called the algebraic codimension of $M$.

THEOREM 2. Let $Y$ be a subvariety of dimension $k$ in $N$-dimensional primary Hopf manifold $H^{N}$. Assume that $a(Y)=0$. Then the number of $(k-1)$-dimensional subvarieties in $Y$ is at most $N$.

Before proving the theorem, we shall make some preparations.

Let $\alpha_{1}, \ldots, \alpha_{N}$ be the eigenvalues of $\tilde{g}$ ((1)). Put $\theta_{j}=\log \alpha_{j}, (0\leq \arg \theta_{j}<2\pi, j=1, 2, \ldots, N)$. Let $K$ be a vector space over the field of rational numbers $\mathbb{Q}$ generated by the elements $2\pi\sqrt{-1}, \theta_{1}, \ldots, \theta_{N}$. Choose a basis $\tau_{0}, \tau_{1}, \ldots, \tau_{\lambda}$ of $K$ so that following conditions may be satisfied:

(i) $\tau_{0}=2\pi\sqrt{-1}$,
(ii) $\{\tau_{1}, \ldots, \tau_{\lambda}\}$ is a subset of $\{\theta_{1}, \ldots, \theta_{N}\}$,
(iii) for any $\nu\geq 1$, $\tau_{\nu}$ is linearly independent to $Q\tau_{0}+Q\tau_{1}+\cdots+Q\tau_{\nu-1}$,
(iv) if $\tau_{\nu}=\theta_{j}, \tau_{\mu}=\theta_{k}$ and $\nu<\mu$, then $j<k$.

It is easy to check that we can choose such a basis. We denote by $\alpha_{\nu}$ the element of $\{\alpha_{1}, \ldots, \alpha_{N}\}$ corresponding to $\tau_{\nu}$. Note that $\tau_{\nu}=\theta_{\nu_{\nu}}=\log \alpha_{\nu_{\nu}} (\nu=1, 2, \ldots, \lambda)$. If the equation
\[ \alpha_{i_{\nu}} = \alpha_{l_{\mu}}^{a_{1}} \cdots \alpha_{i_{\iota}}^{a_{l}} \quad (l < i_{\nu}) \]

holds for some integers \( a_{1}, \ldots, a_{\iota} \), then

\[ \tau_{\nu} = \theta_{:_{\nu}} = \sum_{j=1}^{\iota} a_{;} \theta_{j} + p\tau_{0} \quad (p \in \mathbb{Z}) \]

Since \( \sum_{j=1}^{l} a_{j} \theta_{\dot{f}} \) is written by a linear combination of \( \tau_{0}, \tau_{1}, \ldots, \tau_{\nu-1} \), this is absurd. Therefore \( \alpha_{i_{\nu}} \) has no such relations. Hence by (1),

\[ z'_{i_{\nu}} = \alpha_{i_{\nu}} z_{i} \quad (\nu = 1, 2, \ldots, \lambda) \]

PROOF OF THEOREM 2. We may assume that \( Y \) can not be contained any primary Hopf manifold of dimension less than \( N \). Let \( D \) be a subvariety of codimension 1 in \( Y \). By Lemma 4, \( \mathcal{D} \) is contained in the zero locus of a non-constant holomorphic function \( f \) on \( \mathcal{Y} \) such that \( \bar{g}*f = \alpha f \) \( (0 < |\alpha| < 1) \). There exist some integers \( m, m_{1}, \ldots, m_{\lambda} \) such that

\[ \alpha^{m} = \alpha_{t_{1}}^{m_{1}} \cdots \alpha_{i_{\lambda}}^{m_{\lambda}}. \]

Put

\[ h = z_{1}^{m_{1}} \cdots z_{i_{\lambda}}^{m_{\lambda}}. \]

Since \( Y \) is not contained in any lower dimensional primary Hopf manifold, \( h \) is not equal to zero on \( \mathcal{Y} \). Hence both \( f^{m} \) and \( h \) are eigenfunctions of \( \bar{g}* \) of which the eigenvalues are the same \( \alpha^{m} \). Then \( h/f^{m} \) defines a non-zero meromorphic function on \( Y \). By the assumption \( a(Y) = 0 \), \( h/f^{m} = \) constant \( = c \neq 0 \). Hence we get

\[ (3) \quad h = cf^{m}. \]

Let \( Z_{i_{\nu}} (\nu = 1, \ldots, \lambda) \) be analytic subsets of \( Y \) corresponding to \( \{ z_{i_{\nu}} = 0 \} \cap \mathcal{Y} \). The equation (3) implies that \( D \) is contained in \( \bigcup_{\nu=1}^{\lambda} Z_{i_{\nu}} \). Since \( \lambda \leq N \), this proves the theorem.

Q.E.D.

§ 5. \( C^{*} \)-actions.

PROPOSITION 4. There exists a holomorphic mapping

\[ \bar{\varphi}: C \times C^{N} \longrightarrow C^{N} \quad \omega \]

\[ (t, z) \longmapsto \bar{\varphi}_{t}(z) \]

which satisfies the following properties:

(i) for every \( t \in C \), \( \bar{\varphi}_{t} \) is a holomorphic automorphism of \( C^{N} \) which fixes the origin,
(ii) \( \tilde{\varphi}_{t+s} = \tilde{\varphi}_t \circ \tilde{\varphi}_s \)

(iii) There exists an integer \( n_0 \) such that \( \tilde{\varphi}_1 = \tilde{g}^{n_0} \),

(iv) Every \( \tilde{g} \)-invariant subvarieties in \( C^N \) is \( \tilde{\varphi}_t \)-invariant for all \( t \in C \).

We say that an analytic subset of \( C^N \) is \( \tilde{\varphi} \)-invariant, if it is \( \tilde{\varphi}_t \)-invariant for all \( t \in C \).

**Proof.** Let \( \alpha_{i_1}, \cdots, \alpha_{i_N} \) be the eigenvalues of \( \tilde{g} \) considered in \( \S 4 \). For any eigenvalue \( \alpha_j \) of \( \tilde{g} \), there exist some integers \( m_j, m_{i_j}, \cdots, m_{i_N} \) such that

\[
\alpha_j^{m_j} = \alpha_{i_1}^{m_{i_1}} \cdots \alpha_{i_N}^{m_{i_N}} \quad (j = 1, 2, \ldots, N).
\]

Put \( n_0 = m_1 \cdots m_N \) and \( g_0 = \tilde{g}^{n_0} \). We define

(4) \( \alpha^t_{i_\nu} = \exp t \tau_{\nu} \) \( (t \in C, \nu = 1, 2, \ldots, \lambda) \),

and

(5) \( \alpha_i^{n_0 t} = \exp \left( t n_j \sum_{\nu=1}^{\lambda} m_{i_\nu} \tau_{\nu} \right) \) \( (n_j = n_0 m_j^{-1}, j = 1, 2, \ldots, N) \).

Let \( R(\alpha_{i_1}^{n_0}, \cdots, \alpha_{i_N}^{n_0}) = 1 \) be any relation among the eigenvalues of \( g_0 \), where \( R(u_i, \cdots, u_N) \) is a product of some (possibly negative) powers of \( u_i \) \( (i = 1, 2, \ldots, N) \), \( u_i \) being indeterminates. Now let \( R(u_i, \cdots, u_N) = u_i^{a_1} \cdots u_N^{a_N} \) \( (a_j \in \mathbb{Z}) \). Then, for \( t \in C \),

(6) \[
R(\alpha_i^{n_0 t}, \cdots, \alpha_N^{n_0 t}) = \alpha_{i_1}^{a_1 n_0 t} \cdots \alpha_{i_N}^{a_N n_0 t} = \exp \left( t \sum_{j=1}^{N} a_j n_j \sum_{\nu=1}^{\lambda} m_{i_\nu} \tau_{\nu} \right) = \exp \left( t \sum_{j=1}^{N} \left( \sum_{\nu=1}^{\lambda} a_j n_j m_{i_\nu} \right) \tau_{\nu} \right).
\]

Put \( t = 1 \) in (6). Then we get

\[
\sum_{j=1}^{N} \left( \sum_{\nu=1}^{\lambda} a_j n_j m_{i_\nu} \right) \tau_{\nu} = p \tau_0 \quad (p \in \mathbb{Z}).
\]

Hence we get \( p = 0 \) and \( \sum_{\nu=1}^{\lambda} a_j n_j m_{i_\nu} = 0 \) \( (\nu = 1, 2, \ldots, \lambda) \). Therefore

(7) \( R(\alpha_1^{n_0 t}, \cdots, \alpha_N^{n_0 t}) = 1 \)

for all \( t \in C \). Put \( \beta_j = \alpha_j^{n_0} \). By (4), the \( j \)-th coordinate of the point \( g_0(z) \) is given by

(8) \[
(g_0(z))_j = \beta_j^t \{ z_j + Q_j(n, z_1, \cdots, z_{j-1}) \},
\]
where \( Q_j \) is a polynomial of \( n, z_1, \cdots, z_{j-1} \). Replace \( n \) and \( \beta_j^i \) of (8) by \( t \) and \( \alpha_j^i \beta_i = \beta_j^i \), respectively. Then we get a holomorphic automorphism \( \tilde{\varphi}_t \) of \( C^n \) defined by

\[
(\tilde{\varphi}_t(z))_j = \beta_j^i(z_j + Q_j(t, z_1, \cdots, z_{j-1}))
\]

We shall prove that \( \tilde{\varphi} = \{\tilde{\varphi}_t\}_{t \in C} \) satisfies the desired conditions. The condition (i) and (iii) are clearly satisfied. To prove the condition (ii) is satisfied we put

\[
z = \begin{pmatrix} z_1 \\ \vdots \\ z_N \end{pmatrix}, \quad Q(t, z) = \begin{pmatrix} Q_1(t, z) \\ \cdots \\ Q_N(t, z) \end{pmatrix} \quad \text{and} \quad A^i = \begin{pmatrix} \beta_1^i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \beta_N^i \end{pmatrix}.
\]

We write \( \tilde{\varphi}_t(z) \) as

\[
(9) \quad \tilde{\varphi}_t(z) = A^i(z + Q(t, z)).
\]

Again we put

\[
d(t, s, z) = \tilde{\varphi}_{t+s}(z) - \tilde{\varphi}_t \circ \tilde{\varphi}_s(z).
\]

It is sufficient to prove that \( d(t, s, z) \) vanishes identically. By (9),

\[
d(t, s, z) = A^{i+t}(z + Q(t + s, z)) - A^i(A^s(z + Q(s, z)) + Q(t, A^s(z + Q(s, z))))
\]

\[
= A^{i+t}Q(t + s, z) - A^{i+t}Q(s, z) - A^iQ(t, A^s(z + Q(s, z))).
\]

Let \( Q_j(s, z) = \sum q_{i_1, \cdots, i_{j-1}}(s)z_1^{i_1} \cdots z_j^{i_{j-1}} \) be the \( j \)-th component of \( Q(s, z) \), where \( i_1, \cdots, i_{j-1} \) satisfy \( \beta_1^{i_1} \cdots \beta_j^{i_j-1} = \beta_j \) and \( i_j > 0 \). Then, by (7),

\[
Q_j(t, A^s(z + Q(s, z))) = \sum q_{i_1, \cdots, i_{j-1}}(t)(\beta_1^{i_1}(z_1 + Q_1(s, z))^{i_1} \cdots (z_{j-1} + Q_{j-1}(s, z))^{i_{j-1}}.
\]

Hence we get

\[
(12) \quad A^iQ(t, A^s(z + Q(s, z))) = A^{i+t}Q(t, z + Q(s, z)).
\]

Combining (11) with (12), we obtain

\[
d(t, s, z) = A^{i+t}(Q(t + s, z) - Q(s, z) - Q(t, z + Q(s, z))).
\]

Hence it is sufficient to show that

\[
d_i(t, s, z) = Q(t + s, z) - Q(s, z) - Q(t, z + Q(s, z))
\]

vanishes identically. Note that every component of \( d_i(t, s, z) \) is a poly-
nomial of $t$, $s$, and $z$.

Fix any integer $t=m$. Since $d_i(m, n, z)$ vanishes identically for any $n \in \mathbb{Z}$, the algebraic subset in $C^{N+1}$ defined by

$$\{(s, z) \in C^{N+1} : d_i(m, s, z)=0\}$$

contains infinitely many $N$-dimensional subspaces of $C^{N+1}$. Hence we infer that $d_i(m, s, z)$ vanishes identically for any integer $m$. Again, since $d_i(t, s, z)=0$ for any $m \in \mathbb{Z}$, the algebraic subset in $C^{N+2}$ defined by $d_i(t, s, z)=0$ contains infinitely many $(N+1)$-dimensional subspaces of $C^{N+2}$. Hence we conclude that $d_i$ vanishes identically on $C^{N+2}$. Therefore the condition (ii) is satisfied.

Next we prove that the condition (iv) is satisfied. We need the following

**Lemma 5.** Let $\mathcal{Z}$ be a $\mathcal{F}$- and $\phi$-invariant analytic subvariety in $C^N$. Let $\mathcal{X}$ be a pure 1-codimensional $\mathcal{F}$-invariant analytic subset of $\mathcal{Z}$. Then each irreducible component of $\mathcal{X}$ is $\tilde{\phi}$-invariant.

**Proof.** By Lemma 4, there exists a holomorphic function $f$ on $\mathcal{Z}$ such that $\mathcal{F}^* f = \alpha f$ ($0 < |\alpha| < 1$) and that $f|_x = 0$. Here we shall prove the following equation:

$$(13) \quad \tilde{\phi}^*_i f = \alpha^i f .$$

Once the equation (13) is proved, the lemma is clear. In fact, each irreducible component of $\mathcal{X}$ is an irreducible component of the zero locus of $f$. Since everything continuously varies depending on $t$, (13) implies that the irreducible components of $\mathcal{X}$ is $\tilde{\phi}$-invariant.

We put

$$M(\alpha) = \{h \in \mathcal{O}_\mathcal{Z} : \mathcal{F}^* h = \alpha h\} .$$

Then $M(\alpha)$ is a finite dimensional vector space over $C$ (cf. [2]). Let $\sigma_1, \cdots, \sigma_s$ be a basis of $M(\alpha)$. Put $\sigma_i^t = \sigma_i(\tilde{\phi}_t(z))$ ($i = 1, 2, \cdots, s$). Since $\mathcal{Z}$ is $\tilde{\phi}_t$-invariant, the elements $\sigma_1^t, \cdots, \sigma_s^t$ form another basis of $M(\alpha)$. Hence there exist some constants $c_{ij}(t)$ depending on $t$ such that

$$\sigma_i^t = \sum_{j=1}^s c_{ij}(t) \sigma_j .$$

We claim that $C(t) = (c_{ij}(t))$ is holomorphically dependent on $t$. In fact, we can choose points $z_1, \cdots, z_s \in \mathcal{Z}$ such that
is a non-singular matrix. Then,

$$\left(\begin{array}{lll}
\sigma_1(z_1) & \cdots & \sigma_1(z_s) \\
\vdots & & \vdots \\
\sigma_s(z_1) & \cdots & \sigma_s(z_s)
\end{array}\right) S^{-1} = C(t) .$$

Since the left hand side of (14) is holomorphically dependent on $t$, $C(t)$ is holomorphic.

It is easy to see that $\{C(t)\}_{t \in \mathbb{C}}$ is a 1-parameter subgroup of $\text{GL}(s, \mathbb{C})$, satisfying the equality,

$$C(n) = a^n I \quad (n \in \mathbb{Z}) .$$

Hence there exist a matrix $A$ which has the Jordan canonical form and a non-singular matrix $P$ such that

$$C(t) = P^{-1} \exp(tA)P .$$

By (15), $A$ is a diagonal matrix. Put $P^{-1} \sigma_j = \tau_j \ (j = 1, 2, \cdots, s)$. Then,

$$\tau_j^t = (\text{ext } ta_j) \tau_j \quad (j = 1, 2, \cdots, s) ,$$

where $a_1, \cdots, a_s$ are the diagonal components of $A$. Comparing the initial terms of the both sides of (16), we get

$$\exp ta_j = \exp \sum_{\nu=1}^{\lambda} t n_{i_{\nu}} \tau_{\nu} \quad (j = 1, 2, \cdots, s) ,$$

for some integers $n_{i_{\nu}}$. Letting $t = 1$, we get

$$\alpha = \exp a_j = \exp \sum_{\nu=1}^{\lambda} n_{i_{\nu}} \tau_{\nu} \quad (j = 1, 2, \cdots, s) ! ,$$

Hence for any $i$ and $j$,

$$\sum_{\nu=1}^{\lambda} (n_{j_{\nu}} - n_{i_{\nu}}) \tau_{\nu} = p_{ij} \tau_0 ,$$

choosing some integers $p_{ij}$. Since $\tau_0, \tau_1, \cdots, \tau_2$ are linearly independent over $\mathbb{Q}$, this implies that $n_{i_{\nu}} = n_{j_{\nu}}$ and $p_{ij} = 0$. Hence $\exp ta_j = \exp ta_i$ for any $i$ and $j$. Therefore $C(t)$ is a scalar matrix:

$$C(t) = \alpha^t I \quad (\alpha^t = \exp ta_i) .$$
Since \( f \in M(\alpha) \), \( f \) can be expressed as
\[
f = c_1 \tau_1 + \cdots + c_\pi \tau_\pi \quad (c_\pi \in C).
\]
Then \( \bar{\varphi}^*_t f = \sum_j c_j \bar{\varphi}^*_t \tau_j = \alpha^t \sum c_j \tau_j = \alpha^t f \).

Q.E.D.

Proof of (iv). By Lemma 5 [2], there exists a sequence \( \{ \mathcal{W}_j; j = 0, 1, \cdots, \pi \} \) of \( \mathcal{g} \)-invariant subvarieties of \( C^n \) such that \( \mathcal{W}_0 = \) a given \( \mathcal{g} \)-invariant subvariety \( \mathcal{W} \), \( \mathcal{W}_j \subset \mathcal{W}_{j+1} \), \( \dim \mathcal{W}_j + 1 = \dim \mathcal{W}_{j+1} \) and \( \mathcal{W}_\pi = C^n \) (\( \pi = N - \dim \mathcal{W}_0 \)). Since \( C^n \) is obviously \( \mathcal{g} \)- and \( \mathcal{g} \)-invariant, we infer that \( \mathcal{W} \) is \( \mathcal{g} \)-invariant by the previous lemma.

Q.E.D.

As a corollary, we obtain

**Theorem 3.** For any primary Hopf manifold \( H^N \), there exists another primary Hopf manifold \( H'^N \) with following properties:

(1) \( H'^N \) is a finite cyclic unramified covering of \( H^N \),
(2) \( H^N \) has a free \( C^* \)-action \( \varphi = \{ \varphi_\tau \}_{\tau} \) such that every positive dimensional subvariety in \( H'^N \) is \( \varphi \)-invariant.

**Proof.** Let \( H' = C^N - \{ O \}/\langle \mathcal{g}^n \rangle \). Then everything is clear from Proposition 4.

**Corollary.** The Euler number of a submanifold of a Hopf manifold is equal to 0.

**Proof.** By Theorem 3, every submanifold of a Hopf manifold has a finite unramified covering which admits a free \( S^1 \)-action. Hence the Euler number vanishes.

Q.E.D.


Let \( Y \) be a \( n \)-dimensional \( (n \geq 2) \) subvariety of a primary Hopf manifold \( H^N \). Take another primary Hopf manifold \( H'^N \) of Theorem 3. Let \( \omega: H'^N \rightarrow H^N \) be the covering map. We denote by \( Y' \) a connected component of \( \omega^{-1}(Y) \).

**Theorem 4.** The algebraic dimension of \( Y \) is \( n - 1 \) if and only if the \( C^* \)-action \( \varphi \) on \( Y' \) reduces to a complex torus action.

**Proof.** Assume that \( a(Y) = n - 1 \). Since \( a(Y') = a(Y) = n - 1 \), \( Y' \) has an \( (n - 1) \)-dimensional algebraic family of elliptic curves.

The moduli of curves depends continuously on the parameters. Hence, by Proposition 3, the moduli are constant. Since every curve in \( Y \) is \( \mathcal{g} \)-invariant, the \( C^* \)-action reduces to a complex torus action on the open dense subset of \( Y' \) and therefore on the whole \( Y' \).
Conversely, assume that $\mathcal{P}$ reduces to a complex torus action $\mathcal{V}$ on $Y'$. Then $\mathcal{V}'$ is an affine variety in $C^n$ with the $C^*$-action $\mathcal{V}'$ induced by $\mathcal{P}$. Moreover the action $\mathcal{V}$ is compatible with $g'$, where $g'$ is a contracting automorphism to $O$ of $C^n$ defining $H'$. It is not difficult to check that the $C^*$-action $\mathcal{V}'$ on $Y'$ is algebraic. (Construct a contracting automorphism on $C \times \mathcal{V}' \times \mathcal{V}'$ which leaves invariant the closure $\overline{\Gamma}$ of the graph $\Gamma$ of $\mathcal{V}$, where $\overline{\Gamma}$ is an analytic subset of $C \times \mathcal{V}' \times \mathcal{V}'$. Use the result of [2] to show that $\overline{\Gamma}$ is an algebraic subset of $C \times \mathcal{V}' \times \mathcal{V}'$.) Hence, by Proposition (1.1.3) in Orlik-Wagreich [5], there is an embedding $j: \mathcal{V}' \to C^{N'}$ for some $N'$ and a $C^*$-action $\mathcal{V}'$ on $C^{N'}$ such that $j(\mathcal{V}')$ is $\mathcal{V}'$-invariant and that $\mathcal{V}'$ induces $\mathcal{V}$ on $\mathcal{V}'$. Moreover, by a suitable choice of coordinates $(z_1, \cdots, z_N)$ on $C^{N'}$, the action $\mathcal{V}'$ on $C^{N'}$ can be written as

$$\tilde{\mathcal{V}}'(\rho, (z_1, \cdots, z_N)) = (\rho^{q_1}z_1, \cdots, \rho^{q_N}z_N),$$

where the $q_i$'s are positive integers. There exists a constant $\alpha$ such that $\tilde{\mathcal{V}}_{\alpha}$ induces $g'$ on $\mathcal{V}'$. Then $Y' = \mathcal{V}' - (O)/\langle g' \rangle$ can be considered as a submanifold of $C^{N'} - (O)/\langle \tilde{\mathcal{V}}_{\alpha} \rangle$.

The following theorem is known.

**Theorem (Ueno [8]).** Let $M$ be a subvariety of a compact complex variety $M_0$. Then

$$\dim M_1 - a(M_1) \leq \dim M_0 - a(M_0).$$

(18)

Now it is clear that $a(C^{N'} - (O)/\langle \mathcal{V}' \rangle) = N' - 1$. Hence, by (18), we get $a(Y') \geq \dim Y' - 1$. Since $a(Y') < \dim Y'$, we obtain $a(Y') = a(Y) = n - 1$.

Q.E.D.

**Remark 5.** Topologically, any submanifold of a Hopf manifold is diffeomorphic to a fibre bundle over a 1-dimensional circle of which the transition function has a finite order as an element of the diffeomorphism group of the fibre. This can be seen without difficulty from Theorem 3.

**Remark 6.** A compact complex surface $S$ is a submanifold of a Hopf manifold if and only if $S$ is a relatively minimal surface of class $VI_0$, $VII_0$-elliptic or a Hopf surface (see [3] for the proof of the "if" part). Let $S$ be a submanifold of a Hopf manifold. It is clear by Proposition 3 that $S$ is relatively minimal. By Theorem 1, $S$ is not algebraic. Hence $a(S) \leq 1$. Assume that $a(S) = 1$. Then, by Theorem 1, there exists a flat line bundle $L$ on $S$ such that the mapping $\Phi_L: S \to P^*$ defined by the linear system $|L|$ gives an algebraic reduction of $S$ which is defined everywhere. Put $\Delta = \Phi_L(S)$. Let $\eta$ be the line bundle on $\Delta$ associated to
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a hyperplane section of $\Delta$. Then we have $\Phi_{L}^{*}\eta=L$. We note that every fibre of $\Phi_{L}: S\rightarrow\Delta$ is a non-singular elliptic curve (Proposition 3). We indicate by $b_{i}(M)$ the $i$-th Betti number of a manifold $M$. It is clear that $b_{i}(\Delta)\leq b_{i}(S)\leq b_{i}(\Delta)+2$. Assume first that $b_{i}(\Delta)=b_{i}(S)$. Since $L$ is a flat line bundle on $S$, $L$ is raised from a group representation $\rho$ of $H_{1}(S, Z)$ into $C^{*}$. Let $m$ be a certain positive integer such that $\rho^{m}$ is trivial on the torsion part of $H_{1}(S, Z)$. Then, in view of $b_{i}(\Delta)=b_{i}(S)$, there exists a flat line bundle $\xi$ on $\Delta$ such that $\Phi_{L}^{*}\xi=L^{*}$. Hence we get $\Phi_{L}^{*}\xi=\Phi_{L}^{*}\eta^{m}$. Since $\Phi_{L}^{*}: H^{1}(\Delta, O^{*})\rightarrow H^{1}(S, O^{*})$ is an injection, this implies that the ample line bundle $\eta$ on $\Delta$ is flat. This is absurd. Hence we get $b_{i}(\Delta)<b_{i}(S)$. Next assume that $b_{i}(S)=b_{i}(\Delta)+2$. By Corollary to Theorem 3, we get $b_{i}(S)=2b_{i}(\Delta)+2$. This implies that the dual of the homology class represented by a general fibre is a Betti base of $H^{2}(S, Z)$. This contradicts Theorem 1. Hence we conclude that $b_{i}(S)=b_{i}(\Delta)+1$. Therefore $b_{i}(S)$ is odd. Hence $S$ is either a surface of VI$_{o}$ or VII$_{o}$-elliptic. Consider the case $a(S)=0$. By the classification theory of surfaces [4], a relatively minimal surface with no non-constant meromorphic functions and vanishing Euler number is either a complex torus or a surface of VII$_{o}$. A complex torus has a positive algebraic dimension if it contains a divisor. Hence by Proposition 3 we infer that $S$ is of VII$_{o}$-class. Moreover $b_{1}(S)=1$ and $b_{2}(S)=0$. Hence, by Theorem 34 [4], $S$ is a Hopf surface.

References


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