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A Generalization of the Fourier-Borel Transformation for the Analytic Functionals with non Convex Carrier

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Introduction

Let K be a compact set of $\boldsymbol{C}$. For an analytic functional with carrier in K, $T\in\mathscr{O}'(K)$, we define the Fourier-Borel transformation by

$$\mathscr{F}(T)(u)=\langle T_z, \exp(uz)\rangle . \tag{0.1}$$

If K is convex, it is classical that the Fourier-Borel transformation establishes a linear isomorphism of the space $\mathscr{O}'(K)$ onto the space of the entire functions of exponential type in K, $\mathrm{Exp}(\boldsymbol{C}, K)$. (Polya's representation, see for example Chapter 5 of Boas [1]. For the general theory of the Fourier-Borel transformation, see Martineau [6].) If K is not convex, this theorem is false. We shall consider in this paper the case where K is an annulus with center at the origin. Let $\lambda\neq 0$ be a fixed complex number. For $T\in\mathscr{O}'(K)$, we define the transformation $\mathscr{F}_\lambda$ by

$$\mathscr{F}_\lambda(T)(u, v)=\left\langle T_z, \exp\left(\lambda\left(uz+\frac{v}{z}\right)\right)\right\rangle . \tag{0.2}$$

This simple transformation $\mathscr{F}_\lambda$ generalizes the Fourier-Borel transformation in the case of annulus and we can determine the image of $\mathscr{O}'(K)$ under the transformation $\mathscr{F}_\lambda$ (Theorem 4.2). (Kiselman [4] and Martineau [7] considered another kind of generalizations of the Fourier-Borel transformation.)

On the other hand, let S^{n-1} be the $n-1$ dimensional sphere and $\mathscr{B}(S^{n-1})=\mathscr{A}'(S^{n-1})$ the space of hyperfunctions (analytic functionals) on the sphere. For $T\in\mathscr{B}(S^{n-1})$, Hashizume, Kowata, Minemura and Okamoto [2] defined the transformation $\mathscr{P}_\lambda$ by

$$\mathscr{P}_\lambda(T)(x)=\langle T_\omega, \exp(i\lambda\langle x, \omega\rangle)\rangle . \tag{0.3}$$

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They constructed a space $\tilde{\mathscr{B}}(S^{n-1})$ strictly larger than $\mathscr{B}(S^{n-1})$ and claimed $\mathscr{P}_\lambda$ can be extended to the space $\tilde{\mathscr{B}}(S^{n-1})$. They proved the image of $\tilde{\mathscr{B}}(S^{n-1})$ under the transformation $\mathscr{P}_\lambda$ is precisely the space of all C^∞ functions on $\boldsymbol{R}^n$ satisfying the differential equation

$$(\Delta+\lambda^2)f=0\,, \tag{0.4}$$

where Δ is the Laplacian on $\boldsymbol{R}^n$. This result can be understood in the frame work of the Ehrenpreis-Palamodov fundamental principl. I am interested in the following questions:

What is the space $\tilde{\mathscr{B}}(S^{n-1})$ and what is the $\mathscr{P}_\lambda$-image of $\mathscr{B}(S^{n-1})$? When $n=2$, that is, in the case of circle, Helgason [3] gave a meaning to the space $\tilde{\mathscr{B}}(S^1)$ as the space of "entire functionals". (We can give a meaning to $\tilde{\mathscr{B}}(S^{n-1})$ for general n. See our forthcoming paper [8].) An answer to the second question can be given, in the case $n=2$, by our knowledge on the transformation $\mathscr{F}_\lambda$ (Theorem 7.3(ii)).

The plan of this paper is as follows: §§1 and 2 are preliminary studies on the spaces of analytic functions and analytic functionals related to the unit circle S^1. In §1 we define the space $\mathscr{O}(S^1)$ of real analytic functions on the unit circle S^1 and its subspaces, especially the space $\mathscr{O}(\boldsymbol{C}^*)$ of holomorphic functions on $\boldsymbol{C}^*$ and the space $\mathrm{Exp}(\boldsymbol{C}^*)$ of holomorphic functions of exponential type on $\boldsymbol{C}^*$. We characterize these spaces by the growth conditions on their Laurent coefficients. In §2 we define the spaces of analytic functionals by the duality. $\mathscr{O}'(S^1)$ is the space of analytic functionals on the circle and $\mathrm{Exp}'(\boldsymbol{C}^*)$ is the space of "entire functionals" of Helgason [3]. We characterize also these spaces by the growth conditions of their Laurent coefficients.

In §3, we study the space $\mathscr{O}_{(\lambda)}(\boldsymbol{C}^2)$ of the entire functions $F(u, v)$ satisfying the differential equation $(\partial^2/\partial u\partial v)F=\lambda^2F$. Our main remark is that the function $F\in\mathscr{O}_{(\lambda)}(\boldsymbol{C}^2)$ is completely defined by its restrictions $F(u, 0)$ and $F(0, v)$ (Theorem 3.1). In §4 we define the transformation $\mathscr{F}_\lambda$ for the analytic functionals by the formula (0.2) and determine the $\mathscr{F}_\lambda$-images of the spaces of analytic functionals introduced in §2. In §5 we determine the $\mathscr{F}_\lambda$-images of the spaces of analytic functions introduced in §1. But the description of the $\mathscr{F}_\lambda$-images becomes more complicated than in §4. In §6, we shall sum up the properties of the function F of $\mathscr{O}_{(\lambda)}(\boldsymbol{C}^2)$.

In the final section §7, we apply our preceeding results to the study of the transformations $\mathscr{P}_\lambda$. We can determine, among others, the $\mathscr{P}_\lambda$-images of $\mathrm{Exp}'(\tilde{S}^1)=\mathrm{Exp}'(\boldsymbol{C}^*)$, $\mathscr{O}'(\tilde{S}^1)=\mathscr{O}'(\boldsymbol{C}^*)$ and $\mathscr{B}(S^1)=\mathscr{O}'(K_{1,1})$.

This paper was written during my stay in France in the academic year 1978/1979. The discussions with French mathematicians, especially C. C. Chou and J. Faraut, were very informative. I am very grateful to them.

§ 1. Analytic functions on an annulus.

Let $A>0$ and $B>0$ satisfy $AB\geqslant 1$ (resp. $AB>1$). Consider the annulus:

$$K_{A,B}=\{z\in \boldsymbol{C};\ B^{-1}\leqslant|z|\leqslant A\},\ (\text{resp. } K^{\circ}_{A,B}=\{z\in \boldsymbol{C};\ B^{-1}<|z|<A\})\ . \tag{1.1}$$

Let ε be a sufficiently small positive number. $\mathcal{O}_b(K_{A,B}(\varepsilon))$(resp. $\mathcal{O}_b(K_{A,B}(-\varepsilon))$) denotes the space of all continuous functions on $K_{A,B}(\varepsilon)$(resp. $K_{A,B}(-\varepsilon)$) which are holomorphic in its interior, where

$$\begin{aligned} K_{A,B}(\varepsilon)&=\{z\in \boldsymbol{C};\ B^{-1}(1+\varepsilon)^{-1}\leqslant|z|\leqslant A(1+\varepsilon)\}\\ (\text{resp. } K_{A,B}(-\varepsilon)&=\{z\in \boldsymbol{C};\ B^{-1}(1-\varepsilon)^{-1}\leqslant|z|\leqslant A(1-\varepsilon)\})\ . \end{aligned} \tag{1.2}$$

It is clear that the space $\mathcal{O}_b(K_{A,B}(\varepsilon))$(resp. $\mathcal{O}_b(K_{A,B}(-\varepsilon))$), equipped with the norm

$$\begin{aligned} \|f\|_{\varepsilon}&=\sup\{|f(z)|;\ z\in K_{A,B}(\varepsilon)\}\\ (\text{resp. } \|f\|_{-\varepsilon}&=\sup\{|f(z)|;\ z\in K_{A,B}(-\varepsilon)\})\ , \end{aligned} \tag{1.3}$$

is a Banach space. We define the *DFS* space $\mathcal{O}(K_{A,B})$ of germs of holomorphic functions on $K_{A,B}$ as follows:

$$\mathcal{O}(K_{A,B})=\lim_{\varepsilon>0}\operatorname{ind}\mathcal{O}_b(K_{A,B}(\varepsilon))\ . \tag{1.4}$$

The *FS* space $\mathcal{O}(K^{\circ}_{A,B})$ of holomorphic functions on the domain $K^{\circ}_{A,B}$ is defined as follows:

$$\mathcal{O}(K^{\circ}_{A,B})=\lim_{\varepsilon>0}\operatorname{proj}\mathcal{O}_b(K_{A,B}(-\varepsilon))\ . \tag{1.4'}$$

PROPOSITION 1.1 (Cauchy-Hedamard). *Let $f(z)=\sum_{k=-\infty}^{\infty}a_k z^k$ be the Laurent expansion of $f\in\mathcal{O}(K_{A,B})$(resp. $\mathcal{O}(K^{\circ}_{A,B})$), then we have*

$$\begin{aligned} &\limsup_{k\to\infty}\sqrt[k]{|a_k|}<A^{-1}\ ,\quad \limsup_{k\to-\infty}\sqrt[|k|]{|a_k|}<B^{-1}\ , \\ (\text{resp. (1.5')}\ &\limsup_{k\to\infty}\sqrt[k]{|a_k|}\leqslant A^{-1}\ ,\quad \limsup_{k\to-\infty}\sqrt[|k|]{|a_k|}\leqslant B^{-1})\ . \end{aligned} \tag{1.5}$$

Conversely, if a sequence a_k, $k\in\boldsymbol{Z}$, satisfies the conditions (1.5) (resp. (1.5′)), *then the series $\sum_{k=-\infty}^{\infty}a_k z^k$ converges to a unique function $f\in\mathcal{O}(K_{A,B})$*(resp. $\mathcal{O}(K^{\circ}_{A,B})$) *in the topology of $\mathcal{O}(K_{A,B})$*(resp. $\mathcal{O}(K^{\circ}_{A,B})$).

PROOF. Suppose $f \in \mathcal{O}(K_{A,B})$. Then there exists $\varepsilon>0$ such that $f \in \mathcal{O}_b(K_{A,B}(\varepsilon))$. As we have

$$a_k = \frac{1}{2\pi i}\oint_{|z|=c} f(z)\frac{dz}{z^{k+1}} \tag{1.6}$$

for any c with $B^{-1}(1+\varepsilon)^{-1} \leqslant c \leqslant A(1+\varepsilon)$, we have $|a_k| \leqq \|f\|_\varepsilon A^{-k}(1+\varepsilon)^{-k}$ for $k \geqq 0$ and $|a_k| \leqslant \|f\|_\varepsilon B^k(1+\varepsilon)^k$ for $k<0$. Hence we obtain (1.5).

Conversely if we have (1.5), there exists $\varepsilon>0$ and $N \geqslant 0$ such that

$$\sqrt[k]{|a_k|} \leqslant A^{-1}(1+2\varepsilon)^{-1} \quad \text{for} \quad k \geqslant N .$$

Therefore $\sum_{k=0}^{\infty} a_k z^k$ converges uniformly in the disc $\{z;\, |z| \leqslant A(1+\varepsilon)\}$ and define a holomorphic function there. Similarly, $\sum_{k=-\infty}^{-1} a_k z^k$ converges uniformly in $\{z;\, |z| \geqslant B^{-1}(1+\varepsilon)^{-1}\}$ for some $\varepsilon>0$. Therefore $\sum_{k=-\infty}^{\infty} a_k z^k$ converges uniformly in $K_{A,B}(\varepsilon)$ for some $\varepsilon>0$ to a function $f \in \mathcal{O}_b(K_{A,B}(\varepsilon))$. The proof for the case $\mathcal{O}(K_{A,B}^{\circ})$ is similar. q.e.d.

COROLLARY 1. *Let* $A \geqslant 1$ (resp. $A>1$). *Then* $f(z)=\sum_{k=-\infty}^{\infty} a_k z^k \in \mathcal{O}(K_{A;A})$(resp. $\mathcal{O}(K_{A\,A}^{\circ})$) *is equivalent to*

$$\limsup_{k\to\pm\infty} \sqrt[|k|]{|a_k|} < A^{-1} \quad (\text{resp. } \limsup_{k\to\pm\infty} \sqrt[|k|]{|a_k|} \leqq A^{-1}) . \tag{1.7}$$

Now we denote by $\mathcal{O}(\boldsymbol{C}^*)$ the Fréchet space of all holomorphic functions on $\boldsymbol{C}^*=\boldsymbol{C}\backslash(0)$. If $A_1 B_1 \geqslant 1$, $A_1<A$ and $B_1<B$, the inclusions

$$\mathcal{O}(K_{A,B}) \subset \mathcal{O}(K_{A,B}^{\circ}) \subset \mathcal{O}(K_{A_1,B_1}) \tag{1.8}$$

are defined by the restriction mappings. Taking the projective limit tending $A \to \infty$ and $B \to \infty$, we have

$$\mathcal{O}(\boldsymbol{C}^*) = \lim \operatorname{proj} \mathcal{O}(K_{A,B}) = \lim \operatorname{proj} \mathcal{O}(K_{A,B}^{\circ}) .$$

Therefore we have

COROLLARY 2. $f(z)=\sum_{k=-\infty}^{\infty} a_k z^k \in \mathcal{O}(\boldsymbol{C}^*)$ *is equivalent to*

$$\limsup_{k\to\pm\infty} \sqrt[|k|]{|a_k|} = 0 . \tag{1.9}$$

The holomorphic function $f(z)$ on $\boldsymbol{C}^*$ is, by definition, of exponential type if

(1.10) there exist $M>0$ and $C \geqslant 0$ such that $|f(z)| \leqslant C \exp\left(M\left(|z|+\frac{1}{|z|}\right)\right)$.

$\operatorname{Exp}(\boldsymbol{C}^*)$ is the space of all such functions. The topology of $\operatorname{Exp}(\boldsymbol{C})^*$ is the inductive limit topology of the Banach spaces X_M, where

$$X_M=\left\{f\in\mathcal{O}(C^*);\ \sup_{z\in C^*}|f(z)|\exp\left(-M\left(|z|+\frac{1}{|z|}\right)\right)<\infty\right\}. \tag{1.11}$$

PROPOSITION 1.2 (Helgason [3]). *Let $f(z)=\sum_{k=-\infty}^{\infty}a_kz^k$ be the Laurent expansion of $f\in\mathrm{Exp}(C^*)$. Then we have*

$$\limsup_{k\to\pm\infty}\sqrt[|k|]{|k|!\,|a_k|}<\infty. \tag{1.12}$$

Conversely if a sequence a_k, $k\in Z$, satisfies the condition (1.12), *then the series $\sum_{k=-\infty}^{\infty}a_kz^k$ converges to a unique function $f\in\mathrm{Exp}(C^*)$ in the topology of* $\mathrm{Exp}(C^*)$.

In order to prove Proposition 1.2, we need the following well-known fact (see for example Boas [1]).

LEMMA 1.1. *Let $F(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n$ be an entire function of one complex variable ζ and $M\geqslant 0$. Then the following two conditions are equivalent:*

(1.13) *F is of exponential type $\leqslant M$,* i.e., *for any $\varepsilon>0$ there exists $C_\varepsilon\geqslant 0$ such that $|F(\zeta)|\leqslant C_\varepsilon\exp((M+\varepsilon)|\zeta|)$ for $\zeta\in C$.*

$$\limsup_{n\to\infty}\sqrt[n]{n!\,|a_n|}\leqslant M. \tag{1.14}$$

PROOF OF PROPOSITION 1.2. Suppose $f\in\mathcal{O}(C^*)$ satisfies (1.10). Then by (1.6) we get $|a_k|\leqslant Cr^{-k}\exp(M(r+1/r))$ for all r with $0<r<\infty$. But by an elementary calculus, we have

$$\min\left\{r^{-k}\exp\left(M\left(r+\frac{1}{r}\right)\right);\ 0<r<\infty\right\}=\left(\frac{2M}{|k|+\sqrt{k^2+4M^2}}\right)^{|k|}\exp\left(\sqrt{k^2+4M^2}\right).$$

Therefore we get, by Stirling's formula,

$$\begin{aligned}|k|!\,|a_k|&\sim 2\sqrt{\pi}\,|k|^{|k|+1/2}\exp(-|k|)|a_k|\\&\leqslant 2\sqrt{\pi}\,|k|^{1/2}C\left(\frac{2M|k|}{|k|+\sqrt{k^2+4M^2}}\right)^{|k|}\exp\left(\sqrt{k^2+4M^2}-|k|\right),\end{aligned}$$

from which we can conclude (1.12).

Conversely if we have (1.12), by Lemma 1.1, $\sum_{k=0}^{\infty}a_kz^k$ and $\sum_{k=-\infty}^{-1}a_kz^k$ are of exponential type. Therefore $\sum_{k=-\infty}^{\infty}a_kz^k$ is also of exponential type. The convergence in the topology of $\mathrm{Exp}(C^*)$ can be checked easily.

q.e.d.

At last, let us denote by $P(C^*)$ the space of all polynomials of z and z^{-1}, i.e., finite Laurent series:

$$P(C^*)=\left\{\sum_{k=-N}^{N} a_k z^k;\ a_k\in C,\ N=0, 1, 2, \cdots\right\}. \tag{1.15}$$

The topology of $P(C^*)$ is defined as the inductive limit of the finite dimensional vector space $P_N(C^*)=\{\sum_{k=-N}^{N} a_k z^k;\ a_k\in C\}$.

PROPOSITION 1.3. *Suppose $A_1B_1\geqslant 1$, $A_1<A$ and $B_1<B$. Then the following chain of inclusion relations is valid*:

$$P(C^*)\subset \mathrm{Exp}(C^*)\subset \mathcal{O}(C^*)\subset \mathcal{O}(K_{A,B})\subset \mathcal{O}(\overset{\circ}{K}_{A,B})\subset \mathcal{O}(K_{A_1,B})\subset \mathcal{O}(K_{1,1}), \tag{1.16}$$

the last inclusion taking place only if $A_1\geqslant 1$ and $B_1\geqslant 1$. The space $P(C^*)$ is dense in any of other spaces.

The proof is almost trivial.

§2. Analytic functionals with carrier in an annulus.

We shall denote by $\mathcal{O}'(K_{A,B})$(resp. $\mathcal{O}'(K^\circ_{A,B})$, $\mathcal{O}'(K_1)$, $\mathcal{O}'(C)$, $\mathrm{Exp}'(C^*)$, $P'(C^*)$) the dual space of $\mathcal{O}(K_{A,B})$(resp. $\mathcal{O}(K^\circ_{A,B})$, $\mathcal{O}(K_1)$, $\mathcal{O}(C^*)$, $\mathrm{Exp}(C^*)$, $P(C^*)$). By Proposition 1.3, we have the following inclusion relations:

$$P'(C^*)\supset \mathrm{Exp}'(C^*)\supset \mathcal{O}'(C^*)\supset \mathcal{O}'(K_{A,B})\supset \mathcal{O}'(K^\circ_{A,B})\supset \mathcal{O}'(K_{A_1,B_1})\supset \mathcal{O}'(K_1), \tag{2.1}$$

where $A_1B_1\geqslant 1$, $A_1<A$ and $B_1<B$ and the last inclusion takes place only if $A_1\geqslant 1$ and $B_1\geqslant 1$. We shall call an element T of $\mathcal{O}'(C^*)$ an analytic functional on C^* and an element T of $\mathrm{Exp}'(C^*)$ an entire functional on C^*(see Helgason [3]). If T is in $\mathcal{O}'(K_{A,B})$, T is said to have a carrier in $K_{A,B}$. Remark that

$$\begin{aligned}\mathcal{O}'(C^*)&=\lim\operatorname{ind}\{\mathcal{O}'(K_{A,B});\ A>0,\ B>0,\ AB\geqslant 1\}\\ &=\lim\operatorname{ind}\{\mathcal{O}'(K^\circ_{A,B});\ A>0,\ B>0,\ AB>1\}.\end{aligned}$$

We shall denote by $\langle\ ,\ \rangle$ the canonical inner product of duality. The Laurent coefficients c_k of $T\in P'(C^*)$ are defined as follows:

$$c_k=\langle T_z, z^{-k}\rangle \quad \text{for} \quad k\in Z. \tag{2.2}$$

We have clearly

$$\langle T, f\rangle=\sum_{k=-N}^{N} c_{-k}a_k \quad \text{for} \quad f(z)=\sum_{k=-N}^{N} a_k z^k\in P(C^*). \tag{2.3}$$

The formal Laurent series $\sum_{k=-\infty}^{\infty} c_k z^k$ is called the (formal) Laurent expansion of $T \in P'(\boldsymbol{C}^*)$.

LEMMA 2.1. *Let* $T \in \mathscr{O}'(K_{A,B})$(resp. $\mathscr{O}'(K^\circ_{A,B})$, $\mathscr{O}'(\boldsymbol{C}^*)$, $\mathrm{Exp}'(\boldsymbol{C}^*)$) *and* $c_k = \langle T_z, z^{-k}\rangle$, $k \in \boldsymbol{Z}$. *Then we have*

$$\langle T, f\rangle = \sum_{k=-\infty}^{\infty} c_{-k}a_k \tag{2.4}$$

for any $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \in \mathscr{O}(K_{A,B})$(*resp.* $\mathscr{O}(K^\circ_{A,B})$, $\mathscr{O}(\boldsymbol{C}^*)$, $\mathrm{Exp}(\boldsymbol{C}^*)$).

PROOF. By Propositions 1.1 and 1.2, $f_N(z) = \sum_{k=-N}^{N} a_k z^k$ converges to f as $N \to \infty$ in the topology of $\mathscr{O}(K_{A,B})$(resp. $\mathscr{O}(K^\circ_{A,B})$, $\mathscr{O}(\boldsymbol{C}^*)$, $\mathrm{Exp}(\boldsymbol{C}^*)$). Therefore, by (2.3), $\langle T, f\rangle = \lim_{N\to\infty}\langle T, f_N\rangle = \lim_{N\to\infty}\sum_{k=-N}^{N} c_{-k}a_k$. q.e.d.

Suppose $K_{A_1,B_1} \cap K_{A,B} \neq \emptyset$ (resp. $K^\circ_{A_1,B_1} \cap K_{A,B} \neq \emptyset$). We define, for $g(z) = \sum_{k=-\infty}^{\infty} b_k z^k \in \mathscr{O}(K_{A_1,B_1})$ (resp. $\mathscr{O}(K^\circ_{A_1,B_1})$) and $f(z) = \sum_{k=-\infty}^{\infty} a_k z^k \in \mathscr{O}(K_{A,B})$

$$(g, f) = \frac{1}{2\pi i}\oint_{|z|=c} g(z)f(z)\,\frac{dz}{z} \tag{2.5}$$

where $\max(B_1^{-1}, B^{-1}) \leqslant c \leqslant \min(A_1, A)$ (resp. $\max(B_1^{-1}, B^{-1}) < c < \min(A_1, A)$). Then, by Cauchy's integral theorem, (2.5) is defined independently of c and (g, f) is a bilinear form on $\mathscr{O}(K_{A_1,B_1}) \times \mathscr{O}(K_{A,B})$(resp. $\mathscr{O}(K^\circ_{A_1,B_1}) \times \mathscr{O}(K_{A,B})$). The mapping $T_g\colon f \mapsto (g, f)$ is a continuous linear functional on $\mathscr{O}(K_{A,B})$. By an elementary calculus, we have

$$(g, f) = \left(\sum_{k=-\infty}^{\infty} b_k z^k, \sum_{k=-\infty}^{\infty} a_k z^k\right) = \sum_{k=-\infty}^{\infty} b_{-k}a_k\ . \tag{2.6}$$

Especially the Laurent coefficients of the functional T_g are equal to those of the function g. Identifying the function g with the functional T_g we can consider

$$\mathscr{O}(K_{A_1,B_1}) \subset \mathscr{O}'(K_{A,B})(\text{resp. } \mathscr{O}(K^\circ_{A_1,B_1}) \subset \mathscr{O}'(K_{A,B}))\ . \tag{2.7}$$

Therefore we can graft the chain of inclusions (2.1) on (1.16). We get especially the following proposition:

PROPOSITION 2.1. *Let* $A > 1$ *and put* $K_A = K_{A,A}$. *Then we have the following inclusion relations*:

$$\begin{aligned} P(\boldsymbol{C}^*) &\subset \mathrm{Exp}(\boldsymbol{C}^*) \subset \mathscr{O}(\boldsymbol{C}^*) \subset \mathscr{O}(K_A) \subset \mathscr{O}(K^\circ_A) \subset \mathscr{O}(K_1) \\ &\subset \mathscr{O}'(K_1) \subset \mathscr{O}'(K^\circ_A) \subset \mathscr{O}'(K_A) \subset \mathscr{O}'(\boldsymbol{C}^*) \subset \mathrm{Exp}'(\boldsymbol{C}^*) \subset P'(\boldsymbol{C}^*)\ . \end{aligned} \tag{2.8}$$

If $f \in P(\boldsymbol{C}^*)$, then, for any formal Laurent series $g(z) = \sum_{k=-\infty}^{\infty} b_k z^k$, we can define (g, f) by (2.6). And it is clear the space $P'(\boldsymbol{C}^*)$ coincides with the space of all formal Laurent series.

PROPOSITION 2.2. *Let c_k be the Laurent coefficients of $T\in\mathcal{O}'(K_{A,B})$* (resp. $\mathcal{O}'(K^\circ_{A,B})$). *Then we have*

$$T_z=\sum_{k=-\infty}^{\infty} c_k z^k \tag{2.9}$$

in the weak topology of $\mathcal{O}'(K_{A,B})$(resp. $\mathcal{O}'(K^\circ_{A,B})$). *We have also*

$$\limsup_{k\to\infty} \sqrt[k]{|c_k|}\leqslant B\,,\quad \limsup_{k\to-\infty} \sqrt[|k|]{|c_k|}\leqslant A \tag{2.10}$$

$$(\text{resp. } (2.10')\quad \limsup_{k\to\infty} \sqrt[k]{|c_k|}< B\,,\quad \limsup_{k\to-\infty} \sqrt[|k|]{|c_k|}< A)\,.$$

Conversely, if a formal Laurent series satisfies the condition (2.10) (resp. (2.10′)), *then it converges to a functional $T\in\mathcal{O}'(K_{A,B})$*(resp. $\mathcal{O}'(K^\circ_{A,B})$) *in the weak topology of $\mathcal{O}'(K_{A,B})$*(resp. $\mathcal{O}'(K^\circ_{A,B})$).

PROOF. Suppose $T\in\mathcal{O}'(K_{A,B})$, $c_k=\langle T, z^{-k}\rangle$, $k\in\boldsymbol{Z}$. Consider the (formal) Laurent expansion of T: $\sum_{k=-\infty}^{\infty} c_k z^k$. Then by Lemma 2.1, the sequence $\sum_{k=-N}^{N} c_k z^k$ is convergent to $T_z\in\mathcal{O}'(K_{A,B})$ in the weak topology, i.e., we have (2.9). Now by the continuity of T, for all $\varepsilon>0$, there exists $C_\varepsilon\geqslant 0$ such that

$$|\langle T, f\rangle|\leqslant C_\varepsilon\|f\|_\varepsilon \quad\text{for}\quad f\in\mathcal{O}_b(K_{A,B}(\varepsilon))\,. \tag{2.11}$$

In particular, we have

$$|c_k|\leqslant C_\varepsilon \sup\{|z|^{-k};\ B^{-1}(1+\varepsilon)^{-1}\leqslant|z|\leqslant A(1+\varepsilon)\}\,.$$

Therefore we have $|c_k|\leqslant C_\varepsilon B^k(1+\varepsilon)^k$ for $k\geqslant 0$ and $|c_k|\leqslant C_\varepsilon A^{|k|}(1+\varepsilon)^{|k|}$ for $k<0$. Hence we get (2.10).

Conversely, if (2.10) is valid, $\sum_{k=-\infty}^{\infty} c_{-k}a_k$ converges for any $f(z)=\sum_{k=-\infty}^{\infty} a_k z^k\in\mathcal{O}(K_{A,B})$ because of (1.5). By the proof of Proposition 1.1, it is clear that $f\mapsto\sum_{k=-\infty}^{\infty} c_{-k}a_k$ is continuous on every $\mathcal{O}_b(K_{A,B}(\varepsilon))$. The proof for $\mathcal{O}'(K^\circ_{A,B})$ is similar and is omitted. q.e.d.

COROLLARY 1. *Let $A\geqslant 1$*(resp. $A>1$)$\cdot$ $T_z=\sum_{k=-\infty}^{\infty} c_k z^k\in\mathcal{O}'(K_{A,A})$(resp. $\mathcal{O}'(K^\circ_{A,A})$) *is equivalent to*

$$\limsup_{k\to\pm\infty} \sqrt[|k|]{|c_k|}\leqslant A\quad (\text{resp. } \limsup_{k\to\pm\infty} \sqrt[|k|]{|c_k|}< A)\,. \tag{2.12}$$

COROLLARY 2. *$T_z=\sum_{k=-\infty}^{\infty} c_k z^k\in\mathcal{O}'(C^*)$ is equivalent to*

$$\limsup_{k\to\pm\infty} \sqrt[|k|]{|c_k|}<\infty\,. \tag{2.13}$$

PROPOSITION 2.3 (Helgason [3]). *Let $c_k=\langle T_z, z^{-k}\rangle$ be the Laurent* *coefficients of an entire functional* $T\in \mathrm{Exp}'(\boldsymbol{C}^*)$. *Then we have*

(2.14) $$\limsup_{k\to\pm\infty} \sqrt[|k|]{|c_k|/|k|!}=0 .$$

Conversely, if we have (2.14), *the series* $\sum_{k=-\infty}^{\infty} c_k z^k$ *converges in the weak topology to an entire functional* $T\in \mathrm{Exp}'(\boldsymbol{C}^*)$ *for which we have* $\langle T_z, z^{-k}\rangle = c_k$ *for* $k\in \boldsymbol{Z}$.

Proof is similar to that of Proposition 2.2 and is omitted.

REMARK. Let $T\in \mathcal{O}'(K_{A,B})$ and $c_k=\langle T, z^{-k}\rangle$, $k\in\boldsymbol{Z}$. By (2.10), $\varphi_1(z)=\sum_{k=1}^{\infty} c_k z^k$ is holomorphic in $\{z; |z|<B^{-1}\}$ and $\varphi_2(z)=\sum_{k=-\infty}^{0} c_k z^k$ is holomorphic in $\{z; |z|>A\}$. Let $f(z)=\sum_{k=-\infty}^{\infty} a_k z^k\in\mathcal{O}(K_{A,B})$. Then we have

(2.15) $$\begin{aligned}\langle T, f\rangle &= \sum_{k=-\infty}^{-1} c_{-k}a_k+\sum_{k=0}^{\infty} c_{-k}a_k\\ &=\frac{1}{2\pi i}\oint_{|z|=B^{-1}(1+\varepsilon)^{-1}} \varphi_1(z)f(z)\frac{dz}{z}\\ &\quad+\frac{1}{2\pi i}\oint_{|z|=A(1+\varepsilon)} \varphi_2(z)f(z)\frac{dz}{z}\end{aligned}$$

with a sufficiently small $\varepsilon>0$. If we put

$$\varphi(z)=\begin{cases}\ \ \varphi_1(z) & \text{for } |z|<B^{-1}\\ -\varphi_2(z) & \text{for } |z|>A ,\end{cases}$$

then $\varphi\in\mathcal{O}(\boldsymbol{C}\backslash K_{A,B})$ and

(2.16) $$\langle T, f\rangle=\frac{-1}{2\pi i}\oint_{\partial K_{A,B}(\varepsilon)} \varphi(z)f(z)\frac{dz}{z} .$$

This is nothing but Köthe's duality [5].

§ 3. Entire functions which satisfy $(\partial^2/\partial u\partial v)F(u, v)=\lambda^2 F(u, v)$.

Suppose that

(3.1) $$F(u, v)=\sum_{n,m=0}^{\infty} a_{n,m}u^n v^m$$

is an entire function of two complex variables $(u, v)\in\boldsymbol{C}^2$ and that $F(u, v)$ satisfies the differential equation

(3.2) $$\frac{\partial^2}{\partial u\partial v}F(u, v)=\lambda^2 F(u, v) ,$$

where $\lambda \neq 0$ is a constant complex number. Let us denote by $\mathcal{O}_{(\lambda)}(C^2)$ the space of all such functions. For example, if $AB=\lambda^2$, then $\exp(Au+Bv) \in \mathcal{O}_{(\lambda)}(C^2)$. As we have

$$\frac{\partial^2}{\partial u \partial v}F(u, v)=\sum_{n,m=0}^{\infty} nma_{n,m}u^{n-1}v^{n-1}$$

$$=\sum_{n,m=0}^{\infty}(n+1)(m+1)a_{n+1,m+1}u^n v^m ,$$

the condition (3.2) is equivalent to the condition (3.3) on the Taylor coefficients of F:

$$\lambda^2 a_{n,m}=(n+1)(m+1)a_{n+1,m+1} \quad \text{for} \quad n, m=0, 1, 2, \cdots . \tag{3.3}$$

Hence we have

$$\begin{aligned} a_{n,n}&=\frac{\lambda^{2n}}{n!\, n!}a_{0,0} && \text{for} \quad n=0, 1, 2, \cdots \\ a_{n,n+p}&=\frac{p!\lambda^{2n}}{n!\,(n+p)!}a_{0,p} && \text{for} \quad n, p=0, 1, 2, \cdots \\ a_{n+p,n}&=\frac{p!\,\lambda^{2n}}{(n+p)!\, n!}a_{p,0} && \text{for} \quad n, p=0, 1, 2, \cdots . \end{aligned} \tag{3.4}$$

On the other hand, the Bessel functions are defined as follows:

$$J_\nu(z)=(z/2)^\nu \sum_{n=0}^{\infty}\frac{(-1)^n(z/2)^{2n}}{n!\,\Gamma(\nu+n+1)}, \quad \nu \neq -1, -2, -3, \cdots . \tag{3.5}$$

Therefore we have

$$(iz)^{-\nu}J_\nu(2iz)=\sum_{n=0}^{\infty}\frac{z^{2n}}{n!\,\Gamma(\nu+n+1)}, \quad \nu \neq -1, -2, -3, \cdots . \tag{3.6}$$

Remark the functions

$$(i\sqrt{z})^{-\nu}J_\nu(2i\sqrt{z})=\sum_{n=0}^{\infty}\frac{z^n}{n!\,\Gamma(\nu+n+1)}, \quad \nu \neq -1, -2, -3, \cdots$$

are entire functions of z.

Using the formulas (3.4), we can sum up as follows:

$$\sum_{n=0}^{\infty} a_{n,n}u^n v^n=a_{0,0}\sum_{n=0}^{\infty}\lambda^{2n}\frac{(uv)^n}{n!\, n!}=a_{0,0}J_0(2i\sqrt{uv}) , \tag{3.7}$$

$$(3.8)\qquad \sum_{n=0}^{\infty} a_{n+p,n}u^{n+p}v^{n} = a_{p,0}u^{p}p!\sum_{n=0}^{\infty}\frac{\lambda^{2n}(uv)^{n}}{(n+p)!\,n!} = a_{p,0}u^{p}p!(i\lambda\sqrt{uv})^{-p}J_{p}(2i\lambda\sqrt{uv})$$

and similarly

$$(3.9)\qquad \sum_{n=0}^{\infty} a_{n,n+p}u^{n}v^{n+p} = a_{0,p}v^{p}p!(i\lambda\sqrt{uv})^{-p}J_{p}(2i\lambda\sqrt{uv})\ .$$

Therefore we have proved the following proposition.

PROPOSITION 3.1. *If an entire function* $F(u, v)=\sum_{n,m=0}^{\infty} a_{n,m}u^{n}v^{m}$ *satisfies the differential equation* (3.2), *it takes the following form*:

$$(3.10)\qquad F(u, v) = a_{0,0}J_{0}(2i\lambda\sqrt{uv}) + \sum_{p=1}^{\infty}(a_{p,0}u^{p}+a_{0,p}v^{p})p!(i\lambda\sqrt{uv})^{-p}J_{p}(2i\lambda\sqrt{uv})\ .$$

THEOREM 3.1. *If* $f(z)$ *and* $g(z)$ *are two entire functions of one complex variable with* $f(0)=g(0)$, *then there exists a unique function* $F(u, v)\in\mathcal{O}_{(\lambda)}(C^{2})$ *such that* $F(u, 0)=f(u)$ *and* $F(0, v)=g(v)$.

PROOF. The uniqueness of the function $F(u, v)$ is clear from Proposition 3.1. Put $f(z)=\sum_{p=0}^{\infty} a_{p,0}z^{p}$, $g(z)=\sum_{q=0}^{\infty} a_{0,q}z^{q}$. We have only to prove the uniform convergence of the right hand term of (3.10) on every compact set of C^{2}. Now we have, by the definition formula (3.5), the following majoration (see Lemma 1 of [2]):

$$(3.11)\qquad |J_{\nu}(z)| \leqslant \frac{|z/2|^{\nu}}{\Gamma(\nu+1)}\exp(|z/2|^{2})\ .$$

Therefore

$$\sum_{p=1}^{\infty}(a_{p,0}u^{p}+a_{0,p}v^{p})p!(i\lambda\sqrt{uv})^{-p}J_{p}(2i\lambda\sqrt{uv}) \leqslant \sum_{p=1}^{\infty}(|a_{p,0}|\,|u|^{p}+|a_{0,p}|\,|v|^{p})\exp(|\lambda|^{2}|uv|)\ ,$$

which proves the theorem. q.e.d.

COROLLARY. *To give a function* $F\in\mathcal{O}_{(\lambda)}(C^{2})$ *is equivalent to give a series* $a_{0,0}, a_{p,0}, a_{0,q}$ $p, q=1, 2, 3, \cdots$ *such that*

$$(3.12)\qquad \limsup_{p\to\infty}\sqrt[p]{|a_{p,0}|}=0\ ,\quad \limsup_{q\to\infty}\sqrt[q]{|a_{0,q}|}=0\ .$$

§ 4. The transformation $\mathscr{F}_\lambda$ of analytic functionals.

DEFINITION 4.1. As the function of z, $\exp(\lambda(uz+v/z))$ is in $\mathrm{Exp}(C^*)$ for all $(u, v)\in C^2$, we can put for $T\in \mathrm{Exp}'(C^*)$,

$$F_\lambda(u, v)=\left\langle T_z, \exp\left(\lambda\left(uz+\frac{v}{z}\right)\right)\right\rangle, \tag{4.1}$$

where λ is a fixed complex number. We write by $\mathscr{F}_\lambda$ the transformation $T\mapsto F_\lambda$.

Remark that the mapping $z\mapsto 1/z$ is an analytic diffeomorphism of C^* onto itself. For $T_z\in \mathrm{Exp}'(C^*)$, we define $T_{1/z}\in \mathrm{Exp}'(C^*)$ by

$$\langle T_{1/z}, f(z)\rangle=\langle T_z, f(1/z)\rangle . \tag{4.2}$$

Then we have

$$\mathscr{F}_\lambda(T_z)(u, v)=\mathscr{F}_\lambda(T_{1/z})(v, u) . \tag{4.3}$$

THEOREM 4.1. *Suppose* $\lambda\neq 0$. *Let* $c_k=\langle T_z, z^{-k}\rangle$ *be the Laurent coefficients of* $T\in \mathrm{Exp}'(C^*)$. *Then the function* F_λ *defined by* (4.1) *is expressed as follows*:

$$F_\lambda(u, v)=c_0J_0(2i\lambda\sqrt{uv})+\sum_{p=1}^{\infty}(c_{-p}u^p+c_pv^p)(i\sqrt{uv})^{-p}J_p(2i\lambda\sqrt{uv}) . \tag{4.3}$$

The transformation $\mathscr{F}_\lambda$ *maps bijectively* $\mathrm{Exp}'(C^*)$ *onto* $\mathcal{O}_{(\lambda)}(C^2)$:

$$\mathscr{F}_\lambda.\ \mathrm{Exp}'(C^*) \xrightarrow{\sim} \mathcal{O}_{(\lambda)}(C^2) . \tag{4.4}$$

PROOF. Because $f(u, v)=\exp(\lambda(uz+v/z))$ satisfies the differential equation (3.2) and that the functional T commutes with $\partial^2/\partial u\partial v$, the function F belongs to $\mathcal{O}_{(\lambda)}(C^2)$.

From the formula

$$\exp\left(\lambda\left(uz+\frac{v}{z}\right)\right)=\sum_{n=0}^{\infty}\frac{\lambda^n u^n z^n}{n!}\sum_{m=0}^{\infty}\frac{\lambda^m v^m z^{-m}}{m!}=\sum_{n,m=0}^{\infty}\frac{u^n v^m}{n!m!}\lambda^{n+m}z^{n-m} ,$$

we get

$$F_\lambda(u, v)=<T_z, \exp\left(\lambda\left(uz+\frac{v}{z}\right)\right)>=\sum_{n,m=0}^{\infty}\frac{\lambda^{n+m}}{n!m!}c_{m-n}u^n v^m .$$

Therefore the Taylor coefficients $a_{n,m}$ of F_λ are given by

$$a_{n,m}=\frac{\lambda^{n+m}}{n!m!}c_{m-n} .$$

In particular, we have

$$(4.5)\qquad a_{0,0}=c_0,\ a_{p,0}=\frac{\lambda^p}{p!}c_{-p} \quad\text{and}\quad a_{0,p}=\frac{\lambda^p}{p!}c_p \quad\text{for}\quad p=1, 2, \cdots .$$

Replacing $a_{0,0}$, $a_{p,0}$ and $a_{0,p}$ in the formula (3.10) by (4.5), we obtain (4.3).

If $F_\lambda=0$, then, the Taylor coefficients of F_λ being 0, the Laurent coefficients of T all vanish by (4.5). Therefore $T=0$, which proves that $\mathscr{F}_\lambda$ is one-to-one. The surjectivity of $\mathscr{F}_\lambda$ results from Corollary to Theorem 3.1 and Proposition 2.3. q.e.d.

COROLLARY.

$$(4.6)\qquad \mathscr{F}_\lambda(z^k)(u, v)=\begin{cases} J_0(2i\lambda\sqrt{uv}) & \textit{for}\quad k=0 \\ v^k(i\sqrt{uv})^{-k}J_k(2i\lambda\sqrt{uv}) & \textit{for}\quad k>0 \\ u^{|k|}(i\sqrt{uv})^{-|k|}J_{|k|}(2i\lambda\sqrt{uv}) & \textit{for}\quad k<0 . \end{cases}$$

LEMMA 4.1. *Suppose* $AB\geqslant 1$(resp. $AB>1$). *Let* $T\in\mathscr{O}'(K_{A,B})$(resp. $\mathscr{O}'(K^\circ_{A,B})$). *The function* $F_\lambda(u, v)$ *defined by* (4.1) *satisfies the following condition*:

(4.7) *For any* $\varepsilon>0$, *there exists* $C_\varepsilon\geqslant 0$ *such that*

$$|F_\lambda(u, v)|\leqslant C_\varepsilon \exp(|\lambda|(1+\varepsilon)(A|u|+B|v|))$$

(resp. (4.7′). *There exist* $\varepsilon>0$ *and* $C\geqslant 0$ *such that*

$$|F_\lambda(u, v)|\leqslant C \exp(|\lambda|(1-\varepsilon)(A|u|+B|v|))) .$$

PROOF. By the continuity of $T\in\mathscr{O}'(K_{A,B})$, for any $\varepsilon>0$, we can find $C_\varepsilon\geqslant 0$ such that

$$\begin{aligned} |F_\lambda(u, v)| &\leqslant C_\varepsilon \sup\left\{\exp\left(\operatorname{Re}\lambda\left(uz+\frac{v}{z}\right)\right);\ B^{-1}(1+\varepsilon)^{-1}\leqslant|z|\leqslant A(1+\varepsilon)\right\} \\ &\leqslant C_\varepsilon \exp(|\lambda|(1+\varepsilon)(A|u|+B|v|)) . \end{aligned}$$

The proof of (4.7′) is similar. q.e.d.

DEFINITION 4.2. we define for $A\geqslant 0$ and $B\geqslant 0$

(4.8) $\operatorname{Exp}(C^2; (A, B))=\{F\in\mathscr{O}(C^2)$; for any $\varepsilon>0$ there exists $C_\varepsilon\geqslant 0$ such that $|F(u, v)|\leqslant C_\varepsilon \exp((A+\varepsilon)|u|+(B+\varepsilon)|v|)\}$.

We define for $A>0$ and $B>0$

(4.8′) $\mathrm{Exp}(C^2; (A, B)^\circ)=\{F\in\mathcal{O}(C^2)$; there exist $\varepsilon>0$ and $C\geqslant 0$ such that $|F(u, v)|\leqslant C\exp((A-\varepsilon)|u|+(B-\varepsilon)|v|)\}$.

We put also

$$\mathrm{Exp}(C^2)=\lim\mathrm{ind}\{\mathrm{Exp}(C^2; (A, B)); A\geqslant 0, B\geqslant 0\} \tag{4.9}$$
$$=\lim\mathrm{ind}\{\mathrm{Exp}(C^2; (A, B)^\circ); A>0, B>0\} .$$

If $F\in\mathrm{Exp}(C^2)$, we say that F is of exponential type. If $F\in\mathrm{Exp}(C^2; (A, B))$, we say that F is of exponential type (A, B). We put further

$$\mathrm{Exp}_{(\lambda)}(C^2; (A, B))=\mathcal{O}_{(\lambda)}(C^2)\cap\mathrm{Exp}(C^2; (A, B)) , \tag{4.10}$$

$$\mathrm{Exp}_{(\lambda)}(C^2; (A, B)^\circ)=\mathcal{O}_{(\lambda)}(C^2)\cap\mathrm{Exp}(C^2; (A, B)^\circ) , \tag{4.10′}$$

$$\mathrm{Exp}_{(\lambda)}(C^2)=\mathcal{O}_{(\lambda)}(C^2)\cap\mathrm{Exp}(C^2) . \tag{4.11}$$

THEOREM 4.2. *Suppose* $AB\geqslant 1$(resp. $AB>1$). *The transformation* $\mathcal{F}_\lambda: T\mapsto F_\lambda$, *defined by* (4.1), *establishes a linear isomorphism of* $\mathcal{O}'(K_{A,B})$ *onto* $\mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|B))$(resp. *of* $\mathcal{O}'(K^\circ_{A,B})$ *onto* $\mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|B)^\circ)$):

$$\mathcal{F}_\lambda: \mathcal{O}'(K_{A,B}) \xrightarrow{\sim} \mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|B)) \qquad (AB\geqslant 1) , \tag{4.12}$$

$$\mathcal{F}_\lambda: \mathcal{O}'(K^\circ_{A,B}) \xrightarrow{\sim} \mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|B)^\circ) \qquad (AB>1) . \tag{4.12′}$$

PROOF. We prove only the surjectivity in the case of $\mathcal{O}'(K_{A,B})$. Suppose $F\in\mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|B))$ is given. We put $F(u, v)=\sum_{n,m=0}^\infty a_{n,m}u^n v^m$. Because F is of exponential type $(|\lambda|A, |\lambda|B)$, $F(u, 0)$ (resp. $F(0, v)$) satisfies the condition (1.13) of Lemma 1.1 with $M=|\lambda|A$ (resp. $M=|\lambda|B$). By Lemma 1.1, we have

$$\limsup_{n\to\infty}\sqrt[n]{n!\,|a_{n,0}|}\leqslant|\lambda|A , \quad \limsup_{n\to 8}\sqrt[n]{n!\,|a_{0,n}|}\leqslant|\lambda|B . \tag{4.13}$$

Therefore, if we define c_p, $p\in Z$ by (4.5), c_p's satisfy

$$\limsup_{p\to-\infty}\sqrt[|p|]{|c_p|}\leqslant B , \quad \limsup_{p\to\infty}\sqrt[p]{|c_{-p}|}\leqslant A . \tag{4.14}$$

By Proposition 2.2, $T_z=\sum_{p=-\infty}^\infty c_p z^p$ belongs to $\mathcal{O}'(K_{A,B})$. It is clear by Theorem 4.1 that $\mathcal{F}_\lambda T=F$, which proves the surjectivity of the transformation $\mathcal{F}_\lambda$. q.e.d.

COROLLARY 1.

$$\mathcal{F}_\lambda: \mathcal{O}'(K_{A,A}) \xrightarrow{\sim} \mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|A)) \qquad \textit{if}\ \ A\geqslant 1 , \tag{4.15}$$

$$(4.15') \qquad \mathscr{F}_\lambda\colon \mathscr{O}'(K^\circ_{A,A}) \xrightarrow{\sim} \mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|A, |\lambda|A)^\circ) \qquad \textit{if} \quad A>1 .$$

COROLLARY 2.

$$(4.16) \qquad \mathscr{F}_\lambda\colon \mathscr{O}'(C^*) \xrightarrow{\sim} \mathrm{Exp}_{(\lambda)}(C^2) .$$

§ 5. The transformation $\mathscr{F}_\lambda$ of analytic functions.

Now we consider the transformation $\mathscr{F}_\lambda$ for an analytic function $f \in \mathscr{O}(K^\circ_{A,B})$, $AB>1$. λ denotes always a fixed nonzero complex number. By the definition

$$(5.1) \qquad F_\lambda(u, v)=\mathscr{F}_\lambda(f)(u, v)=\frac{1}{2\pi i}\oint_{|z|=c} f(z)\exp\left(\lambda\left(uz+\frac{v}{z}\right)\right)\frac{dz}{z}$$

for any c with $B^{-1}<c<A$. Therefore for any sufficiently small $\varepsilon>0$, we have

$$|F_\lambda(u, v)|\leqslant \|f\|_{-\varepsilon}\exp\left(|\lambda|\left(c|u|+\frac{|v|}{c}\right)\right)$$

for any c with $B^{-1}(1-\varepsilon)^{-1}\leqslant c\leqslant A(1-\varepsilon)$. Now the function of c, $0<c<\infty$, $c|u|+|v|/c$ takes its minimum value $2\sqrt{|uv|}$ at $c=\sqrt{(|v|/|u|)}$, provided $|uv|\neq 0$.

Therefore if we put $C_\varepsilon=\|f\|_{-\varepsilon}$, the function $F(u, v)=F_\lambda(u, v)$ satisfies the majoration:

$$(5.2) \qquad \begin{cases} |F(u, v)|\leqslant C_\varepsilon \exp\left(|\lambda|\left(\dfrac{|u|}{B(1-\varepsilon)}+B(1-\varepsilon)|v|\right)\right) & \text{if} \quad \sqrt{|v|}\leqslant\dfrac{\sqrt{|u|}}{B(1-\varepsilon)} \\ |F(u, v)|\leqslant C_\varepsilon \exp(|\lambda|2\sqrt{|uv|}) \quad \text{if} \quad \dfrac{\sqrt{|u|}}{B(1-\varepsilon)}\leqslant\sqrt{|v|}\leqslant A(1-\varepsilon)\sqrt{|u|} \\ |F(u, v)|\leqslant C_\varepsilon \exp\left(|\lambda|\left(A(1-\varepsilon)|u|+\dfrac{|v|}{A(1-\varepsilon)}\right)\right) \\ \qquad\qquad\qquad\qquad \text{if} \quad A(1-\varepsilon)\sqrt{|u|}\leqslant\sqrt{|v|} . \end{cases}$$

DEFINITION 5.1. If $AB>1$, we denote by $\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2; (|\lambda|B^{-1}, |\lambda|A^{-1}))$ the space of all $F\in\mathscr{O}_{(\lambda)}(C^2)$ for which, for any $\varepsilon>0$ there exists $C_\varepsilon\geqslant 0$ such that the majoration (5.2) is valid. Remark that

$$(5.3) \qquad \mathrm{Exp}_{(\lambda)}(C^2; (|\lambda|B^{-1}, |\lambda|A^{-1}))\subset\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2; (|\lambda|B^{-1}, |\lambda|A^{-1})) ,$$

because we have $A^{-1}(1-\varepsilon)^{-1}<B(1-\varepsilon)$ and $B^{-1}(1-\varepsilon)^{-1}<A(1-\varepsilon)$ for sufficiently small $\varepsilon>0$.

If $AB\geqslant 1$, we denote by $\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2; (|\lambda|B^{-1}, |\lambda|A^{-1})^\circ)$ the space of all $F\in\mathscr{O}_{(\lambda)}(C^2)$ for which, for some $\varepsilon>0$ and $C\geqslant 0$ the majoration (5.2′) is valid:

$$(5.2')\quad \begin{cases} |F(u,v)|\leqslant C\exp\Big(|\lambda|\Big(\frac{|u|}{B(1+\varepsilon)}+B(1+\varepsilon)|v|\Big)\Big) & \text{if } \sqrt{|v|}\leqslant\frac{\sqrt{|u|}}{B(1+\varepsilon)} \\ |F(u,v)|\leqslant C\exp(|\lambda|2\sqrt{|uv|}) & \text{if } \frac{\sqrt{|u|}}{B(1+\varepsilon)}\leqslant\sqrt{|v|}\leqslant A(1+\varepsilon)\sqrt{|u|} \\ |F(u,v)|\leqslant C\exp\Big(|\lambda|\Big(A(1+\varepsilon)|u|+\frac{|v|}{A(1+\varepsilon)}\Big)\Big) & \\ & \text{if } A(1+\varepsilon)\sqrt{|u|}\leqslant\sqrt{|v|} \end{cases}$$

In this case we have

$$(5.3')\qquad \mathrm{Exp}_{(\lambda)}(C^2;\ (|\lambda|B^{-1},\ |\lambda|A^{-1})^\circ)\subset \widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|B^{-1},\ |\lambda|A^{-1})^\circ)\ ,$$

because we have $A^{-1}(1+\varepsilon)^{-1}\leqslant B(1+\varepsilon)$ and $B^{-1}(1+\varepsilon)^{-1}\leqslant A(1+\varepsilon)$ for any $\varepsilon>0$.

Finally we denote by $\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (0,0))$ the space of all $F\in\mathcal{O}_{(\lambda)}(C^2)$ for which, for any $\varepsilon>0$ there exists $C_\varepsilon\geqslant 0$ such that the majoration (5.4) is valid:

$$(5.4)\quad \begin{cases} |F(u,v)|\leqslant C_\varepsilon\exp(\varepsilon|u|+\varepsilon^{-1}|v|) & \text{if } \sqrt{|v|}\leqslant\varepsilon\sqrt{|u|}\ , \\ |F(u,v)|\leqslant C_\varepsilon\exp(2\sqrt{|uv|}) & \text{if } \varepsilon\sqrt{|u|}<\sqrt{|v|}<\varepsilon^{-1}\sqrt{|u|}\ , \\ |F(u,v)|\leqslant C_\varepsilon\exp(\varepsilon^{-1}|u|+\varepsilon|v|) & \text{if } \varepsilon^{-1}\sqrt{|u|}\leqslant\sqrt{|v|}\ . \end{cases}$$

In this case we have

$$(5.5)\qquad \mathrm{Exp}_{(\lambda)}(C^2;\ (0,0))\subset\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (0,0))\ .$$

THEOREM 5.1. *Suppose* $AB>1$(resp. $AB\geqslant 1$). *The transformation* $\mathcal{F}_\lambda$ *establishes a linear isomorphism:*

$$(5.6)\qquad \mathcal{F}_\lambda\colon \mathcal{O}(K^\circ_{A,B})\xrightarrow{\ \sim\ }\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|B^{-1},\ |\lambda|A^{-1}))$$

$$\text{(resp. (5.6')}\quad \mathcal{F}_\lambda\colon \mathcal{O}(K_{A,B})\xrightarrow{\ \sim\ }\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|B^{-1},\ |\lambda|A^{-1})^\circ))\ .$$

PROOF. We have only to prove the surjectivity. If

$$F\in\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|B^{-1},\ |\lambda|A^{-1}))\ ,$$

then $F(u,0)$(resp. $F(0,v)$) is of exponential type $\leqslant|\lambda|B^{-1}$(resp. $\leqslant|\lambda|A^{-1}$) (see Lemma 1.1). Therefore we can argue as in the proof of Theorem 4.2 using Proposition 1.1. q.e.d.

COROLLARY 1.

$$(5.7)\qquad \mathscr{F}_\lambda\colon \mathscr{O}(K^\circ_{A,A}) \xrightarrow{\sim} \widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|A^{-1},\ |\lambda|A^{-1})) \qquad \text{if}\quad A>1\ .$$

$$(5.7')\qquad \mathscr{F}_\lambda\colon \mathscr{O}(K_{A,A}) \xrightarrow{\sim} \widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (|\lambda|A^{-1},\ |\lambda|A^{-1})^\circ) \qquad \text{if}\quad A\geqslant 1\ .$$

COROLLARY 2.

$$(5.8)\qquad \mathscr{F}_\lambda\colon \mathscr{O}(C^*) \xrightarrow{\sim} \widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ (0,\ 0))\ .$$

Now we are going to determine the $\mathscr{F}_\lambda$-image of Exp(C^*). Suppose $F\in \mathrm{Exp}(C^*)$ satisfies (1.10). Then by (5.1), we have

$$\begin{aligned} |F_\lambda(u,\ v)| &= |\mathscr{F}_\lambda(f)(u,\ v)| \\ &\leqslant C\exp\Big(M\Big(c+\frac{1}{c}\Big)\Big)\exp\Big(|\lambda|\Big(c|u|+\frac{|v|}{c}\Big)\Big) \\ &= C\exp\Big((M+|\lambda|\,|u|)c+(M+|\lambda|\,|v|)\frac{1}{c}\Big) \end{aligned}$$

for every c with $0<c<\infty$. Therefore the function $F(u,\ v)=F_\lambda(u,\ v)$ satisfies the following estimate:

$$(5.9)\qquad |F(u,\ v)|\leqslant C\exp(2\sqrt{(M+|\lambda|\,|u|)(M+|\lambda|\,|v|)})\ .$$

DEFINITION 5.2. Let us denote by $\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ 1/2)$ the subspace of $\mathscr{O}_{(\lambda)}(C^2)$ whose elements satisfy (5.9) with some $M>0$ and $C\geqslant 0$.

THEOREM 5.2. *The transformation $\mathscr{F}_\lambda$ establishes a linear isomorphism of the space* Exp(C^*) *onto the space* $\widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ 1/2)$:

$$(5.10)\qquad \mathscr{F}_\lambda\colon \mathrm{Exp}(C^*) \xrightarrow{\sim} \widetilde{\mathrm{Exp}}_{(\lambda)}(C^2;\ 1/2)\ .$$

PROOF. We have only to prove the surjectivity. Suppose $F_\lambda(u,\ v)$ satisfies (5.9). Then, in particular,

$$|F_\lambda(u,\ 0)|\leqslant C\exp(2\sqrt{M(M+|\lambda|\,|u|)})\leqslant Ce^{2M}\exp(2\sqrt{M|\lambda|\,|u|})\ .$$

If we write $F_\lambda(u,\ v)=\sum_{n,m=0}^\infty a_{n,m}u^n v^m$, then

$$|a_{n,0}|\leqslant Ce^{2M}r^{-n}\exp(2\sqrt{M|\lambda|r})$$

for every $r>0$. Therefore, putting $r=n^2/M|\lambda|$, we get

$$|a_{n,0}|\leqslant Ce^{2M}\Big(\frac{n^2}{M|\lambda|}\Big)^{-n}\exp(2n)\ .$$

Similarly we have

$$|a_{0,m}| \leqslant Ce^{2M}\left(\frac{m^2}{M|\lambda|}\right)^{-m} \exp(2m) .$$

We put $c_{-p}=(p!/\lambda^p)a_{p,0}$ and $c_p=(p!/\lambda^p)a_{0,p}$ by (4.5). Now we have

$$\begin{aligned}\limsup_{p\to\infty} \sqrt[p]{p!\,|c_{-p}|} &= |\lambda|^{-1} \limsup_{p\to\infty} \sqrt[p]{(p!)^2|a_{p,0}|} \\ &\leqslant |\lambda|^{-1} \limsup_{p\to\infty} \left\{\left(2\sqrt{\pi}p^{p+1/2}e^{-p}\right)^2 Ce^{2M}\left(\frac{p^2}{M|\lambda|}\right)^{-p} \exp(2p)\right\}^{1/p} \\ &= |\lambda|^{-1} \limsup_{p\to\infty} (2\sqrt{\pi})^{2/p} p^{1/p} (Ce^{2M})^{1/p} \left(\frac{1}{M|\lambda|}\right)^{-1} \leqslant |\lambda|^{-1} M|\lambda| = M ,\end{aligned}$$

where we used Stirling's formula. Therefore by Lemma 1.1,

$$\left|\sum_{p=1}^{\infty} c_{-p}z^{-p}\right| \leqslant C_1 \exp(2M|z|^{-1}) \quad \text{for} \quad z\neq 0 .$$

Similarly we have

$$\left|\sum_{p=0}^{\infty} c_p z^p\right| \leqslant C_2 \exp(2M|z|) .$$

Therefore we have

$$\begin{aligned}\left|\sum_{p=-\infty}^{\infty} c_p z^p\right| &\leqslant C_1 \exp(2M|z|^{-1}) + C_2 \exp(2M|z|) \\ &\leqslant C \exp(2M(|z|^{-1}+|z|))\end{aligned}$$

with some constants C_1, C_2 and $C\geqslant 0$, which proves the holomorphic function on C^*, $\sum_{p=-\infty}^{\infty} c_p z^p$, is of exponential type. It is clear by (4.5) that the $\mathscr{F}_\lambda$-image of the function $\sum_{p=-\infty}^{\infty} c_p z^p$ is equal to F_λ. q.e.d.

§ 6. The properties of the functions in $\mathscr{O}_{(\lambda)}(C^2)$.

Looking at the results of §§4 and 5, we can give some precisions to Theorem 3.1.

THEOREM 6.1. *Suppose* $F(u, v) \in \mathscr{O}_{(\lambda)}(C^2)$.

(i) *If* $F(u, 0)$ *and* $F(0, v)$ *are of exponential type,* $F(u, v)$ *is of exponential type.*

(ii) *Suppose* $AB\geqslant 1$. *If* $F(u, 0)$ *is of exponential type* $\leqslant|\lambda|A$ *and* $F(0, v)$ *is of exponential type* $\leqslant|\lambda|B$, $F(u, v)$ *is of exponential type* $(|\lambda|A, |\lambda|B)$.

(iii) *Suppose* $AB<1$. *If* $F(u, 0)$ *is of exponential type* $\leqslant|\lambda|A$ *and* $F(0, v)$ *is of exponential type* $\leqslant|\lambda|B$, $F(u, v)$ *is in the space*

$$\widetilde{\mathrm{Exp}}_{(\lambda)}(\boldsymbol{C}^2; (|\lambda|A, |\lambda|B)) ,$$

(*which is seemingly strictly larger than the space* $\mathrm{Exp}_{(\lambda)}(\boldsymbol{C}^2; (|\lambda|A, |\lambda|B))$).

(iv) *If* $F(u, 0)$ *and* $F(0, v)$ *are of minimal exponential type, then* $F(u, v)$ *belongs to* $\widetilde{\mathrm{Exp}}_{(\lambda)}(\boldsymbol{C}^2; (0, 0))$.

(v) *If* $F(u, 0)$ *and* $F(0, v)$ *are of order* $1/2$, *then* $F(u, v)$ *belongs to the space* $\widetilde{\mathrm{Exp}}_{(\lambda)}(\boldsymbol{C}^2; 1/2)$.

(vi) *If* $F(u, 0)=F(0, v)=a$(*constant*), *then*

$$F(u, v)=aJ_0(2i\lambda\sqrt{uv}) . \tag{6.1}$$

§ 7. The transformation $\mathscr{P}_\lambda$.

Now we consider the unit circle $S^1=\{(x_1, x_2)\in \boldsymbol{R}^2;\ x_1^2+x_2^2=1\}$. We shall denote by $\mathscr{A}(S^1)$ the space of all real-analytic functions on S^1. Putting $z=x_1+ix_2$, we have

$$S^1=\{(x_1, x_2)\in \boldsymbol{R}^2;\ x_1^2+x_2^2=1\}=\{z\in \boldsymbol{C};\ |z|=1\}=K_{1,1} . \tag{7.1}$$

The spaces $\mathscr{A}(S^1)$ and $\mathscr{O}(K_{1,1})$ are equal. We shall denote by $\mathscr{B}(S^1)$ the space of all hyperfunctions on S^1, that is, by definition, $\mathscr{B}(S^1)=\mathscr{A}'(S^1)=\mathscr{O}'(K_{1,1})$. If we fix $(\zeta_1, \zeta_2)\in \boldsymbol{C}^2$, the function $\exp(i\lambda(\zeta_1x_1+\zeta_2x_2))$ is real-analytic on S^1. We shall define, for $T\in\mathscr{B}(S^1)$,

$$G_\lambda(\zeta_1, \zeta_2)=\langle T_{x_1,x_2}, \exp(i\lambda(\zeta_1x_1+\zeta_2x_2))\rangle . \tag{7.2}$$

The transformation $T\mapsto G_\lambda$ will be denoted by $\mathscr{P}_\lambda$.

As we have $z\bar{z}=1$ on $K_{1,1}$,

$$x_1=\frac{z+z^{-1}}{2} , \quad x_2=\frac{z-z^{-1}}{2i} \quad \text{on} \quad S^1 . \tag{7.3}$$

Therefore we have

$$\begin{aligned} G_\lambda(\zeta_1, \zeta_2) &= < T_z, \exp\left(i\lambda\left(\zeta_1\frac{z+z^{-1}}{2}+\zeta_2\frac{z-z^{-1}}{2i}\right)\right) \\ &= \mathscr{F}_\lambda(T)(i(\zeta_1-i\zeta_2)/2,\ i(\zeta_1+i\zeta_2)/2) . \end{aligned} \tag{7.4}$$

By this formula the transformation $\mathscr{P}_\lambda$ can be extended to $\mathrm{Exp}'(\boldsymbol{C}^*)$, consequently to any of its subspaces.

Let us denote by $\tilde{S}^1$ the complexified circle, that is,

$$\tilde{S}^1=\{(z_1, z_2)\in \boldsymbol{C}^2;\ z_1^2+z_2^2=1\} . \tag{7.5}$$

It is clear that $\tilde{S}^1$ and $\boldsymbol{C}^*$ are complex-analytically diffeomorphic by

$z_1+iz_2=z$. Therefore the spaces $\mathcal{O}(\boldsymbol{C}^*)$, $\mathrm{Exp}(\boldsymbol{C}^*)$ and others can be considered to be the spaces of functions on the complexified circle $\tilde{S}^1$. We may note $\mathcal{O}(\tilde{S}^1)=\mathcal{O}(\boldsymbol{C}^*)$, $\mathrm{Exp}(\tilde{S}^1)=\mathrm{Exp}(\boldsymbol{C}^*)$ etc.

THEOREM 7.1. *Suppose $c_k=\langle T_z, z^{-k}\rangle$, $k\in\boldsymbol{Z}$, are the Laurent coefficients of $T\in\mathrm{Exp}'(\tilde{S}^1)$. Then the function $G_\lambda(\zeta_1, \zeta_2)$ defined by* (7.2) *or* (7.4) *can be expressed as follows:*

$$G_\lambda(\zeta_1, \zeta_2)=c_0J_0(\lambda\sqrt{\zeta_1^2+\zeta_2^2}) + \sum_{p=1}^{\infty} i^p(c_{-p}(\zeta_1-i\zeta_2)^p+c_p(\zeta_1+i\zeta_2)^p)(\zeta_1^2+\zeta_2^2)^{-p/2}J_p(\lambda\sqrt{\zeta_1^2+\zeta_2^2})\ . \tag{7.6}$$

PROOF. Let us consider the following change of variables:

$$u=i(\zeta_1-i\zeta_2)/2\ ,\quad v=i(\zeta_1+i\zeta_2)/2\ , \tag{7.7}$$

$$\zeta_1=-i(u+v)\ ,\quad \zeta_2=u-v\ . \tag{7.8}$$

Then we have

$$-2i\sqrt{uv}=\sqrt{\zeta_1^2+\zeta_2^2}\ . \tag{7.9}$$

Therefore the formula (7.6) is derived from (4.5) of Theorem 4.1, if we take into account the formula (7.4). q.e.d.

COROLLARY. (Hashizume-Kowata-Minemura-Okamoto [2]). *Let $G_\lambda(\xi_1, \xi_2)$ denote the restriction of the function $G_\lambda(\zeta_1, \zeta_2)$ on $\boldsymbol{R}^2$. If we put $\xi_1+i\xi_2=re^{i\varphi}$, $r\geqslant 0$, $\varphi\in\boldsymbol{R}$, we have*

$$G_\lambda(re^{i\varphi})=c_0J_0(\lambda r)+\sum_{p=1}^{\infty} i^p(c_{-p}e^{-ip\varphi}+c_pe^{ip\varphi})J_p(\lambda r)\ . \tag{7.10}$$

(cf. Lemma 2 of [2], where the formulas (7.6) and (7.10) are given even in the higher dimensional case. See also [8].)

THEOREM 7.2 (Helgason [3]). *The transformation $\mathcal{P}_\lambda$ is a linear isomorphism of* $\mathrm{Exp}'(\tilde{S}^1)$ *onto the space $\mathcal{O}_\lambda(\boldsymbol{C}^2)$ of the entire functions G on $\boldsymbol{C}^2$ which satisfy the differential equation*

$$\left(\frac{\partial^2}{\partial\zeta_1^2}+\frac{\partial^2}{\partial\zeta_2^2}+\lambda^2\right)G(\zeta_1, \zeta_2)=0\ . \tag{7.11}$$

PROOF. Under the change of variables (7.7) and (7.8), we have

$$\frac{\partial^2}{\partial\zeta_1^2}+\frac{\partial^2}{\partial\zeta_2^2}=-\frac{\partial^2}{\partial u\partial v}\ .$$

Therefore the theorem is a reformulation of Theorem 4.1. q.e.d.

REMARK. Let us denote by $C_\lambda^\infty(\boldsymbol{R}^2)$ the space of C^∞ functions g on $\boldsymbol{R}^2$ satisfying the differential equation

$$\left(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}+\lambda^2\right)g(\xi_1, \xi_2)=0 .$$

Now the Laplacian is an elliptic differential operator with constant coefficients, the restriction $\mathcal{O}_\lambda(\boldsymbol{C}^2)\to C_\lambda^\infty(\boldsymbol{R}^2)$ is bijective. Therefore we can say the transformation $\mathscr{P}_\lambda$ establishes a linear isomorphism of $\mathrm{Exp}'(\tilde{S}^1)$ onto $C_\lambda^\infty(\boldsymbol{R}^2)$. The space $\tilde{\mathscr{B}}(\tilde{S}^1)$ defined by [2] is nothing but the space $\mathrm{Exp}'(\tilde{S}^1)$ of entire functionals (Helagason [3]).

The $\mathscr{P}_\lambda$-images of subspaces of $\mathrm{Exp}'(\tilde{S}^1)$ can be described by the simple transcription of the results on the transformation $\mathscr{F}_\lambda$. For example, we have

THEOREM 7.3. (i) *The function* $G\in\mathcal{O}_\lambda(\boldsymbol{C}^2)$ *is in the* $\mathscr{P}_\lambda$*-image of* $\mathcal{O}'(\tilde{S}^1)$ *if and only if the function* G *is of exponential type.*

(ii) *The function* $G\in\mathcal{O}_\lambda(\boldsymbol{C}^2)$ *is in the* $\mathscr{P}_\lambda$*-image of the space of hyperfunctions* $\mathscr{B}(S^1)$, *if and only if, for any* $\varepsilon>0$ *there exists* $C_\varepsilon\geq 0$ *such that*

$$|G(\xi_1, \xi_2)|\leq C_\varepsilon \exp\left(\lambda(1+\varepsilon)\left(\frac{|\zeta_1-i\zeta_2|}{2}+\frac{|\zeta_1+i\zeta_2|}{2}\right)\right) .$$

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