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The Riemann-Hilbert Problem in Several Complex Variables II

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Introduction

In the preceding paper [6], the author proved that, in a *two-dimensional* connected Stein manifold X satisfying the condition $H^2(X, \boldsymbol{Z})=0$, one can solve the Riemann-Hilbert problem without apparent singularities for an arbitrary divisor D and an arbitrary representation of $\pi_1(X-D, *)$ into $GL_q(\boldsymbol{C})$. The purpose of the present paper is to give an example of the Riemann-Hilbert problemwhich cannot be solved without apparent singularities *by the same method as in the two-dimensional case.* More precisely, let S be a 3-dimensional polydisc; then, by a result of H. Lindel [7], there exists a special divisor D of S such that we can construct a flat vector bundle V of rank q over $S-D$ satisfying the following conditions:

1) There exists an integrable holomorphic connection ∇ on $\mathscr{O}(V)$ such that $\operatorname{Ker}\nabla=\boldsymbol{C}(V)$ where $\boldsymbol{C}(V)$ is the sheaf of germs of locally constant sections of V.

2) $\mathscr{O}(V)$ is extended to a locally free analytic sheaf $\mathscr{H}$ on $S-\operatorname{Sing}(D)$ on which ∇ is the meromorphic connection with logarithmic poles along $D\cap(S-\operatorname{Sing}(D))$. The eigenvalues $\alpha_1, \cdots, \alpha_q$ of the residue of ∇ at any point of $D-\operatorname{Sing}(D)$ are rational numbers and satisfy the inequalities $0\leqq\alpha_i<1$ for $i=1, \cdots, q$.

3) $\mathscr{H}$ is extended *uniquely* to a coherent analytic sheaf $\tilde{\mathscr{H}}$ on S satisfying $\tilde{\mathscr{H}}^{[1]}=\tilde{\mathscr{H}}$, but $\mathscr{H}$ *cannot be extended to any locally free analytic sheaf on* S, where $\tilde{\mathscr{H}}^{[1]}$ is the *first absolute gap-sheaf* of $\tilde{\mathscr{H}}$ (for the definition of absolute gap-sheaves, see [9]).

It seems to the author that if, in three dimension, one wants to solve the Riemann-Hilbert problem without apparent singularities even in the local sense, one should study in detail the Manin extension (See 1.2.) and the structure of vector bundles which are meromorphic along a divisor (see [3]), and should take deeper consideration on the equation

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of L. Schlesinger and Lappo-Danilevski (see the papers of K. Aomoto [1], [2]).

§1. Analytic covers and connections with logarithmic poles.

1.1. Let X be an n-dimensional *normal* analytic space and let S be an n-dimensional connected complex manifold. Suppose that a *finite* holomorphic mapping $f: X \to S$ is given; then there is a divisor D of S such that $f: X^* \to S^*$ is an *unramified* covering of the sheet number q, where we put $X^* := X - f^{-1}(D)$ and $S^* := S - D$. We denote by C_{X^*} the constant sheaf on X^* with coefficients in C and by $\mathcal{O}_{X^*}$ the structural sheaf of the complex manifold X^*. Then we can consider two sheaves on S^*; one is the direct image $f_*(C_{X^*}) =: V$ of C_{X^*} which is a locally constant sheaf with coefficients in C^q and the other is the direct image $f_*(\mathcal{O}_{X^*})$ of $\mathcal{O}_{X^*}$ which is a locally free analytic sheaf of rank q. It is easy to see that $f_*(\mathcal{O}_{X^*}) = V \otimes_C \mathcal{O}_{S^*}$. It follows that there exists a unique integrable holomorphic connection

$$\nabla : f_*(\mathcal{O}_{X^*}) \longrightarrow \Omega^1_{S^*} \otimes_{\mathcal{O}_{S^*}} f_*(\mathcal{O}_{X^*})$$

satisfying the condition $\operatorname{Ker} \nabla = V$. Put $S' := S - \operatorname{Sing}(D)$ where $\operatorname{Sing}(D)$ is the singular locus of the divisor D. We write $X' := f^{-1}(S')$. Then $f: X' \to S'$ is a finite holomorphic mapping and from an elementary fact about analytic covers (see [4]), it follows that X' does not have any singular points. For later applications, we recall the following standard results about analytic local C-algebras (for the proof, see [5]).

LEMMA 1. *Let A and B be n-dimensional analytic local C-algebras. Suppose that A is regular and that a finite homomorphism $\varphi: A \to B$ is given. Then B is a free A-module of finite rank if and only if B is a Macaulay ring.*

X' being non-singular, the local ring $\mathcal{O}_{X',x}$ at any point $x \in X'$ is regular; hence $\mathcal{O}_{X',x}$ is a Macaulay ring. Since $f: X' \to S'$ is a finite holomorphic mapping and S' is a complex manifold, it follows from Lemma 1 that the direct image $f_*(\mathcal{O}_{X'})$ of the structural sheaf $\mathcal{O}_{X'}$ of X' is a *locally free analytic sheaf on S' of rank q*. In the rest of 1.1. we shall prove the following

THEOREM 1. *The connection ∇ on $f_*(\mathcal{O}_{X^*})$ is extended to the meromorphic connection $\tilde{\nabla}$ on $f_*(\mathcal{O}_{X'})$ with logarithmic poles along $D \cap S'$. The eigenvalues $\alpha_1, \cdots, \alpha_q$ of the residue of $\tilde{\nabla}$ at an arbitrary point of $D \cap S'$ are rational numbers and satisfy the inequalities $0 \leqq \alpha_i < 1$ for $i = 1, \cdots, q$.*

The problem is local on $D\cap S'$. For an arbitrary point x of $D\cap S'$, we can take a small polydisc centered at x such that the following conditions are satisfied:

1) $U\subset S$ and $U\cap \mathrm{Sing}(D)=\emptyset$.

2) There exists an open polydisc Δ in $\boldsymbol{C}^n(z_1, \cdots, z_n)$ centered at the origin and a complex manifold W (*not necessarily connected*) where $(z_1, \cdots, z_n)$ is the coordinate system of $\boldsymbol{C}^n$.

3) There exists a finite holomorphic mapping $\tau\colon W\to\Delta$ with the critical locus $A:=\{z\in\Delta \mid z_n=0\}$.

4) The following diagram is commutative and the *horizontal arrows are biholomorphic mappings*:

$$\begin{tikzcd}
f^{-1}(U) \arrow[r] \arrow[d, "f"'] & W \arrow[d, "\tau"] \\
U \arrow[r] & \Delta \\
D\cap U \arrow[r] \arrow[u, hook, "i"] & A \arrow[u, hook, "j"']
\end{tikzcd}$$

where i and j are the inclusion mappings.

We put $W^*:=W-\tau^{-1}(A)$ and $\Delta^*:=\Delta-A$. By the condition 4), the problem is reduced to showing that the holomorphic connection ∇ on $\tau_*(\mathcal{O}_{W^*})$ with $\mathrm{Ker}\,\nabla=\tau_*(\boldsymbol{C}_{W^*})$ is extended to the meromorphic connection $\tilde{\nabla}$ on $\tau_*(\mathcal{O}_W)$ with logarithmic poles along A. If $W=\bigcup_{i=1}^k W_i$ is the decomposition of W into connected components, then the direct images $\tau_*(\boldsymbol{C}_{W^*})$, $\tau_*(\mathcal{O}_{W^*})$ and $\tau_*(\mathcal{O}_W)$ are decomposed into the direct sums $\tau_*(\boldsymbol{C}_{W^*})=\bigoplus_{i=1}^k \tau_*(\boldsymbol{C}_{W_i^*})$, $\tau_*(\mathcal{O}_{W^*})=\bigoplus_{i=1}^k \tau_*(\mathcal{O}_{W_i^*})$ and $\tau_*(\mathcal{O}_W)=\bigoplus_{i=1}^k \tau_*(\mathcal{O}_{W_i})$ where $W_i^*=W_i-\tau^{-1}(A)\cap W_i$. So we have reduced proving Theorem 1 to the case where W is connected. When W is connected, we can regard the analytic cover $\tau\colon W\to\Delta$ as follows, by a well-known fact about analytic covers (see [4]); let Y be an n-dimensional non-singular affine algebraic variety in $\boldsymbol{C}^{n+1}(z_1, \cdots, z_n, w)$ defined by the equation $w^q-z_n=0$ and let $p\colon \boldsymbol{C}^{n+1}\to\boldsymbol{C}^n(z)$ be the natural projection. Put $W:=p^{-1}(\Delta)\cap Y$ and let $\tau\colon W\to\Delta$ be the holomorphic map induced by the projection p. It is obvious that τ is a finite holomorphic mapping with the critical locus $A=\{z\in\Delta \mid z_n=0\}$; this is our model of the analytic cover $\tau\colon W\to\Delta$. Hence *it is sufficient to prove Theorem* 1 *in the above situation.*

Let $a\in A$ and take a small polydisc N centered at a in Δ. Let s be a section of $\tau_*(\mathcal{O}_W)$ on N; then by the definition of direct image, there is a holomorphic function g on $\tau^{-1}(N)$ which corresponds to s under the isomorphism $\Gamma(\tau^{-1}(N), \mathcal{O}_W) \simeq \Gamma(N, \tau_*(\mathcal{O}_W))$. Since $\tau^{-1}(N)$ is a closed complex submanifold of $p^{-1}(N) = N \times C$ and since $N \times C$ is a Stein manifold, there exists, by Theorem B on Stein manifolds, a holomorphic function $G(z, w)$ on $N \times C$ such that the restriction $G|\tau^{-1}(N)$ of G to $\tau^{-1}(N)$ coincides with g. For an abitrary $z \in N$, we see that the number of the roots of the equation $w^q - z_n = 0$ is always equal to q (properly counted with multiplicities). Hence by the *division theorem of Oka* (see [8], p. 109), G can be written *in a unique manner* in the form

$$(1) \qquad G(z, w) = (w^q - z_n)Q(z, w) + H(z, w)$$

where Q is holomorphic in $N \times C$ and H has the following form:

$$H(z, w) = a_0(z) + a_1(z)w + \cdots + a_{q-1}(z)w^{q-1}$$

with each $a_i(z)$ holomorphic on N. It is obvious that $G|\tau^{-1}(N) = H|\tau^{-1}(N)$; hence, putting $w^k|\tau^{-1}(N) = s_k$ $(k = 0, \cdots, q-1)$, we have that

$$g = a_0(z)s_0 + a_1(z)s_1 + \cdots + a_{q-1}(z)s_{q-1} .$$

Since s_k $(k = 0, \cdots, q-1)$ can be regarded as a section of $\tau_*(\mathcal{O}_W)$ over N, we obtain the following:

$$s = a_0(z)s_0 + \cdots + a_{q-1}(z)s_{q-1} ;$$

here we are identifying g with the section $s \in \Gamma(N, \tau_*(\mathcal{O}_W))$. The uniqueness of the expression (1) shows that the sections $s_0, \cdots, s_{q-1}$ are linearly independent over $\Gamma(N, \mathcal{O}_N)$. Since N is an arbitrary small polydisc centered at a, it follows that, putting $e_k = w^k| W$ $(k = 0, \cdots, q-1)$, the set $(e_0, \cdots, e_{q-1})$ is a basis of the locally free analytic sheaf $\tau_*(\mathcal{O}_W)$ over Δ.

We will express explicitly the locally constant sheaf $\tau_*(C_{W^*})$ over Δ^* by means of the basis $(e_0, \cdots, e_{q-1})$. Let b be an arbitrary point of Δ^* and let $N(b)$ be a small polydisc centered at b in Δ^*. Let $\tau^{-1}(N(b)) = \bigcup_{i=1}^q N_i$ be the decomposition of $\tau^{-1}(N(b))$ into connected components and we fix over $N(b)$ a branch $(z_n)^{1/q}$ of the many-valued holomorphic function defined by the equation $w^q - z_n = 0$. Then by changing the indices of N_i, if necessary, we can identify the restriction $w|N_i$ of w to N_i with $\zeta^{i-1}(z_n)^{1/q}$ $(i = 1, \cdots, q)$, where $\zeta = \exp(2\pi i/q)$. Since $(e_0, \cdots, e_{q-1})$ is a basis of $\tau_*(\mathcal{O}_W)$ over $N(b)$, it follows that, for any section v of $\tau_*(C_{W^*})$ over $N(b)$, there exist holomorphic functions $b_0(z), \cdots, b_{q-1}(z)$ on $N(b)$ such that

$$v=b_0(z)e_0+\cdots+b_{q-1}(z)e_{q-1} .$$

Observing that $\tau^1(N(b))=\bigcup_{i=1}^q N_i$, we have

$$v|N_i=b_0(z)(e_0|N_i)+\cdots+b_{q-1}(z)(e_{q-1}|N_i)$$
$$\text{for} \quad i=1, \cdots, q .$$

Since we have identified $e_1|N_i$ with $\zeta^{i-1}(z_n)^{1/q}$ for $i=1, \cdots, q$, we obtain the following relations, putting $v|N_i=v_{i-1}\in \boldsymbol{C}$:

$$\begin{aligned}(2) \qquad v_i=b_0(z)+b_1(z)\zeta^{i-1}(z_n)^{1/q}+\cdots\\ +b_{q-1}(z)\zeta^{(q-1)(i-1)}(z_n)^{(q-1)/q} \qquad (i=1, \cdots, q) .\end{aligned}$$

If we put $\hat{b}_i(z)=b_i(z)(z_n)^{i/q}$ $(i=0, \cdots, q-1)$, we can rewrite (2) in matrix notations in the following form:

$$A\begin{pmatrix}\hat{b}_0\\ \hat{b}_1\\ \vdots\\ \hat{b}_{q-1}\end{pmatrix}=\begin{pmatrix}v_0\\ v_1\\ \vdots\\ v_{q-1}\end{pmatrix},$$

where

$$(3) \qquad A=\begin{pmatrix}1 & 1 & \cdots & 1\\ 1 & \zeta & \cdots & \zeta^{(q-1)}\\ & & \cdots & \\ 1 & \zeta^{q-1} & \cdots & \zeta^{(q-1)(q-1)}\end{pmatrix}.$$

The determinant of the matrix A is non-zero by a result of van der Monde; hence we see that, in order for the section v of $\tau_*(\mathcal{O}_{W^*})$ to be *constant* on $N(b)$, it is necessary and sufficient that the function $(\hat{b}_0(z), \cdots, \hat{b}_{q-1}(z))$ is constant on $N(b)$. This means that, when we use the basis $(e_0, \cdots, e_{q-1})$ of $\tau_*(\mathcal{O}_W)$ over $N(b)$, any section v of $\tau_*(\boldsymbol{C}_{W^*})$ over $N(b)$ can be written in the following form:

$$v=c_0e_0+c_1z_n^{-1/q}e_1+\cdots+c_{q-1}z_n^{-(q-1)/q}e_{q-1}$$

where $c_0, \cdots, c_{q-1}$ are arbitrary constants: $c_i\in \boldsymbol{C}$. v is a horizontal section of ∇. So, writing $\nabla e_i=\sum_{i=0}^{q-1}\omega_{ji}e_j$, we have

$$0=\nabla v=\sum_{i=0}^{q-1}c_iz_n^{-i/q}\nabla e_i+\sum_{i=0}^{q-1}c_i\left(-\frac{i}{q}\frac{dz_n}{z_n}\right)z_n^{-i/q}e_i ;$$

hence by an elementary computation, we conclude that the connection matrix $\Gamma=(\omega_{ij})$ is written in the following form:

$$\Gamma=\begin{pmatrix} 0 & & & \\ & 1/q & & 0 \\ & 0 & \ddots & \\ & & & (q-1)/q \end{pmatrix}\frac{dz_n}{z_n}.$$

This formula shows that ∇ is the meromorphic connection on $\tau_*(\mathcal{O}_W)$ with logarithmic poles along A and the eigenvalues of the residue of ∇ are $0, 1/q, \cdots, (q-1)/q$. *This completes the proof of Theorem* 1.

Q.E.D.

1.2. Let M be an arbitrary connected complex manifold and let D be a *normal crossing divisor*. Suppose that a flat vector bundle V of rank q on $M-D$ is given; then there is a unique holomorphic integrable connection ∇ on $\mathcal{O}(V)$ such that $\operatorname{Ker}\nabla=\boldsymbol{C}(V)$. As is well-known, Deligne-Manin [3] proved that the vector bundle V is extended uniquely to a holomorphic vector bundle $\tilde{V}$ on which ∇ is the meromorphic connection with logarithmic poles along D. Moreover the eigenvalues $\alpha_1, \cdots, \alpha_q$ of the residue of ∇ at any point of D satisfy the inequalities $0\leqq\operatorname{Re}\alpha_i<1$. We shall call such an extension of the flat vector bundle V the *Manin extension* of V. Turning to our situation, let the notations be the same as those in Theorem 1. From Theorem 1, it follows that the locally free analytic sheaf $f_*(\mathcal{O}_{X'})$ is the Manin extension of the flat vector bundle $f_*(\mathcal{O}_{X^*})$. Hence we have the following:

COROLLARY 1 TO THEOREM 1. *$f_*(\mathcal{O}_{X'})$ is the Manin extension of the flat vector bundle $f_*(\mathcal{O}_{X^*})$.*

Let X be a normal complex space and S be an n-dimensional connected complex manifold. Let $f\colon X\to S$ be a finite holomorphic mapping with the critical locus D. *We suppose that D is normal crossing.* Since $f_*(\mathcal{O}_{X^*})=:V$ is a flat vector bundle on $S-D$ with the integrable holomorphic connection ∇ such that $\operatorname{Ker}\nabla=\boldsymbol{C}(V)$, it follows from the result of Deligne-Manin quoted above that there exists the Manin extension $\tilde{V}$ of V which is locally free on S. By the definition of the $(n-2)$-th absolute gap-sheaf and the continuation theorem of Hartogs, we have $\tilde{V}^{[n-2]}=\tilde{V}$ where $\tilde{V}^{[n-2]}$ is the $(n-2)$-th absolute gap-sheaf of $\tilde{V}$. On the other hand, from the Corollary 1 to Theorem 1, it follows that $f_*(\mathcal{O}_{X'})$ is the Manin extension of V on S'. Since Manin extension is unique, we have $\tilde{V}|S'=f_*(\mathcal{O}_{X'})$. X is normal and $f\colon X\to S$ is a finite holomorphic mapping; hence by Hartogs' continuation theorem, we see that $(f_*(\mathcal{O}_X))^{[n-2]}=f_*(\mathcal{O}_X)$. Sing (D) being of codimension at least two and $f_*(\mathcal{O}_X)$ coherent on S, we conclude, by a result of Y.-T. Siu ([9], p. 202), that two coherent extensions $\tilde{V}$ and $f_*(\mathcal{O}_X)$ of $\tilde{V}|S'=f_*(\mathcal{O}_{X'})$ on S are *isomorphic*; therefore it follows that $f_*(\mathcal{O}_X)$ is *locally free* on S. Hence we obtain the following:

COROLLARY 2 TO THEOREM 1. *Let $f\colon X\to S$ be as above. We suppose that the critical locus D of $f\colon X\to S$ is normal crossing; then the direct image $f_*(\mathcal{O}_X)$ is a locally free analytic sheaf on S.*

§2. An example to the Riemann-Hilbert problem.

H. Lindel [7] gave the following example; let X be an analytic space defined by the following equations in C^6 $(x_0, x_1, x_2, y_0, y_1, y_2)$, $x_iy_j-x_jy_i=0$ $(i, j\neq 0, 1, 2, i\neq j)$, $\sum_{i=0}^2 x_i^3=0$, $\sum_{i=0}^2 x_i^2y_i=0$, $\sum_{i=0}^2 x_iy_i^2=0$, $\sum_{i=0}^2 y_i^3=0$. X is a 3-dimensional analytic space with the only isolated singular point $x_0=(0, \cdots, 0)$. Then X is *normal*, but the local ring $\mathcal{O}_{X,x_0}$ of X at x_0 is *not a Macaulay ring*. By a well-known local theory of analytic spaces, there exists a finite holomorphic mapping $f\colon X\to S=C^3$. From Lemma 1, it follows that the direct image $f_*(\mathcal{O}_X)$ is *not a locally free analytic sheaf on* S. Let D be a critical locus of $f\colon X\to S$ and put $S':=S-\mathrm{Sing}(D)$. We write $X':=f^{-1}(S')$. By Corollary 1 to Theorem 1, we see that the direct image $f_*(\mathcal{O}_{X'})$ is the Manin extension of the flat vector bundle $f_*(\mathcal{O}_{X^*})$. If the locally free analytic sheaf $f_*(\mathcal{O}_{X'})$ could be extended to a locally free analytic sheaf $\mathcal{L}$ on S, then by the same reason as in the proof of Corollary 2 to Theorem 1, we would have $f_*(\mathcal{O}_X)=\mathcal{L}$. Since $f_*(\mathcal{O}_X)$ is not locally free, this is contradiction. Thus $f_*(\mathcal{O}_{X'})$ cannot be extended to a locally free analytic sheaf on S. Hence we have the following:

THEOREM 2. *There exists a special divisor D of C^3 and a certain flat vector bundle V on C^3-D such that the Manin extension of V on $C^3-\mathrm{Sing}(D)$ cannot be extended to a locally free analytic sheaf on C^3.*

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