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An Analogue of the Theorem of Paley-Wiener Type on the Universal Covering Group of de Sitter Group

Masaichi MAMIUDA

Waseda University

Introduction

The purpose of this paper is to prove the theorem of Paley-Wiener type on the universal covering group of de Sitter group (Theorem 3.4). Theorems of this type on semisimple Lie groups have been proved in several cases: L. Ehrenpreis and F. I. Mautner [5] is the first that proved theorems of Paley-Wiener type for SL(2, $\boldsymbol{R}$). For a general non-compact semisimple Lie group, this is not proved but there are some papers on theorems of similar type; [17] for compact groups, [7], [11] for symmetric spaces and [1], [14], [16], for certain another function spaces. And recently some characterizations of the Fourier image of C^{∞}-function with compact support on a rank 1 semisimple Lie group were given in [2], [6], [12]. But in the case when G is the universal covering group of de Sitter group we can give a more explicit characterization than that of [2], [6], [12].

The techniques used in the proof of Theorem 3.4 are similar to that of [11]. But, in our case, functions which appear in the proof have more singularities. Most difficulties of the proof are the arguments on these singularities. Theorem 3.4 of [13] is the key result. By using this theorem, we can reduce the proof to the case where above singularities are absent.

We devide this paper into three parts. In the first section, we give the realization of representations of G and establish certain elementary properties of the matrix coefficients of these representations. We need these properties to describe and to prove Theorem 3.4. In the second section, we give the definition of the Fourier transform and Plancherel formula. In section three, we state the main theorem (Theorem 3.4) and prove this.

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§1. Some properties of representations $R_{s,n}$, $T_{p,n}$.

Through this paper, we use the same notations as in [13]. So, G is the universal covering group of de Sitter group and K, M, A, N, $\bar{N}$ are the subgroups of G defined in §1 of [13], and we write $g=k(g)a_{t(g)}n_{x(g)}$ for the Iwasawa decomposition of $g\in G$ etc., let $\hat{K}$, $\hat{M}$ be the sets introduced in §1 of [13] (p. 116). Then for each $(n', n'')\in\hat{K}$ (resp. $n\in\hat{M}$) we can correspond to the irreducible unitary representation $\tau=\tau^{n',n''}$ of K (resp. σ^n of M) on $V_\tau=V^{n',n''}$ (resp. V^n). For simplicity we denote this by $\tau\in\hat{K}$ (resp. $\sigma^n\in\hat{M}$). Moreover, for each $n\in\hat{M}$ we put

$$\hat{K}(n)=\{\tau\in K;\ \text{the restriction of } \tau \text{ to } M \text{ contains } \sigma^n\}\ .$$

In this section, we give the realization of principal series $R_{s,n}$ and discrete series $T_{p,n}$ of G and establish several facts about $R_{s,n}$, $T_{p,n}$ which are needed in later arguments.

1.1. Realization of representations $R_{s,n}$, $T_{p,n}$.

For each $n\in\hat{M}$, we denote by $\mathscr{H}_n^\infty$ the space of all C^∞-functions of K into V^n which satisfies the functional equation;

$$f(km)=\sigma^n(m^{-1})f(k) \quad \text{for} \quad k\in K\ , \quad m\in M\ .$$

Then $\mathscr{H}_n^\infty$ is a pre-Hilbert space with the Hermitian form

$$(f, f')_n=\int_K (f(k), f'(k))_{V^n}dk\ .$$

Let $\mathscr{H}_n$ be the completion of $\mathscr{H}_n^\infty$. Then the principal series representation $R_{s,n}$ of G on $\mathscr{H}_n$ is given by

$$R_{s,n}(x)f(k)=e^{-(s+3/2)t(x^{-1}k)}f(k(x^{-1}k))\ ,$$

where $x\in G$, $k\in K$, $f\in\mathscr{H}_n$ and s is a complex number. Thus principal series representations are parametrized by the set

$$\hat{G}_c=\{(s, n);\ s \text{ is a complex number and } 2n \text{ is a non-negative integer}\}\ .$$

Let

$$\hat{G}_d=\{(p, n);\ 2p,\ 2n,\ n-p \text{ are integers with } 1\leqq|p|\leqq n\}\ .$$

Then for each $(p, n)\in\hat{G}_d$ we corresponds to a discrete series representation $T_{p,n}$ of G on a Hilbert space $\mathscr{H}_{p,n}$ as follows;

$$T_{p,n}=\begin{cases} T^{n\ 0;p} & \text{if } p>0 \\ T^{0\ n;-p} & \text{if } p<0 , \end{cases}$$

where $T^{n\ 0;p}$, $T^{0\ n;-p}$ are the realizations of discrete series of G given in [18] (p. 399).

Let $\mathscr{H}_n(\tau)$ (resp. $\mathscr{H}_{p,n}(\tau)$) be the space of all K-finite vectors of type τ under $R_{s,n}$ (resp. $T_{p,n}$) $(\tau \in K)$ and $R_n(\tau)$ (resp. $T_{p,n}(\tau)$) be the orthogonal projection of $\mathscr{H}_n$ (resp. $\mathscr{H}_{p,n}$) onto $\mathscr{H}_n(\tau)$ (resp. $\mathscr{H}_{p,n}(\tau)$). Then $R_n(\tau)$ (resp. $T_{p,n}(\tau)$) are mutually orthogonal. Let $P_n(\tau)$ $(\tau \in \hat{K}(n))$ be a linear endomorphism from V_τ onto V^n which is defined in § 1 of [13] and put

$$f^n_{\tau v}(k)=P_n(\tau)(\tau(k^{-1})v) \quad \text{for} \quad k \in K,\ v \in V_\tau .$$

Then $f^n_{\tau v} \in \mathscr{H}_n(\tau)$. Furthermore, by simple calculations, we have

$$(f^n_{\tau v},\ f^n_{\tau v})_n=(2n+1)/d(\tau) ,$$

where $d(\tau)$ is the degree of τ. Hence the mapping $I_{\tau n}v \mapsto ((2n+1)/d(\tau))^{-1/2} f^n_{\tau v}$ is an isometry from V_τ into $\mathscr{H}_n(\tau)$. Since the restriction of $R_{s,n}$ to K is the representation which is induced from σ^n, we obtain the next lemma from the Frobenius's reciprocity theorem.

LEMMA 1.1. *$\mathscr{H}_n(\tau) \neq 0$ if and only if $\tau \in \hat{K}(n)$, and when $\tau \in \hat{K}(n)$ the mapping $I_{\tau n}$ is an isometry from V_τ onto $\mathscr{H}_n(\tau)$ which satisfies*

$$R_{s,n}(k)I_{\tau n}=I_{\tau n}\tau(k) \quad \text{for any} \quad k \in K .$$

For $\tau_i \in \hat{K}(n)$ $(i=1, 2)$ let $v_n=P_n(\tau_1)*P_n(\tau_2)$, $E(s,\ v_n,\ x)$ be the same as in § 3 of [13] and V_i be the representation space of $\tau_i (i=1, 2)$. Then we may identify $\mathscr{H}_n(\tau_i)$ with V_i under the isometry $I_{\tau_i n}$ and regard $R_n(\tau_1)R_{s,n}(x)R_n(\tau_2)$ as a linear endomorphism from V_2 into V_1. Moreover, by simple calculations, we have the following formula

$$(R_{s,n}(x)f^n_{\tau_2 v_2},\ f^n_{\tau_1 v_1})_n=(E(s,\ v_n,\ x)v_2,\ v_1)_{V_1} , \quad \text{for} \quad v_i \in V_i \ (i=1, 2) .$$

Consequently, we have the next lemma.

LEMMA 1.2. $(d(\tau_1)d(\tau_2)/(2n+1)^2)^{1/2}E(s,\ v_n,\ x)=R_n(\tau_1)R_{s,n}(x)R_n(\tau_2)$.

1.2. Infinitesimal operators.

Let $\mathfrak{K}$ be the complex universal enveloping algebra of the Lie algebra $\mathfrak{k}$ of $\mathfrak{K}$ and $c(\tau)$ be the eigenvalues of the Casimir operator $\omega_{\mathfrak{k}}$ of $\mathfrak{K}$ under the representation $\tau \in \hat{K}$. Let $|\tau|=1+|c(\tau)|(\tau \in \hat{K})$.

LEMMA 1.3. *Let $\tau \in \hat{K}$ and $D \in \mathfrak{K}$. Then there exists a positive con-*

stant c and a non-negative integer j depending only on D such that

$$\|\tau(D)\| \leqq c|\tau|^j$$

where $\|\ \|$ is the operator norm.

PROOF. For each $D \in \mathfrak{k}$, there is a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$ which contains D. Let $\lambda_1, \cdots, \lambda_m$ be the weights of τ with respect to $\mathfrak{h}$. Then

$$\|\tau(D)\| = \max\{\lambda_i(D)| \quad i=1, \cdots, m\} .$$

But it follows from the general representation theory of compact Lie groups that there is a positive number c satisfying

$$|\lambda_i(D)| \leqq c|\tau| \quad i=1, \cdots, m ,$$

and c depends only on D. So, this lemma is valid for $D \in \mathfrak{k}$. For general $D \in \mathfrak{K}$, using the Poincare-Birkhoff-Witt's theorem and the induction on degree of D, this lemma is proved.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G and denote the Killing form by B. For each $X \in \mathfrak{p}$, we define the function q_X on K by

$$q_X(k) = B(\mathrm{Ad}\,(k^{-1})X, H)/B(H, H) \quad (k \in K)$$

where H is the element of the Lie algebra $\mathfrak{a}$ of A so that $a_t = \exp(tH)$. Then it is clear that $q_X \in \mathscr{H}_0^\infty$. Since $\mathfrak{p}$ is an $\mathrm{Ad}(K)$-stable subspace of $\mathfrak{g}$ and B is a positive definite $\mathrm{Ad}(K)$-invariant symmetric bilinear form on $\mathfrak{p}$,

$$B(\mathrm{Ad}(k^{-1})X, Y)^2 \leqq B(\mathrm{Ad}(k^{-1})X, \mathrm{Ad}(k^{-1})X)B(Y, Y) = B(X, X)B(Y, Y)$$

for any $X, Y \in \mathfrak{p}$. Thus

$$|q_X(k)|^2 \leqq B(X, X)/B(H, H) , \quad \text{for any} \quad X \in \mathfrak{p} , \quad k \in K .$$

Let $Q_n(X)$ be the bounded linear operator on $\mathscr{H}_n$ given by

$$Q_n(X)f(k) = q_X(k)f(k) , \quad f \in \mathscr{H}_n \quad \text{and} \quad k \in K .$$

Then

$$\|Q_n(X)\|^2 \leqq B(X, X)/B(H, H) .$$

Let $\mathfrak{G}$ be the complex universal enveloping algebra of $\mathfrak{g}$ and U be a continuous representation of G on a Hilbert space $\mathscr{H}$. Then U can be extended to the differential representation of $\mathfrak{G}$ on the space of all

C^∞-vectors of $\mathscr{H}$. we denote this representation by the same symbol.

The following lemma is preved by E. Thieleker in [19] (Lemma 1, Lemma 4).

LEMMA 1.4. *Let $X \in \mathfrak{p}$ and $(s, n) \in \hat{G}_c$. Then*

$$R_{s,n}(X) = sQ_n(X) + (R_{s,n}(\omega_t)Q_n(X) - Q_n(X)R_{s,n}(\omega_t))/2 ,$$

and for $\tau_i = \tau^{n_i', n_i''}$ $((n_i', n_i'') \in \hat{K})$ $i=1, 2$

$$R_n(\tau_1)R_{s,n}(X)R_n(\tau_2) \neq 0 \quad \textit{if and only if} \quad \tau_i \in \hat{K}(n) \quad i=1, 2$$

and

$$|n_1' - n_2''| + |n_1'' - n_2''| \leqq 1 .$$

PROPOSITION 1.5. *Let $D \in \mathfrak{G}$, $\tau_i \in \hat{K}(n)$ $i=1, 2$. Then,*

1) *if τ_1 (resp. τ_2) is fixed, then except finitely many τ_2 (resp. τ_1) $R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)=0$.*

2) *When $R_n(\tau_1)R_{s,n}(D)R_n(\tau_2) \neq 0$, there are non-negative integers j, k, m and a positive constant c depending only on D such that*

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\| \leqq c(1+|s|)^j |\tau_1|^k |\tau_2|^m .$$

PROOF. First we note that for any $\tau_i \in \hat{K}$ $(i=1, 2)$ there is an integer $j>0$ satisfying $|\tau_1| < |\tau_2|^j$. Since

$$R_n(\tau_1)R_{s,n}(DD')R_n(\tau_2) = \sum_\tau R_n(\tau_1)R_{s,n}(D)R_n(\tau)R_n(\tau)R_{s,n}(D')R_n(\tau_2)$$

for any $D, D' \in \mathfrak{G}$, we may consider only the case $D \in \mathfrak{G}$. When $D \in \mathfrak{k}$, it follows from Lemma 1.1 that

$$R_n(\tau_1)R_{s,n}(D)R_n(\tau_2) = \begin{cases} \tau_1(D) & \text{if } \tau_1 \text{ is equivalent to } \tau_2 \\ 0 & \text{if } \tau_1 \text{ is not equivalent to } \tau_2 . \end{cases}$$

Hence, the first statement is valid for $D \in \mathfrak{k}$ and Lemma 1.3 implies the second statement. When $D \in \mathfrak{p}$, the frst statement is proved from Lemma 1.4. Since $R_n(\tau)R_{s,n}(\omega_t) = c(\tau)R_n(\tau)$, we have from Lemma 1.4 that

$$R_n(\tau_1)R_{s,n}(D)R_n(\tau_2) = (s + (c(\tau_1) - c(\tau_2))/2)R_n(\tau_1)Q_n(D)R_n(\tau_2) .$$

Hence,

$$\begin{aligned} \|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\| &\leqq (|s| + (|\tau_1| + |\tau_2|)/2)\, \|R_n(\tau_1)Q_n(D)R_n(\tau_2)\| \\ &\leqq 2(1+|s|)|\tau_1|\,|\tau_2|\, \|Q_n(D)\| \\ &\leqq 2(B(D, D)/B(H, H))^{1/2}(1+|s|)|\tau_1|\,|\tau_2| . \end{aligned}$$

So, Proposition 1.5 is proved.

COROLLARY 1.6. *Let* $D\in\mathfrak{G}$, $(\tau_i, V_i)\in\hat{K}$ $(i=1, 2)$ *be satisfying that* $R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\neq 0$. *For each basis* $\{f_j'; 1\leqq j\leqq d(\tau_1)\}$, $\{f_j''; 1\leqq j\leqq d(\tau_2)\}$ *of* $\mathscr{H}_n(\tau_1)$, $\mathscr{H}_n(\tau_2)$, *we put*

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1=\sum_{j,k}|(R_{s,n}(D)f_k'', f_j')_{V_n}| .$$

Then there are integers i, j, k *and a constant* $c>0$ *such that*

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1\leqq c(1+|s|)^i|\tau_1|^j|\tau_2|^k .$$

PROOF. Since

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1\leqq d(\tau_1)d(\tau_2)\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\| ,$$

Proposition 1.5 implies

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1\leqq c'(1+|s|)^{i'}|\tau_1|^{j'}|\tau_2|^{k'}d(\tau_1)d(\tau_2)$$

for some integers i', j', k' and a constant $c'>0$. While, if $\tau=\tau^{n',n''}$, then $d(\tau)=(n'+n''+1)(n'-n''+1)$ and $|\tau|=1+(n'+1)^2+(n'')^2-1=(n'+1)^2+(n'')^2$. Hence $d(\tau)\leqq|\tau|$ for any $\tau\in\hat{K}$. Therefore, if $c=c'$, $i=i'$, $j=j'+1$, $k=k'+1$ then

$$\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1\leqq c(1+|s|)^i|\tau_1|^j|\tau_2|^k .$$

Thus our lemma is proved.

COROLLARY 1.7. *Let* D, (τ_i, V_i) $i=1, 2$ *be the same as Corollary* 1.6. *Then there are integers* i, j, k *and a constant* $c>0$ *such that*

$$|\mathrm{trace}(R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)A)|\leqq c\,\|A\|(1+|s|)^i|\tau_1|^j|\tau_2|^k$$

for every linear endomorphism A *of* V_1 *and* V_2.

PROOF. Since

$$|\mathrm{trace}(R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)A)|\leqq\|A\|\,\|R_n(\tau_1)R_{s,n}(D)R_n(\tau_2)\|_1 ,$$

Corollary 1.7 is a simple consequence of Corollary 1.6.

1.3. Linear independency.

For each $(\tau_i, V_i)\in\hat{K}(n)$, $i=1, 2$ we identify $f(x)=R_n(\tau_1)R_{s,n}(x)R_n(\tau_2)$ with a linear endomorphism on V_2 into V_1 by the isomorphism of Lemma 1.1.

LEMMA 1.8. *Let* $\{v_j^i; 1\leqq j\leqq d(\tau_i)\}$ *be a basis of* V_i $(i=1, 2)$. *If* f *is not identically zero. Then* $f_{ij}(x)=(f(x)v_j^2, v_i^1)$; $1\leqq i\leqq d(\tau_1)$, $1\leqq j\leqq d(\tau_2)$ *are* *linearly independent.*

PROOF. First we prove that for each non-zero vectors $v^i \in V_i$ ($i=1, 2$) the function $(f(x)v^2, v^1)$ is not identically zero. Fix non-zero vectors $v^i \in V_i$ ($i=1, 2$). Then since τ_1, τ_2 are irreducible unitary representations of K, for each $u^i \in V_i$, we can find constants c_j^i and elements k_j^i of K ($i=1, 2, j=1, \cdots, m$) such that

$$u^i = \sum_j c_j^i \tau_i(k_j^i) v^i \ , \quad i=1, 2 \ .$$

Hence $(f(x)u^2, u^1) = \sum_{j,j'} c_j^2 \bar{c}_{j'}^1 (f(x)\tau_2(k_j^2)v^2, \tau_1(k_{j'}^1)v^1)$. But

$$(f(x)\tau_2(k_j^2)v^2, \tau_1(k_{j'}^1)v^1) = (f((k_{j'}^1)^{-1}xk_j^2)v^2, v^1) \ .$$

Therefore, if $(f(x)v^2, v^1)$ is identically zero then $f(x)$ is also identically zero. Thus $(f(x)v^2, v^1)$ is not identically zero if $v^i(i=1, 2)$ are non-zero vectors and $f(x)$ is not identically zero. Since the linear independency of f_{ij} is stable under the change of basis, we may assume that $\{v_j^i; 1 \leqq j \leqq d(\tau_i)\}$ ($i=1, 2$) are orthonormal basis. Then, for each $k_1, k_2 \in K$, $x \in G$,

$$f_{ij}(x) = (f(x)\tau_2(k_2)v_j^2, \tau_1((k_1)^{-1})v_i^1) = \sum_{i'j'} (\tau_2(k_2)v_j^2, v_{j'}^2)(\tau_1(k_1)v_{i'}^1, v_i^1) f_{i'j'}(x) \ .$$

From the orthogonal relation of spherical functions on compact group, we have the formula

$$\int_K \int_K f_{ij}(k_1 x k_2)(\tau_1((k_1)^{-1})v_{i'}^1, v_{i'}^1)(\tau_2((k_2)^{-1})v_{j'}^2, v_{j'}^2) dk_1 dk_2$$
$$= (d(\tau_1)d(\tau_2))^{-1} f_{i'j'}(x) \delta_{ii'} \delta_{jj'}$$

where δ is Kronecker's symbol.

Now we put $\tilde{f}(x) = \sum_{ij} a_{ij} f_{ij}(x)$ (a_{ij} are complex numbers), then

$$(d(\tau_1)d(\tau_2))^{-1} a_{ij} f_{ij}(x) = \int_K \int_K f(k_1 x k_2)(\tau_1((k_1)^{-1})v_i^1, v_i^1)(\tau_2((k_2)^{-1})v_j^2, v_j^2) dk_1 dk_2 \ .$$

If $f(x)=0$ for all $x \in G$, then $a_{ij} f_{ij}(x)=0$ for all $x \in G$ and $1 \leqq i \leqq d(\tau_1)$, $1 \leqq j \leqq d(\tau_2)$. Since f_{ij} is not identically zero, we have that

$$a_{ij} = 0 \quad i=1, \cdots, d(\tau_1) \ , \quad j=1, \cdots, d(\tau_2) \ .$$

Thus Lemma 1.8 is proved.

Let V be a finite dimensional Hilbert space and denote by $C^\infty(G: V)$ the space of all C^∞-functions of G into V. Then elements of $\mathfrak{G}$ act on $C^\infty(G: V)$ from both left and right as differential operators. We write

Df, fD ($f \in C^\infty(G: V)$ $D \in \mathfrak{G}$) for these actions. We also use Harish-Chandra's notations $f(D; x)$, $f(x; D)$.

Let $\mathfrak{Z}$ be the center of $\mathfrak{G}$ and $\chi_{s,n}$ be the infinitesimal character of $R_{s,n}((s, n) \in \hat{G}_c)$. Namely $\chi_{s,n}$ is an algebra homomorphism of $\mathfrak{Z}$ into the complex field C such that $R_{s,n}(Z)\varphi = \chi_{s,n}(Z)\varphi$ for all $\varphi \in \mathscr{H}_n^\infty$ and $Z \in \mathfrak{Z}$. Then there is a Casimir element $\omega \in \mathfrak{Z}$ such that $\chi_{s,n}(\omega) = s^2 - (3/2)^2 + n(n+1)$ and $\omega + 2\omega_t$ is an elliptic differential operator on G. ([21])

LEMMA 1.9. *Let the functions $f_{ij}(x)$, $f(x)$ be the same as in Lemma 1.8. Then these functions are all real analytic on G.*

PROOF. Since $f(\omega; x) = f(x; \omega) = \chi_{s,n}(\omega)f(x)$ and $f(\omega_t; x) = c(\tau_1)f(x)$ (also, $f(x; \omega_t) = c(\tau_2)f(x)$), f is an eigen function of the elliptic differential operator $\omega + 2\omega_t$. Hence $f(x)$, $f_{ij}(x)$ are all real analytic on G.

LEMMA 1.10. *Let $(s_i, n_i) \in \hat{G}_c$ and $(\tau_j, V_j) \in \hat{K}$ $(i=1, \cdots, m,\ j=1, 2)$ satisfy that $f^i(x) = R_{n_i}(\tau_1)R_{s_i,n_i}(x)R_{n_i}(\tau_2)$ is not identically vanished for $i=1, \cdots, m$. Choose a basis v_j^i, $j=1, \cdots, d(\tau_i)$ of V_i $(i=1, 2)$ respectively and put $\chi_i = \chi_{s_i,n_i}$, $i=1, \cdots, m$ and let*

$$f_{jk}^i(x) = (f^i(x)v_k^2, v_j^1);\ i=1, \cdots, m,\ j=1, \cdots, d(\tau_1),\ k=1, \cdots, d(\tau_2)\ .$$

If $\chi_i \neq \chi_{i'}$ for $i \neq i'$, then f_{jk}^i $i=1, \cdots, m$, $j=1, \cdots, d(\tau_1)$, $k=1, \cdots, d(\tau_2)$ are linearly independent real analytic functions on G.

PROOF. The analyticity is already proved in Lemma 1.9. For any complex numbers a_{jk}^i $(i=1, \cdots, m,\ j=1, \cdots, d(\tau_1),\ k=1, \cdots, d(\tau_2))$ we write $F^i(x) = \sum_{jk} a_{jk}^i f_{jk}^i(x)$ and $F(x) = \sum_i F^i(x)$. First we prove the following; $F(x)=0$ implies $F^i(x)=0$ $i=1, \cdots, m$. We use the induction on m. When $m=1$, our assertion is trivial. Since $f^i(x; Z) = \chi_i(Z)f^i(x)$, $F^i(x; Z) = \chi_i(Z)F^i(x)$. Hence $F(x; Z) = \sum_i \chi_i(Z)F^i(x)$. So,

$$F(x; Z) - \chi_m(Z)F(x) = \sum_1^{m-1} (\chi_i(Z) - \chi_m(Z))F^i(x)\ .$$

From the assumption on χ_i, we can select an element $Z_0 \in \mathfrak{Z}$ such that

$$\chi_i(Z_0) \neq \chi_{i'}(Z_0) \quad \text{if} \quad i \neq i'\ .$$

$F(x)=0$ implies that $\sum_1^{m-1} (\chi_i(Z_0) - \chi_m(Z_0))F^i(x) = 0$. Since $\chi_i(Z_0) \neq \chi_m(Z_0)$, the induction hypothesis implies that $F^i(x)=0$ $i=1, \cdots, m-1$. Thus $F^m(x)=0$. We have from Lemma 1.9 that $F^i(x)=0$ for all $x \in G$ implies $a_{jk}^i=0$ $i=1, \cdots, m$, $j=1, \cdots, d(\tau_1)$, $k=1, \cdots, d(\tau_2)$. Hence Lemma 1.10 is proved.

1.4. Intertwining operators for $R_{s,n}$.

Let $d\bar{n}$ be the Haar measure on $\bar{N}$ normalized by $\int_{\bar{N}} e^{-3t(\bar{n})}d\bar{n}=1$. For each $(s, n)\in\hat{G}_c$ with $\mathrm{Re}(s)>0$, we consider the following operator;

$$A_{s,n}f(k)=\int_{\bar{N}} e^{-(s+3/2)t(\bar{n})}f(kwk(\bar{n}))dk ,$$

where w is an element of the center of K which is defined in § 1 of [13] (p. 116). Then it is well-known that the above integral is absolutely convergent for all $f\in\mathscr{H}_n$ and the mapping $f\to A_{s,n}f$ is a bounded linear operator on $\mathscr{H}_n$. Moreover for each $f\in\mathscr{H}_n^\infty$ the mapping $s\mapsto A_{s,n}f$ is a $\mathscr{H}_n$-valued holomorphic function on $\{s;\ s$ is a complex number with $\mathrm{Re}(s)>0\}$ and can be extended meromorphically onto $\boldsymbol{C}$.

We have the next lemma from Proposition 5.1 of [13].

LEMMA 1.11. *Let* $\tau=\tau^{n',n''}\in\hat{K}(n)$. *Then for each* $f\in\mathscr{H}_n(\tau)$ $A_{s,n}f=c_n(s,\tau)f$ *as meromorphic functions of* s, *where*

$$c_n(s,\tau)=\frac{\varepsilon^{2n}2^{-2s+3}\Gamma(2s)\Gamma(-s+3/2+n')\Gamma(-s+1/2-|n''|)}{\Gamma(-s+3/2+n)\Gamma(-s+1/2-n)\Gamma(s+3/2+n')\Gamma(s+1/2-|n''|)}$$

$$\varepsilon=\begin{cases}1 & \text{if } n''\geqq 0\\ -1 & \text{if } n''<0\ .\end{cases}$$

For any $x\in G$ there is an unique non-negative number t such that $x\in Ka_tK$ (Lemma 2.1 of [13]). We write $|x|=t$, then $|x^{-1}|=|x|$.

LEMMA 1.12. *Fix* $x\in G$ *and* n. *Then the mapping* $s\mapsto R_{s,n}(x)$ *is a holomorphic function on* $\boldsymbol{C}$ *into the Banach algebra of all bounded linear operators on* $\mathscr{H}_n$. *Moreover*

$$\left\|\left(\frac{\partial}{\partial s}\right)^j R_{s,n}(x)\right\|\leqq|x|^j e^{|\mathrm{Re}(s)||x|}\quad j=0,1,\cdots .$$

PROOF. It is clear that the function $s\mapsto(R_{s,n}(x)f, f')_n$ is holomorphic on $\boldsymbol{C}$ for each $f, f'\in\mathscr{H}_n$ and $x\in G$. Hence the mapping $s\mapsto R_{s,n}(x)$ is holomorphic. Since

$$\left(\left(\frac{\partial}{\partial s}\right)^j R_{s,n}(x)f(k)\right)=(t(x^{-1}k))^j(e^{-(s+3/2)t(x^{-1}k)}f(k(x^{-1}k))) ,$$

$$\left\|\left(\frac{\partial}{\partial s}\right)^j R_{s,n}(x)f(k)\right\|_{V^n}$$
$$\leqq\sup_{k\in K}(|t(x^{-1}k)|^j e^{-\mathrm{Re}(s)t(x^{-1}k)})(e^{-(3/2)t(x^{-1}k)}\|f(k(x^{-1}k))\|_{V^n}) .$$

From the explicit formula of the Iwasawa decomposition (Lemma 1.1 of Chapter II in [18]), we obtain that

$$\sup_{k\in K}(|t(x^{-1}k)|^j e^{-\mathrm{Re}(s)t(x^{-1}k)})=|x|^j e^{|\mathrm{Re}(s)||x|} .$$

Hence,

$$\left\|\left(\frac{\partial}{\partial s}\right)^j R_{s,n}(x)f\right\|^2 \leqq |x|^{2j}e^{2|\mathrm{Re}(s)||x|}\int_K e^{-3t(x^{-1}k)}\|f(k(x^{-1}k))\|_{V^n}^2\, dk .$$

While,

$$\int_K e^{-3t(x^{-1}k)}\|f(k(x^{-1}k))\|_{V^n}^2\, dk=(f, f)_n .$$

Thus

$$\left\|\left(\frac{\partial}{\partial s}\right)^j R_{s,n}(x)f\right\|\leqq|x|^j e^{|\mathrm{Re}(s)||x|}\|f\|_n .$$

So, Lemma 1.12 is proved.

1.5. Results from the subquotient theorem.

For each subset S of $\hat{K}(n)$, we denote by $\mathscr{H}_n(S)_K$ the algebraic direct sum $\sum_{\tau}\mathscr{H}_n(\tau)$ and denote by $\mathscr{H}_n(S)$ the closure of $\mathscr{H}_n(S)_K$ in $\mathscr{H}_n$. We write $(\mathscr{H}_n)_K$ for $\mathscr{H}_n(\hat{K}(n))_K$. Similarly $(\mathscr{H}_{p,n})_K=\sum_{\tau}\mathscr{H}_{p,n}(\tau)$. Now we introduce certain subsets of $\hat{K}(n)$ as follows; let p be a half of an integer satisfying that $n-p$ is an integer.

(1) For $n<p$

$S_1^+(p, n)=\{(n', n'')\in\hat{K}(n); p\leqq n'\}$, $S_1^-(p, n)=\{(n', n'')\in\hat{K}(n); p>n'\}$.

(2) For $n\geqq p$

$S_2^+(p, n)=\{(n', n'')\in\hat{K}(n); n''\geqq p\}$, $S_2^-(p, n)=\{(n', n')\in\hat{K}(n); n''\leqq -p\}$

and

$$S_2^F(p, n)=\{(n', n'')\in\hat{K}(n); |n''|<p\} .$$

The next lemma is reduced from Theorem 2, Theorem 5 of [19] and Theorem 3 of [20] (also [8]).

LEMMA 1.13. *Let p be the same as above.*

(1) *When $p\leqq n$, the subspaces $\mathscr{H}_n(S_2^+(p, n))$, $\mathscr{H}_n(S_2^-(p, n))$ (resp. $\mathscr{H}_n(S_2^+(p, n)\cup S_2^F(p, n))$, $\mathscr{H}_n(S_2^-(p, n)\cup S_2^F(p, n))$, $\mathscr{H}_n(S_2^F(p, n))$) are stable under $R_{p-1/2,n}(x)$ (resp. $R_{-(p-1/2),n}(x)$) for all $x\in G$.*

(2) *When $p>n$, the subspace $\mathscr{H}_n(S_1^+(p, n))$ (resp. $\mathscr{H}_n(S_1^-(p, n))$) is*

stable under $R_{p+1/2,n}(x)$ (*resp.* $R_{-(p+1/2),n}(x)$) *for all* $x \in G$.

(3) *Denote by* $U_{p,n}$ (*resp.* $U_{-p,n}$) *the representation on* $\mathscr{H}_n(S_2^+(p, n))$ (*resp.* $\mathscr{H}_n(S_2^-(p, n))$ *induced from* $R_{p-1/2,n}$ (*resp.* $R_{-(p+1/2),n}$). *Then* $U_{p,n}$ *is infinitesimally equivalent to* $T_{p,n}$ *for each* $(p, n) \in \hat{G}_d$.

(4) *Denote by* $U'_{p,n}$ (*resp.* $U''_{p,n}$) *the representation on* $\mathscr{H}_n(S_1^+(p, n))$ (*resp.* $\mathscr{H}_n(S_2^F(p, n))$) *induced from* $R_{p+1/2,n}$ (*resp.* $R_{-(p-1/2),n}$). *Then* $U'_{p,n}$ *is infinitesimally equivalent to* $U''_{n+1,p}$.

The next corollary is easily proved from Lemma 1.13.

COROLLARY 1.14. *Let* $\tau_i = \tau^{n_i' n_i''} \in \hat{K}(n)$ $i=1, 2$ *and put*

$$f(s, n, x) = R_n(\tau_1) R_{s,n}(x) R_n(\tau_2) .$$

Then $f(s, n, x) = 0$ *for all* $x \in G$ *in the following cases;*

(1) $n_1' < n_2'$ *and* $s = p + 1/2$ *with* $n_1' < p \leqq n_2'$.

(2) $n_2' < n_1'$ *and* $s = -(p+1/2)$ *with* $n_2' < p \leqq n_1'$

(3) $n_1'' n_2'' \geqq 0$, $|n_1''| < |n_2''|$ *and* $s = p - 1/2$ *with* $|n_1''| < p \leqq |n_2''|$

(4) $n_1'' n_2'' \geqq 0$, $|n_2''| < |n_1''|$ *and* $s = -(p-1/2)$ *with* $|n_2''| < p \leqq |n_1''|$

(5) $n_1'' < 0 < n_2''$ *and* $s = p - 1/2$ *with* $n_1'' < p \leqq n_2''$

(6) $n_2'' < 0 < n_1''$, *and* $s = -(p-1/2)$ *with* $n_2' < p \leqq n_1''$.

The infinitesimal equivalence implies the following corollary (Theorem 4.5.5.2 of [21], or [9], [10]).

COROLLARY 1.15.

(1) *Let* $S_2^+(p, n) = S(p, n)$, $S_2^-(p, n) = S(-p, n)$. *Then for each* $(p, n) \in \hat{G}_d$ *there is a closed one-to-one linear operator* $B_{p,n}$ *on* $\mathscr{H}_n(S(p, n))_0$ *onto* $(\mathscr{H}_{p,n})_0$ *such that* $B_{p,n} U_{p,n}(x) v = T_{p,n}(x) B_{p,n} v$ *for all* $v \in \mathscr{H}_n(S(p, n))_0$ *and* $x \in G$, *where* $\mathscr{H}_n(S(p, n))_0$ (*resp.* $(\mathscr{H}_{p,n})_0$) *is a certain* $U_{p,n}$-*stable* (*resp.* $T_{p,n}$-*stable*) *subspace of* $\mathscr{H}_n(S(p, n))$ (*resp.* $\mathscr{H}_{p,n}$) *which contains* $\mathscr{H}_n(S(p, n))_K$ (*resp.* $(\mathscr{H}_{p,n})_K$).

(2) *Let* $2n$, $2p$, $n-p$ *be integers with* $n > p$ *and* $S'(p, n) = S_1^+(p, n) = S_2^F(n+1, p)$. *Then there is a closed one-to-one linear operator* $B'_{p,n}$ *of* $\mathscr{H}_n(S'(p, n))_0$ *onto itself such that* $B'_{p,n} U'_{p,n}(x) v = U''_{n+1,p}(x) B'_{p,n} v$ *for all* $v \in \mathscr{H}_n(S'(p, n))_0$ *and* $x \in G$, *where* $\mathscr{H}_n(S'(p, n))_0$ *is a certain* $U'_{p,n}$-*stable and* $U''_{n+1,p}$-*stable dense subspace of* $\mathscr{H}_n(S'(p, n))$ *which contains* $\mathscr{H}_n(S'(p, n))_K$.

DEFINITION. $F = (F_c, F_d)$ is said to be linked with the principal series if

(1) For each $(s, n) \in \hat{G}_c$ (resp. $(p, n) \in \hat{G}_d$) $F_c(s, n)$ (resp. $F_d(p, n)$) is a bounded linear operator on $\mathscr{H}_n$ (resp. $\mathscr{H}_{p,n}$).

(2) For each n the mapping $s \mapsto F_c(s, n)$ is a holomorphic function

on C into the Banach algebra of all bounded linear operators on $\mathscr{H}_n$.

(3) $\mathscr{H}_n^\infty$ is stable under $F_c(s, n)$ for all $(s, n) \in \hat{G}_c$ and

$$A_{s,n}F_c(s, n)f = F_c(-s, n)A_{s,n}f \quad \text{for all} \quad f \in \mathscr{H}_n^\infty$$

as meromorphic $\mathscr{H}_n$-valued function.

(4) Replacing $R_{s,n}(x)$, $T_{p,n}(x)$ by $F_c(s, n)$, $F_d(p, n)$ respectively, Corollary 1.14 and Corollary 1.15 are valid.

§2. Plancherel formula.

We denote the Banach algebra of all bounded linear operators on $\mathscr{H}_n$ (resp. $\mathscr{H}_{p,n}$) by $\mathscr{L}_n$ (resp. $\mathscr{L}_{p,n}$). Let $A, B \in \mathscr{L}_n$ (resp. $\mathscr{L}_{p,n}$) and at least one of these is of trace class. Then it is well-known that $\langle A, B\rangle_n = \mathrm{trace}(AB)$ (resp. $\langle A, B\rangle_{p,n} = \mathrm{trace}(AB)$) is well-defined.

We denote the space of all C^∞-functions on G with compact support by $\mathscr{D}(G)$ and put

$$\mathscr{D}_r(G) = \{f \in \mathscr{D}(G);\ \text{the support of } f \subset G_r\},$$

where r is a positive constant and $G_r = \{x \leqq G; |x| \leqq r\}$. Now we define the Fourier-Laplace (or simply Fourier) transform $\hat{f} = (\hat{f}_c, \hat{f}_d)$ of $f \in \mathscr{D}(G)$ by

$$\hat{f}_c(s, n) = \int_G f(x)R_{s,n}(x)dx \quad (s, n) \in \hat{G}_c$$

$$\hat{f}_d(p, n) = \int_G f(x)T_{p,n}(x)dx \quad (p, n) \in \hat{G}_d$$

where dx is a Haar measure.

LEMMA 2.1. *The function $c_n(s, \tau)$, $c_n(-s, \tau)$ does not depend on $\tau \in \hat{K}(n)$ and*

$$c_n(s, \tau)c_n(-s, \tau) = \begin{cases} \left(\dfrac{\pi}{16} s((n+1/2)^2 - s^2)\tan\pi s\right)^{-1} & \text{if } n \text{ is an integer}. \\ \left(-\dfrac{\pi}{16} s((n+1/2)^2 - s^2)\cot\pi s\right)^{-1} & \text{if } n \text{ is a half of odd integer}. \end{cases}$$

PROOF. This lemma is easily proved from the explicit formula of $c_n(s, \tau)$ (Lemma 1.11).

The following formula is given in [18] and proved in [15].

LEMMA 2.2 (*Plancherel formula*). *For any $f \in \mathscr{D}(G)$ and $x \in G$*

$$f(x)=\sum_n \int_{\gamma(0)} \langle R_{s,n}(x^{-1}),\ f_c(s,\ n)\rangle_n Q_c(s,\ n)ds$$
$$+\sum_{(p,n)\in\hat{G}_d} \langle T_{p,n}(x^{-1}),\ f_d(p,\ n)\rangle_{p,n} Q_d(p,\ n)\ .$$

Here, there are constants c', c'' depending only on the normalization dx such that

$$Q_c(s,\ n)=c'((2n+1)c_n(s,\ \tau)c_n(-s,\ \tau))^{-1}\ ,$$
$$Q_d(p,\ n)=c''(2n+1)(2\,|p|-1)(n+|p|)(n-|p|+1)/16\pi^2\ .$$

And $\gamma(a)=\{s;\ s$ is a complex number with $\mathrm{Re}(s)=a\}$ *for each real number a.*

§3. Main theorem.

For each $r>0$ let $\mathscr{D}_r(\hat{G})$ be the linear space of all pairs $F=(F_c,\ F_d)$ satisfying the following conditions;

(1) F is linked with the principal series.

(2) For any $\tau_1,\ \tau_2\in\hat{K}$ and non-negative integers $g,\ h,\ i,\ j,\ k$, there are constants $c'>0$, $c''>0$ such that

$$(1+|s|)^g(1+n)^h|\tau_1|^i|\tau_2|^j\ \left\|\left(\frac{\partial}{\partial s}\right)^k R_n(\tau_1)F_c(s,\ n)R_n(\tau_2)\right\|\leqq c'r^k e^{|\mathrm{Re}(s)|r}$$
$$(1+|p|)^g(1+n)^h|\tau_1|^i|\tau_2|^j\|\ T_{p,n}(\tau_1)F_d(p,\ n)T_{p,n}(\tau_2)\,\|\leqq c''\ .$$

We put $\mathscr{D}(\hat{G})=\bigcup_{r>0}\mathscr{D}_r(\hat{G})$.

Proposition 3.1. *Suppose $f\in\mathscr{D}_r(G)$ $(r>0)$. Then $\hat{f}\in\mathscr{D}_r(\hat{G})$.*

Proof. It is easy to prove that $\hat{f}$ is linked with the principal series for any $f\in\mathscr{D}(G)$. To obtain the estimate (2), we need the lemmas below;

Lemma 3.2. *Fix $\tau_i\in\hat{K}$ $(i=1,\ 2)$ and $f\in\mathscr{D}(G)$. Then for any non-negative integers $i,\ j$ there are $D_1,\ D_2\in\mathfrak{K}$ such that*

$$|\tau_1|^i|\tau_2|^j E'_{s,n}(f)=E'_{s,n}(D_1 f D_2)$$
$$|\tau_1|^i|\tau_2|^j E''_{p,n}(f)=E''_{p,n}(D_1 f D_2)$$

where

$$E'_{s,n}(f)=R_{s,n}(\tau_1)f_c(s,\ n)R_n(\tau_2),\ E''_{p,n}(f)=T_{p,n}(\tau_1)f_d(p,\ n)T_{p,n}(\tau_2)\ .$$

Proof. Let $D\to D^{\vee}$ be anti-automorphism of $\mathfrak{G}$ such that $D^{\vee}=-D$ if $D\in\mathfrak{g}$. Then for any $D_1,\ D_2\in\mathfrak{G}$

$$(D_1 f D_2)^{\wedge}_c(s, n) = R_{s,n}(D^{\vee}{}_1)\hat{f}_c(s, n) R_{s,n}(D^{\vee}{}_2) \quad \text{on} \quad (\mathscr{H}_n)_K\,,$$
$$(D_1 f D_2)^{\wedge}_d(p, n) = T_{p,n}(D^{\vee}{}_1)\hat{f}_d(p, n) T_{p,n}(D^{\vee}{}_2) \quad \text{on} \quad (\mathscr{H}_{p,n})_K\,.$$

Since $\omega^{\vee}{}_{\mathfrak{k}} = \omega_{\mathfrak{k}}$ and $R_{s,n}(\omega_{\mathfrak{k}})R_n(\tau) = R_n(\tau)R_{s,n}(\omega_{\mathfrak{k}}) = c(\tau)R_n(\tau)$ (resp. $T_{p,n}(\omega_{\mathfrak{k}}) T_{p,n}(\tau) = T_{p,n}(\tau)T_{p,n}(\omega_{\mathfrak{k}}) = c(\tau)T_{p,n}(\tau)$),

$$|\tau|\, R_n(\tau) = (1 + c(\tau))R_n(\tau) = R_{s,n}((1+\omega_{\mathfrak{k}}))R_n(\tau)$$

(resp. $|\tau| T_{p,n}(\tau) = T_{p,n}((1+\omega_{\mathfrak{k}}))T_{p,n}(\tau)$). Hence $D_1 = (1+\omega_{\mathfrak{k}})^i$, $D_2 = (1+\omega_{\mathfrak{k}})^j$ satisfy this lemma.

Now return to the proof of Proposition 3.1. Let $\chi_{s,n}$ (resp. $\chi'_{p,n}$) be the infinitesimal character of $R_{s,n}$ (resp. $T_{p,n}$). Then

$$(Zf)^{\wedge}_c(s, n) = (fZ)^{\wedge}_c(s, n) = \chi_{s,n}(Z)\hat{f}_c(s, n)$$
$$(Zf)^{\wedge}_d(p, n) = (fZ)^{\wedge}_d(p, n) = \chi'_{p,n}(Z)\hat{f}_d(p, n)\,.$$

Hence,

$$|\chi_{s,n}(Z)| \left\| \left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(f) \right\| = \left\| \left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(Zf) \right\|$$
$$|\chi'_{p,n}(Z)|\, \|E''_{p,n}(f)\| = \|E''_{p,n}(Zf)\| \qquad \text{for all } Z \in \mathfrak{Z}\,.$$

Since for fixed $Z \in \mathfrak{Z}$, $\chi_{s,n}(Z)$ (resp. $\chi'_{p,n}(Z)$) is a polynomial function of (s, n) (resp. (p, n)), for any integers $g \geqq 0$, $h \geqq 0$ there is an element $Z \in \mathfrak{Z}$ such that

$$(1+|s|)^g(1+n)^h \leqq |\chi_{s,n}(Z)|\,, \quad (1+|p|)^g(1+n)^h \leqq |\chi'_{p,n}(Z)|\,.$$

Hence for any $\tau_1, \tau_2 \in \hat{K}$ and non-negative integers g, h, i, j, k, there exist $D_1, D_2 \in \mathfrak{G}$ such that

$$(1+|s|)^g(1+n)^h\, |\tau_1|^i |\tau_2|^j \left\| \left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(f) \right\| \leqq \left\| \left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(D_1 f D_2) \right\|$$
$$(1+|p|)^g(1+n)^h |\tau_1|^i |\tau_2|^j \| E''_{p,n}(f)\| \leqq \|E''_{p,n}(D_1 f D_2)\|$$

Hence the estimate (2) is a consequent of the next lemma.

Lemma 3.3. *Let $E'_{s,n}(f)$, $E''_{p,n}(f)$ be the same as Lemma 3.2. Then*
(1) $\|(\partial/\partial s)^k E'_{s,n}(f)\| \leqq c(f) r^k e^{|\mathrm{Re}(s)| r} \quad k = 0, 1, \cdots$
(2) $\|E''_{p,n}(f)\| \leqq c(f)$
where $c(f) = \int_G |f(x)|\, dx$.

Proof. Since

$$\left\|\left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(f)\right\| \leqq \left\|\left(\frac{\partial}{\partial s}\right)^k \hat{f}_c(s,\, n)\right\| \leqq \int_{G_r} \left\|\left(\frac{\partial}{\partial s}\right)^k R_{s,n}(x)\right\| |f(x)| dx,$$

we have

$$\left\|\left(\frac{\partial}{\partial s}\right)^k E'_{s,n}(f)\right\| \leqq \sup_{x\in G_r} \left(\left\|\left(\frac{\partial}{\partial s}\right)^k R_{s,n}(x)\right\|\right) c(f) \quad \text{if} \quad f \in \mathscr{D}_r(G)$$

So, (1) is concluded from Lemma 1.11. Similarly,

$$\| E''_{p,n}(f)\| \leqq \int_G \| T_{p,n}(x)\| \, |f(x)| dx \,.$$

Since $T_{p,n}$ is a unitary representation, $\| T_{p,n}(x)\| = 1$ for all $x \in G$. Thus this lemma is proved and also Proposition 3.1 is proved.

The next theorem which is an analogue of the theorem of Paley-Wiener type for Spin (4, 1) is the main theorem of this paper.

THEOREM 3.4. *Fourier transform $f \to \hat{f}$ gives a linear isomorphism of $\mathscr{D}(G)$ onto $\mathscr{D}(\hat{G})$, and if $f \in \mathscr{D}_r(G)$ then $f \in \mathscr{D}_r(\hat{G})$.*

It is proved in Proposition 3.1 that Fourier transform is a linear mapping of $\mathscr{D}(G)$ into $\mathscr{D}(\hat{G})$ and if $f \in \mathscr{D}_r(G)$ then $\hat{f} \in \mathscr{D}_r(\hat{G})$. Furthermore Lemma 2.2 shows that Fourier transform is one-to-one.

To begin with, we prove the next proposition.

PROPOSITION 3.5. *For each $F = (F_c, F_d) \in \mathscr{D}(\hat{G})$ the formula*

$$f(x) = \sum_n \int_{\gamma(0)} \langle R_{s,n}(x^{-1}),\, F_c(s,\, n)\rangle_n Q_c(s,\, n) ds + \sum_{(p,n)\in \hat{G}_d} \langle T_{p,n}(x^{-1}),\, F_d(p,\, n)\rangle_{p,n} Q_d(p,\, n) \tag{3.1}$$

is absolutely and uniformly convergent on G. Furthermore, f is a C^∞-function on G and for any $D_1, D_2 \in \mathfrak{G}$

$$|f|_{D_1 D_2} = \sup_{x\in G} |(D_1 f D_2)(x)|$$

is finite.

PROOF. To begin with, we consider the following special case; (3.2) there are irreducible unitary representations τ_1, τ_2 of K such that

$$F_c(s,\, n) = R_n(\tau_1) F_c(s,\, n) R_n(\tau_2) \,, \quad F_d(p,\, n) = T_{p,n}(\tau_1) F_d(p,\, n) T_{p,n}(\tau_2) \,.$$

LEMMA 3.6. *If $F = (F_c, F_d) \in \mathscr{D}(\hat{G})$ satisfies the condition* (3.2), *then Proposition* 3.5 *is valid. Moreover for any $D_1, D_2 \in \mathfrak{G}$ and integers*

$i \geqq 0$, $j \geqq 0$ there exists a constant $c>0$ depending only on D_1, D_2, i, j, F such that $|f|_{D_1D_2} \leqq c|\tau_1|^{-i}|\tau|^{-j}$.

PROOF. If we write

$$\begin{cases} E''_{s,n}(x) = R_n(\tau_2)R_{s,n}(x^{-1})R_n(\tau_1) , \\ E''_{p,n}(x) = T_{p,n}(\tau_2)T_{p,n}(x^{-1})T_{p,n}(\tau_1) , \end{cases} \tag{3.3}$$

then

$$f(x) = \sum_n \int_{\Gamma(0)} \langle E'_{s,n}(x), F_c(s, n)\rangle_n Q_c(s, n)ds + \sum_{(p,n)\in\hat{G}_d} \langle E''_{p,n}(x), F_d(p, n)\rangle_{p,n} Q_d(p, n) .$$

We have from Proposition 1.5 that for any $D \in \mathfrak{G}$ and $\tau \in \hat{K}$ there are non-negative integers i, j, k and a constant $c>0$ such that

$$\max(||R_{s,n}(D)R_n(\tau)||, ||R_n(\tau)R_{s,n}(D)||) \leqq c(1+|s|)^i(1+n)^j|\tau|^k .$$

Since $E'_{s,n}(D_1; x; D_2) = R_n(\tau_2)R_{s,n}(D_2)R_{s,n}(x^{-1})R_{s,n}(D_1)R_n(\tau_1)$, there are non-negative integers h, i, j, k and a constant $c>0$ depending only on D_1, D_2 such that

$$||E'_{s,n}(D_1; x; D_2)|| \leqq c(1+|s|)^h(1+n)^i|\tau_1|^j|\tau_2|^k||R_{s,n}(x^{-1})|| .$$

Hence, by using Lemma 1.1 and Corollary 1.7, we conclude that

$$|\langle E'_{s,n}(D_1; x; D_2), F_c(s, n)\rangle_n| \leqq c(1+|s|)^h(1+n)^i|\tau_1|^j|\tau_2|^k||R_{s,n}(x^{-1})||\,||F_c(s, n)|| .$$

Similarly, we have from Corollary 1.15 that

$$|\langle E''_{p,n}(D_1; x; D_2), F_d(p, n)\rangle_{p,n}| \leqq c(1+|p|)^h(1+n)^i|\tau_1|^j|\tau_2|^k||T_{p,n}(x^{-1})||\,||F_d(p, n)|| .$$

Since $R_{s,n}$ ($(s, n) \in \hat{G}_c$ with $\mathrm{Re}(s)=0$), $T_{p,n}$($(p, n) \in \hat{G}_d$) are unitary representations of G, it is easy to see that the explicit formulas of Q_c, Q_d (Lemma 2.1, Lemma 2.2) and the above estimates imply the following result; if $F=(F_c, F_d) \in \mathscr{D}(\hat{G})$ satisfies (3.2), then for any D_1, $D_2 \in \mathfrak{G}$ and integers $i \geqq 0$, $j \geqq 0$ there is a constant $c>0$ depending only on D_1, D_2, F, i, j such that

$$\left|\sum_n \int_{\Gamma(0)} \langle E'_{s,n}(D_1; x; D_2), F_c(s, n)\rangle_n Q_c(s, n)ds\right|$$

$$+ \quad |\sum_{(p,n)\in\hat{G}_d} \langle E''_{p,n}(D_1; x; D_2),\ F_d(d, n)\rangle_{p,n} Q_d(p, n)|$$
$$\leqq c|\tau_1|^{-i}|\tau_2|^{-j} .$$

Hence Lemma 3.6 is proved.

Now we return to the proof of Proposition 3.5. Let $\{\tau_i; i=1, 2, \cdots\}$ be a complete set of irreducible unitary representations of K which are mutually inequivalent. For each $F=(F_c, F_c)\in\mathscr{D}(\hat{G})$ we put

$$F_{ij}=(F'_{ij}, F''_{i,j}) ,$$

where

$$F'_{ij}(s, n)=R_n(\tau_i)F_c(s, n)R_n(\tau_j) ,$$
$$F''_{ij}(p, n)=T_{p,n}(\tau_i)F_d(p, n)T_{p,n}(\tau_j) .$$

Then F_{ij} satisfies (3.2). Let f_{ij} be the function corresponding to F_{ij} by (3.1). Then for any $D_1, D_2\in\mathfrak{G}$ there is a constant $c>0$ such that

$$|f_{ij}|_{D_1D_2}\leqq c|\tau_i|^{-1}|\tau_j|^{-1} \quad i, j=1, 2, 3, \cdots .$$

Since $|\tau^{n',n''}|=(n'+1)^2+(n'')^2$, $\sum_i|\tau_i|^{-1}$ is finite. Thus for any $D_1, D_2\in\mathfrak{G}$

$$D_1fD_2=\sum_{i,j}(D_1f_{ij}D_2)$$

is absolutely and uniformly convergent on any compact subset of G. So, our proposition is proved.

PROPOSITION 3.7. *If $F=(F_c, F_d)\in\mathscr{D}_r(\hat{G})$ satisfies (3.2), then the support of f is contained in G_r.*

PROOF. Since $R_n(\tau)=0$ if $\tau\notin\hat{K}(n)$, we may regard $F_c(s, n)$ as a linear endomorphism on the representation space V_2 of τ_2 into the representation space V_1 of τ_1 (Lemma 1.1) and we have from Lemma 1.2 that

$$\langle R_{s,n}(x^{-1}), F_c(s, n)\rangle_n=((d(\tau_1)d(\tau_2))^{1/2}/(2n+1))\mathrm{trace}(E(s, v_n, x^{-1})F_c(s, n)) .$$

We write $E'(s, n, x)$ for the spherical function on $G^+=G-K$ which is given by the function $E(s, n, t)$ defined in (6.1) of [13] (see Proposition 2.2 of [13]). Then we have from Theorem 3.1 and Proposition 5.4 of [13] that

$$E(s, v_n, x)=\varepsilon(E'(s, n, x)c_n(s, \tau_1)+E'(-s, n, x)c_n(-s, \tau_2)) \text{ for any } x\in G^+ ,$$

where $\varepsilon=\pm1$. Since

$$A_{s,n}F_c(s,\ n)=F_c(-s,\ n)A_{s,n}\ ,$$
$$c_n(s,\ \tau_1)F_c(s,\ n)=c_n(s,\ \tau_2),\ F_c(-s,\ n)\ . \tag{3.4}$$

Therefore,

$$E(s,\ v_n,\ x^{-1})F_c(s,\ n)$$
$$=(c_n(s,\ \tau_2)E'(s,\ n,\ x^{-1})F_c(-s,\ n)+c_n(-s,\ \tau_2)E'(-s,\ n,\ x^{-1})F_c(s,\ n))\ .$$

Hence,

$$\int_{\gamma(0)}\langle R_{s,n}(x^{-1}),\ F_c(s,\ n)\rangle_n Q_c(s,\ n)ds=c'f_n(x)$$

where c' is a suitable constant and

$$f_n(x)=\int_{\gamma(0)}\text{trace }(E'(s,\ n,\ x^{-1})F_c(-s,\ n))c_n(-s,\ \tau_2)^{-1}ds\ .$$

Let $\tau_i=\tau^{n_i'n_i''}\in\hat{K}(n)$ $i=1, 2$ and put

$$S'=\{-(k+1/2);\ 2k,\ n-k \text{ are integers with max } (|n_1''|,\ |n_2''|)\leqq k<n\}\ ,$$
$$S''=\{-(k+1/2);\ 2k,\ n-k \text{ are integers with } 0\leqq k<|n_2''| \text{ or } n<k\leqq n_2'\}\ .$$

Then the singularities of the function

$$s\longmapsto E'(s,\ n,\ x^{-1})c_n(-s,\ \tau_2)^{-1}$$

on the half space $C_-=\{s\in C;\ \mathrm{Re}(s)\leqq 0\}$ are at most simple poles and the set of singular points are contained in $S'\cup S''$.

At the beginning, we assume that

$$F_c(s,\ n)=0 \quad \text{for } s=k+1/2;\ 2k,\ n-k \text{ are integers with } 0\leqq k<n\ . \tag{3.5}$$

Then it is an easy consequence of the formula (3.4) and Corollary 5.3 of [13] that the assumption (3.5) implies

$$F_c(s,\ n)=0 \text{ for } s=-(k+1/2);\ 2k,\ n-k \text{ are integers with } |n_2''|\leqq k<n\ . \tag{3.6}$$

By using the equivalence relation (2) of Corollary 1.15, we can see that (3.6) implies

$$F_c(s,\ n)=0 \text{ for } s=k+1/2;\ 2k,\ n-k \text{ are integers with } n<k\leqq\min(n_1',\ n_2')\ . \tag{3.7}$$

Consequently we have from Corollary 1.14 that the assumption (3.5) implies

(3.8) $\quad F_c(s, n)=0$ for $s=k+1/2$; $2k$, $n-k$ are integers with $0 \leqq k \leqq n$, and $k \neq n$.

Hence $F_c(-s, n)=0$ for each $s \in S' \cup S''$. Thus the function $s \mapsto E'(s, n, x^{-1})F_c(-s, n)c_n(-s, \tau_2)^{-1}$ is holomorphic on C_-. Furthermore, the relation (1) in Corollary 1.15 conclude that $F_d(p, n)=0$ for all $(p, n) \in \hat{G}_d$ if (3.5) is satisfied. Let

$$\hat{M}(\tau_1, \tau_2)=\{n;\ 2n \text{ is an integer with } n \geqq 0 \text{ and } \tau_1, \tau_2 \in \hat{K}(n)\}\ .$$

Then $F_c(s, n)=0$ for all $(s, n) \in \hat{G}_c$ with $n \notin \hat{M}(\tau_1, \tau_2)$. Since $\hat{M}(\tau_1, \tau_2)$ is a finite set, the function

$$f(s: x)=\sum_{n \in \hat{M}(\tau_1 \tau_2)} \operatorname{trace}(E'(s, n, x^{-1})F_c(-s, n))c_n(-s, \tau_2)$$

on $C_- \times G^+$ is well-defined and for each $x \in G^+$ the function $s \mapsto f(s: x)$ is holomorphic on C_-. By the definition of $E'(s, n, x)$,

$$\begin{aligned} \|E'(s, n, x)\| &= \|e^{(s-3/2)|x|}\sum_{k=0}^{\infty} e^{-k|x|}A_k(s)v_n\| \\ &\leqq e^{(\mathrm{Re}(s)-3/2)|x|}\sum_{k=0}^{\infty} e^{-k|x|}\|A_k(s)\|\,\|v_n\| \end{aligned}$$

for $x \in G^+$ and $s \in C$ with $|\mathrm{Im}(s)|>0$. But we have from (4.2) of [13] that there is a constant $c>0$ such that

$$\|A_k(s)\| \leqq (c/k)\sum_{j=0}^{k-1}\|A_j(s)\| \quad k=1, 2, 3, \cdots, \quad \text{and} \quad |\mathrm{Im}(s)| \geqq 1\ .$$

Consequently we have from Lemma 4.5 of [13] that

$$\|A_k(s)\| \leqq (c)_k/k! \quad \text{for} \quad k=1, 2, 3, \cdots \quad \text{and} \quad |\mathrm{Im}(s)| \geqq 1\ .$$

Hence,

(3.9) $$\|E'(s, n, x)\| \leqq \|v_n\|(1-e^{-|x|})^{-c}e^{(\mathrm{Re}(s)-3/2)|x|}\ .$$

Since

$$\lim_{z \to \infty} e^{-a(\log(z))}\Gamma(z+a)\Gamma(z)^{-1}=1;\ |\arg(z)| \leqq \pi \quad \text{for any} \quad a \in C\ ,$$

there exists a constant $c>0$ and an integer $i \geqq 0$ such that

(3.10) $$|c_n(-s, \tau_2)| \leqq c|s|^i\ .$$

These estimates (3.9), (3.10) are valid for $|\mathrm{Im}(s)| \geqq 1$ or $\mathrm{Re}(s) \leqq -b$, where

b is a suitable positive number. From the estimate for F_c we have that if F satisfies (3.5) then for any integer $i>0$ there are constants c, c' such that

(3.11) $$|f(s: x)| \leqq c'(1+|s|)^{-i}(1-e^{-|x|})^{-c} e^{(\mathrm{Re}(s)-3/2)|x|} e^{|\mathrm{Re}(s)|r}$$ for each $x \in G^+$ and $s \in C_-$.

Hence, by Cauchy's integral formula, we have that

$$f(x)=\int_{\gamma(-a)} c'' f(s: x) ds$$

for any positive number a where c'' is a suitable constant. Therefore from (3.11) we conclude that

(3.12) for any positive number a there is a constant c such that $|f(x)| \leqq c(1-e^{-|x|}) e^{-a(|x|-r)}$.

Hence, if $|x|>r$ then

$$|f(x)| \leqq c(1-e^{-|x|}) \lim_{a \to \infty} e^{-a(|x|-r)}=0 .$$

So, the support of f is contained in G_r.

For general case, we need the following lemma.

LEMMA 3.8. *Let F, τ_1, τ_2 be the same as in Proposition 3.7. Then there is a function $h \in \mathscr{D}_r(\hat{G})$ which satisfies*

1) $$\hat{h}_c(s, n)=F_c(s, n) \quad \textit{for each} \quad (s, n) \in (*) ,$$

where $(*)=\{(k+1/2, n); n \in \hat{M}(\tau_1, \tau_2)$ *and* $n-k$ *is an integer with* $0 \leqq k<n$

2) $$h_c(s, n)=R_n(\tau_1) h_c(s, n) R_n(\tau_2) \quad \textit{for any} \quad (s, n) \in \hat{G}_c .$$

PROOF. Let $(s, n), (s', n') \in (*)$ and $(s, n) \neq (s', n')$. Then it is easy to see that $\chi_{s,n} \neq \chi_{s',n'}$. Let V_n denote the space of all linear endomorphisms on $\mathscr{H}_n(\tau_2)$ into $\mathscr{H}_n(\tau_1)$ and set $V=\bigoplus_{(s,n) \in (*)} V_n$. Then V is a finite dimensional complex Hilbert space. We consider the linear mapping $\Phi: h \to \bigoplus_{(s,n) \in (*)} \hat{h}'_c(s, n)$ of $\mathscr{D}_r(G)$ into V, where $\hat{h}'_c(s, n)= R_n(\tau_1) \hat{h}_c(s, n) R_n(\tau_2)$.

Then Φ is a surjective mapping. To prove this, we suppose that for each $(s, n) \in (*)$ there are $A(s, n) \in V_n$ such that

$$\sum_{(s,n) \in (*)} (\hat{h}'_c(s, n), A(s, n))_{V_n}=0 \quad \text{for all} \quad h \in \mathscr{D}_r(G) ,$$

where $(\ ,\ )_{V_n}$ is an inner product on V_n. Since

$$\sum_{(s,n)\in(*)} (\hat{h}'_c(s, n), A(s, n))_{V_n} = \int_G g(x)h(x)dx ,$$

where

$$g(x) = \sum_{(s,n)\in(*)} (R_n(\tau_1)R_{s,n}(x)R_n(\tau_2), A(s, n))_{V_n} ,$$

$$g(x)=0 \quad \text{if} \quad |x|<r .$$

Hence from the analiticity of g (Lemma 1.9) we have $g=0$ on G. So, from Lemma 1.10 we conclude that $A(s, n)=0$ for all $(s, n)\in(*)$. Thus $(\Phi(h), v)_V=0$ for all $h\in\mathscr{D}_r(G)$ which implies $v=0$. Hence the surjectivity of Φ is proved.

Now we define the functions ξ_1, ξ_2 on K by

$$\xi_i(k)=d(\tau_i)\text{trace}(\tau_i(k)) \quad \text{for } k\in K \quad i=1, 2 ,$$

and put

$$(\xi_1 * h * \xi_2)(x) = \int_K\int_K \xi_1((k')^{-1})\xi_2((k'')^{-1})h(k'xk'')dk'dk'' .$$

Then we know that $(\xi_1*h*\xi_2)\in\mathscr{D}_r(G)$ for $h\in\mathscr{D}_r(G)$ and

$$(\xi_1*h*\xi_2)\hat{\ }_c(s, n)=R_n(\tau_1)\hat{h}_c(s, n)R_n(\tau_2) \quad \text{for all} \quad (s, n)\in\hat{G}_c ,$$
$$(\xi_1*h*\xi_2)\hat{\ }_d(p, n)=T_{p,n}(\tau_1)\hat{h}_d(p, n)T_{p,n}(\tau_2) \quad \text{for all} \quad (p, n)\in\hat{G}_d .$$

Moreover it is easy to see that if $h\in\mathscr{D}_r(G)$ satisfies 1) then $\xi_1*h*\xi_2$ is also satisfies 1). Therefore the surjectivity of Φ implies Lemma 3.8.

Now we return to the proof of Proposition 3.7. For any $F\in\mathscr{D}_r(\hat{G})$, we select a function $h\in\mathscr{D}_r(G)$ so that h satifies the properties 1), 2) of Lemma 3.8 and put $\tilde{F}=F-\hat{h}$. Then $\tilde{F}$ satiesfies the assumption (3.5). Let $\tilde{f}$ be the function corresponding to $\tilde{F}$ by (3.1). Then $\tilde{f}=f-h$. Hence, the assertion in the special case implies that

$$\text{supp}(f)=\text{the support of } f\subset\text{supp}(\tilde{f})\cup\text{supp}(h)\subset G_r .$$

Thus Proposition 3.7 is proved.

LEMMA 3.9. *If $F=(F_c, F_d)\in\mathscr{D}(\hat{G})$, then $F_c(s, n)$ (resp. $F_d(p, n)$) is a Hilbert-Schmidt operator on $\mathscr{H}_n$ (resp. $\mathscr{H}_{p,n}$) for any $(s, n)\in\hat{G}_c$ (resp. $(p, n)\in\hat{G}_d$). Furthemore, denote the Hilbert-Schmidt norm by $\|\ \|_{HS}$. Then*

$$\|F\|^2 = \sum_n\int_{\Gamma(0)} \|F_c(s, n)\|_{HS}^2 Q_c(s, n)ds + \sum_{(p,n)\in\hat{G}_d} \|F_d(p, n)\|_{HS}^2 Q_d(p, n)$$

is convergent.

PROOF. For each $\tau_1, \tau_2 \in \hat{K}$, it is easy to see that

$$\|R_n(\tau_1)F_c(s, n)R_n(\tau_2)\|_{HS}^2 \leq d(\tau_1)d(\tau_2)\|R_n(\tau_1)F_c(s, n)R_n(\tau_2)\|^2$$
$$\leq |\tau_1|\,|\tau_2|\,\|R_n(\tau_1)F_c(s, n)R_n(\tau_2)\|^2 .$$

Hence, for any non-negative integers h, i, j, k and $\tau_1, \tau_2 \in \hat{K}$, there is a constant $c>0$ such that

$$\|R_n(\tau_1)F_c(s, n)R_n(\tau_2)\|_{HS}^2 \leq c(1+|s|)^{-h}(1+n)^{-i}|\tau_1|^{-j}\tau_2^{-k} .$$

Since

$$\|F_c(s, n)\|_{HS}^2 = \sum_{i,j} \|R_n(\tau_i)F_c(s, n)R_n(\tau_j)\|_{HS}^2 ,$$

for any integers $i \geq 0$, $j \geq 0$, there is a constant $c>0$ such that

$$\|F_c(s, n)\|_{HS}^2 \leq c(1+|s|)^{-i}(1+n)^{-j} .$$

Similarly, for any integers $i \geq 0$, $j \geq 0$ there is a constant $c>0$ such that

$$\|F_d(p, n)\|_{HS}^2 \leq c(1+|p|)^{-i}(1+n)^{-j} .$$

Then Lemma 3.9 is an easy consequence of these estimates. Using Lemma 3.9 and Theorem 2 of [1], we obtain the following.

COROLLARY 3.10. *For each $F \in \mathscr{D}(\hat{G})$ there is a square integrable function on G such that $\hat{f}=F$.*

PROOF OF THEOREM 3.4. Theorem 3.4 has been essentially proved, because the injectivity of the Fourier transform is a simple consequence of Lemma 2.2 and Proposition 3.1, and the surjectivity is a simple consequence of Proposition 3.5, Proposition 3.7 and Corollary 3.10: Moreover Proposition 3.7 implies that $f \in \mathscr{D}_r(G)$ if and only if $\hat{f} \in \mathscr{D}_r(\hat{G})$. So, Theorem 3.4 is completely proved.

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Present Address:
Department of Mathematics
School of Sciences and Engineerings
Waseda University
Nishiokubo, Shinjuku-ku, Tokyo 160