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Metrical Theory for a Class of Continued Fraction Transformations and Their Natural Extensions

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Introduction

In this article we consider the class of continued fraction transformations $\{f_\alpha\}$ including the transformations associated with continued fractions to the nearest integer, singular continued fractions and with simple continued fractions. Here f_α, $1/2\leqq\alpha\leqq 1$, is defined by

$$f_\alpha(x)=\left|\frac{1}{x}\right|-\left[\left|\frac{1}{x}\right|+1-\alpha\right] \qquad \text{for} \quad x\neq 0,\ x\in[\alpha-1,\alpha)\ .$$

Many results concerning the metrical theory for the simple continued fractions had been given by Gauss, Lévy, Khintchine, etc., (see Billingsley [1]). On the other hand, the metrical theory of continued fractions to the nearest integer or of singular continued fractions has been discussed by Rieger [7], [8] and [9], in which he obtained among other things the invariant measures for these transformations.

In contrast with $\{f_\alpha\}$, recently Ito and Tanaka [3] considered the class of transformations $\{S_\alpha\}$ including those associated with the restriction to the real axis of Hurwitz' complex continued fractions and of simple continued fractions. Here S_α, $1/2\leqq\alpha\leqq 1$, is defined by

$$S_\alpha(x)=\frac{1}{x}-\left[\frac{1}{x}+1-\alpha\right] \qquad \text{for} \quad x\neq 0, x\in[\alpha-1,\alpha)\ ;$$

they have obtained the absolutely continuous invariant measures and computed entropies $h(S_\alpha)$ with respect to them for the cases of $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$.

In this note, first we will show the convergence of expansions with respect to f_α and some fundamental properties. The essential property of $\{f_\alpha\}$ is that the denominators q_n of the n-th approximants with respect to f_α are always positive in contrast with the case of S_α. Next we will

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construct the natural extension automorphisms of f_α as "skew product transformations" on suitable subsets of R^2 and deduce the absolutely continuous invariant measures ν_α of f_α. (These discussions in §2 correspond to "the method of backward transformation" considered in Nakada, Ito and Tanaka [6], which enables one to deduce the absolutely continuous invariant measure for $S_{1/2}$.) Furthermore we will show the ergodicity, the exactness and other metrical properties of f_α and calculate the entropies $h(f_\alpha)$ with respect to ν_α. We will find $h(f_\alpha)=h(S_\alpha)$ for $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$; on the other hand, $h(S_\alpha)$ are still unknown for $(\sqrt{5}-1)/2<\alpha<1$. Finally we will discuss, in some sense, the uniqueness of orbits of $\{f_\alpha\}$ for a fixed x. The same situation also holds for S_α, $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$; however, it does not hold for $(\sqrt{5}-1)/2<\alpha<1$; this seems to be one of the main reasons why it is difficult to calculate the absolutely continuous invariant measure for those α.

Here we restrict our attention to the case of $1/2\leqq\alpha\leqq1$; however, the same arguments as in §2 also hold for some $\alpha\in[0, 1/2)$. In particular, for $\alpha=0$, the transformation f_0 has the absolutely continuous invariant measure with total mass infinite, but we will discuss these on another occasion.

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§1. Definitions and fundamental properties.

For each α, $1/2\leqq\alpha\leqq1$, we define the transformation f_α of $I_\alpha=[\alpha-1, \alpha)$ onto itself as follows:

$$f_\alpha(x)=\begin{cases}\left|\dfrac{1}{x}\right|-\left[\left|\dfrac{1}{x}\right|+1-\alpha\right] & \text{for } x\neq0\\ 0 & \text{for } x=0\end{cases}$$

where for any real number a, $[a]$ denotes its integral part. If we put for $x\in I_\alpha$

$$a_\alpha(x)=\begin{cases}\left[\left|\dfrac{1}{x}\right|+1-\alpha\right] & \text{for } x\neq0\\ \infty & \text{for } x=0\,,\end{cases}$$

$$\varepsilon(x)=\begin{cases}1 & \text{for } x\geqq0\\ -1 & \text{for } x<0\end{cases}$$

and

$$a_{\alpha,i}(x)=\begin{cases}\varepsilon(f_\alpha^{i-1}(x))a_\alpha(f_\alpha^{i-1}(x)) & \text{if } f_\alpha^{i-1}(x)\neq 0\\ \infty & \text{if } f_\alpha^{i-1}(x)=0\,,\end{cases}$$

then we have the symbolic realization $\{a_{\alpha,i}(x),\ i=1, 2, 3, \cdots\}$ of x by f_α.

First we show the validity of this realization. For any $x\in I_\alpha$ with $x\neq 0$, $f_\alpha(x)\neq 0, \cdots, f_\alpha^{n-1}(x)\neq 0$; it is easy to see

$$x=\frac{\varepsilon_1|}{|a_1}+\frac{\varepsilon_2|}{|a_2}+\cdots+\frac{\varepsilon_n|}{|a_n}+f_\alpha^n(x)\;; \tag{1}$$

here and henceforth we put $\varepsilon_i=\varepsilon(f_\alpha^{i-1}(x))$ and $a_i=|a_{\alpha,i}(x)|$. As in the case of simple continued fractions, we define p_n and q_n by

$$\begin{cases} p_{-1}(x;\alpha)=1\,, \qquad p_0(x;\alpha)=0\,,\\ p_n(x;\alpha)=|a_{\alpha,n}(x)|\cdot p_{n-1}(x;\alpha)+\varepsilon(f_\alpha^{n-1}(x))\cdot p_{n-2}(x;\alpha)\\ q_{-1}(x;\alpha)=0\,, \qquad q_0(x;\alpha)=1\,,\\ q_n(x;\alpha)=|a_{\alpha,n}(x)|\cdot q_{n-1}(x;\alpha)+\varepsilon(f_\alpha^{n-1}(x))\cdot q_{n-2}(x;\alpha)\;; \end{cases} \tag{2}$$

then we have

$$x=\frac{p_n(x;\alpha)+f_\alpha^n(x)\cdot p_{n-1}(x;\alpha)}{q_n(x;\alpha)+f_\alpha^n(x)\cdot q_{n-1}(x;\alpha)} \tag{3}$$

$$p_n(x;\alpha)q_{n+1}(x;\alpha)-p_{n+1}(x;\alpha)q_n(x;\alpha)=\varepsilon_1\varepsilon_2\cdots\varepsilon_{n+1}(-1)^{n-1}\,. \tag{4}$$

We call

$$\frac{p_n(x;\alpha)}{q_n(x;\alpha)}=\frac{\varepsilon_1|}{|a_1}+\frac{\varepsilon_2|}{|a_2}+\cdots+\frac{\varepsilon_n|}{|a_n}$$

the n-th approximant of x with respect to f_α.

LEMMA 1. *For any irrational number $x\in I_\alpha$ and any positive integer n, we have*

$$q_n(x;\alpha)>0\,, \qquad q_{n+1}(x;\alpha)>q_n(x;\alpha)\;;$$

furthermore,

$$p_n(x;\alpha)>0 \text{ holds if and only if } x>0\,.$$

PROOF. If α belongs to $[1/2, (\sqrt{5}-1)/2]$ or $((\sqrt{5}-1)/2, 1]$, then for any positive integer i, $a_{\alpha,i}(x)$ belongs to $\{\pm 2, \pm 3, \pm 4, \cdots\}$ or $\{1, 2, \pm 3, \pm 4, \cdots\}$ respectively. Using this fact and (2) it is easy to prove the assertion of the lemma.

PROPOSITION 1. *For any irrational number* $x\in I_\alpha$,

$$\lim_{n\to\infty}\frac{p_n(x;\alpha)}{q_n(x;\alpha)}=x \qquad \text{for each} \quad \alpha\in\left[\frac{1}{2},1\right].$$

PROOF. If we put $f_\alpha^n(x)=t$, then $|t|<1$. By using (3) and (4)

$$(5)\qquad \left|x-\frac{p_n(x;\alpha)}{q_n(x;\alpha)}\right|=\left|\frac{p_n(x;\alpha)+tp_{n-1}(x;\alpha)}{q_n(x;\alpha)+tq_{n-1}(x;\alpha)}-\frac{p_n(x;\alpha)}{q_n(x;\alpha)}\right|$$
$$=\left|\frac{t\varepsilon_1\varepsilon_2\cdots\varepsilon_n(-1)^n}{q_n(x;\alpha)(q_n(x;\alpha)+tq_{n-1}(x;\alpha))}\right|.$$

Thus it follows from Lemma 1 that

$$\lim_{n\to\infty}\left|x-\frac{p_n(x;\alpha)}{q_n(x;\alpha)}\right|=0.$$

Next let us consider the error of n-th approximant. From (5) and the fact that

$$|a_{\alpha,n+1}(x)|-1+\alpha\leqq\frac{1}{|t|}<|a_{\alpha,n+1}(x)|+\alpha,$$

it follows that

$$(6)\qquad \frac{1}{2q_{n+1}^2(x;\alpha)}<\frac{1}{q_n(x;\alpha)\cdot(q_n(x;\alpha)+q_{n+1}(x;\alpha))}$$
$$\leqq\left|x-\frac{p_n(x;\alpha)}{q_n(x;\alpha)}\right|$$
$$\leqq\frac{1}{q_n(x;\alpha)\cdot(q_{n+1}(x;\alpha)-(1/2)q_n(x;\alpha))}$$
$$\leqq\frac{2}{q_n^2(x;\alpha)}.$$

These inequalities imply that the convergence rate of the n-th approximant is "$\sim q_n^2(x;\alpha)$" as n tends to ∞.

LEMMA 2. *There exists an absolute constant* $\delta_1>0$ *such that for any* $\alpha\in[1/2,1]$ *and any irrational number* $x\in I_\alpha$,

$$\begin{cases} q_n(x;\alpha)>\delta_1\cdot\sqrt{D}^n \\ |p_n(x;\alpha)|>\delta_1\cdot\sqrt{D}^n \end{cases}\qquad \text{for all} \quad n\geqq 1$$

where $D=2+1/2$.

Proof. From (2), we get

$$\begin{aligned} q_{n+1}(x;\alpha) &= |a_{\alpha,n+1}(x)|\cdot|a_{\alpha,n}(x)|\cdot q_{n-1}(x;\alpha) \\ &\quad + |a_{\alpha,n+1}(x)|\cdot \varepsilon(f_\alpha^{n-1}(x))\cdot q_{n-2}(x;\alpha) \\ &\quad + \varepsilon(f_\alpha^{n}(x))\cdot q_{n-1}(x;\alpha)\ . \end{aligned}$$

If $\alpha\in[1/2, (\sqrt{5}-1)/2]$, then

$$|a_{\alpha,n}(x)|\geqq 2$$

and

$$a_{\alpha,n}(x)=-2 \quad \text{implies} \quad a_{\alpha,n+1}(x)\geqq 2\ .$$

Hence by Lemma 1

$$q_{n+1}(x;\alpha)>3\cdot q_{n-1}(x;\alpha)\ . \tag{7}$$

On the other hand, if $\alpha\in((\sqrt{5}-1)/2, 1]$, then

$$a_{\alpha,n}(x)\neq -2$$

and

$$a_{\alpha,n}(x)=1 \quad \text{implies} \quad a_{\alpha,n+1}(x)\geqq 1\ .$$

So for fixed n and α, $\min_x q_n(x;\alpha)$ is given by $\eta=(\sqrt{5}-1)/2$ with $a_{\alpha,i}(\eta)=1$ for any positive integer i. Since

$$\begin{aligned} q_{n+1}(\eta;\alpha) &= q_n(\eta;\alpha)+q_{n-1}(\eta;\alpha) \\ &= 2\cdot q_{n-1}(\eta;\alpha)+q_{n-2}(\eta;\alpha)\ , \end{aligned}$$

we get

$$\begin{aligned} \frac{q_{n+1}(\eta;\alpha)}{q_{n-1}(\eta;\alpha)} &= 2+\frac{q_{n-2}(\eta;\alpha)}{q_{n-1}(\eta;\alpha)} = 2+\frac{q_{n-2}(\eta;\alpha)}{q_{n-2}(\eta;\alpha)+q_{n-3}(\eta;\alpha)} \\ &> 2+\frac{1}{2} \qquad \text{for } n\geqq 3\ . \end{aligned} \tag{8}$$

From (7) and (8) it follows that there exists a $\delta_1'>0$ such that

$$q_n(x;\alpha)\geqq \delta_1'\cdot\left(2+\frac{1}{2}\right)^{n/2}\ .$$

And in the same way, we have $\delta_1''>0$ with

$$|p_n(x;\alpha)|\geqq \delta_1''\cdot\left(2+\frac{1}{2}\right)^{n/2}\ .$$

LEMMA 3. *For any* $\alpha \in [1/2, 1]$ *and irrational number* $x \in I_\alpha$*, there exists an absolute constant* $\delta_2 > 0$ *such that*

$$\left| \log |x| - \log \left| \frac{p_n(x; \alpha)}{q_n(x; \alpha)} \right| \right| \leqq \delta_2 \cdot D^{-n}$$

for all $n \geqq 1$.

PROOF. It follows from (6) and Lemma 2 that

$$\left| \frac{x}{p_n(x; \alpha)/q_n(x; \alpha)} - 1 \right| \leqq \frac{2}{q_n^2(x; \alpha)} \cdot \frac{q_n(x; \alpha)}{|p_n(x; \alpha)|} \leqq \frac{2}{\delta_1^2} \cdot D^{-n} .$$

So the Taylor expansion of $\log(1+x)$ implies the assertion of Lemma 3.

Now let us consider a sequence of integers $(\omega_1, \omega_2, \cdots, \omega_n)$ of length n and define the n-cylinder set of I_α by

$$\langle \omega_1, \omega_2, \cdots, \omega_n \rangle_\alpha = \{x \in I_\alpha;\ a_{\alpha,1}(x) = \omega_1,\ a_{\alpha,2}(x) = \omega_2, \cdots, a_{\alpha,n}(x) = \omega_n\} .$$

If $\langle \omega_1, \omega_2, \cdots, \omega_n \rangle_\alpha \neq \phi$ (a.e.), then we call $(\omega_1, \omega_2, \cdots, \omega_n)$ an admissible sequence of length n with respect to f_α. For any admissible sequence $(\omega_1, \omega_2, \cdots, \omega_n)$ we put

$$\begin{cases} p_m(\omega) = p_m(x; \alpha) \\ q_m(\omega) = q_m(x; \alpha) , \qquad 1 \leqq m \leqq n \end{cases}$$

where $x \in \langle \omega_1, \omega_2, \cdots, \omega_n \rangle_\alpha$. It is easy to see that the n-cylinder set is an interval in I_α and it follows that

$$\begin{aligned} (9) \qquad & m(\langle \omega_1, \omega_2, \cdots, \omega_n \rangle_\alpha) \\ & \leqq \left| \frac{p_n(\omega) + \alpha \cdot p_{n-1}(\omega)}{q_n(\omega) + \alpha \cdot q_{n-1}(\omega)} - \frac{p_n(\omega) + (\alpha - 1) \cdot p_{n-1}(\omega)}{q_n(\omega) + (\alpha - 1) \cdot q_{n-1}(\omega)} \right| \\ & = \frac{\alpha}{(q_n(\omega) + \alpha \cdot q_{n-1}(\omega)) \cdot (q_n(\omega) + (\alpha - 1) \cdot q_{n-1}(\omega))} \end{aligned}$$

for any admissible sequence $(\omega_1, \omega_2, \cdots, \omega_n)$, where $m(\cdot)$ is the Lebesgue measure. It is possible to prove that the validity of the equality in (9) is equivalent to the assertion

$$f_\alpha^n(\langle \omega_1, \omega_2, \cdots, \omega_n \rangle_\alpha) = I_\alpha .$$

NOTES. i) For any rational number $x \in I_\alpha$, there exists $K = K(x; \alpha) > 0$ such that

$$|a_{\alpha,1}(x)|<\infty,\ \cdots,\ |a_{\alpha,K}(x)|<\infty\ ,\qquad |a_{\alpha,n}(x)|=\infty\qquad \text{for all}\quad n>K\ .$$

This is proved by the same argument as in the case of the simple continued fraction transformation and we call such a K the length of the rational number x with respect to f_α.

ii) It follows from (9) and Lemma 2 that cylinder sets generate Borel sets.

§2. Constructions of natural extensions and their invariant measures.

In this section we construct the natural extension T_α to each f_α, $1/2\leqq\alpha\leqq1$, on a suitable subset M_α of R^2. We start by defining M_α, the domain of T_α, and constructing the fundamental partition P_α which will be the generator of T_α. To do this we consider two separate classes of $\alpha\in[1/2, 1]$ for which the constructions of M_α are different. It is convenient to consider $\lim_{x\to\alpha} f_\alpha^n(x)$, so we include α in the domain of f_α in this sense.

Case (i). $(1/2\leqq\alpha\leqq(\sqrt{5}-1)/2)$. For each $\alpha\in[1/2, (\sqrt{5}-1)/2]$, we define

$$R_\alpha(x)=\begin{cases}\left[0, \dfrac{3-\sqrt{5}}{2}\right) & \text{if}\quad x\in\left[\alpha-1, \dfrac{1-2\alpha}{\alpha}\right]\\ \left[0, \dfrac{1}{2}\right) & \text{if}\quad x\in\left(\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}\right)\\ \left[0, \dfrac{\sqrt{5}-1}{2}\right) & \text{if}\quad x\in\left[\dfrac{2\alpha-1}{1-\alpha}, \alpha\right)\end{cases}$$

here if $\alpha=1/2$, then $R_\alpha(0)=[0, (3-\sqrt{5})/2)$ and if $\alpha=(\sqrt{5}-1)/2$, then $R_\alpha(x)=[0, 1/2)$ for all $x\in I_\alpha$. The domain M_α is defined as follows:

$$\begin{aligned}(10)\qquad M_\alpha=&\bigcup_{x\in I_\alpha}(\{x\}\times R_\alpha(x))\\ =&\left(\left[\alpha-1, \frac{1-2\alpha}{\alpha}\right]\times\left[0, \frac{3-\sqrt{5}}{2}\right)\right)\\ &\cup\left(\left(\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}\right)\times\left[0, \frac{1}{2}\right)\right)\\ &\cup\left(\left[\frac{2\alpha-1}{1-\alpha}, \alpha\right]\times\left[0, \frac{\sqrt{5}-1}{2}\right)\right)\\ &(\subset R^2)\ .\end{aligned}$$

The fundamental partition P_α of I_α with respect to f_α is defined by

$$P_\alpha=\{\langle k\rangle_\alpha;\ k=\pm 2,\ \pm 3,\ \pm 4,\ \cdots\}\ ,$$

where

$$\begin{cases}\langle -2\rangle_\alpha=\left[1-\alpha,\ -\dfrac{1}{2+\alpha}\right), & \langle 2\rangle_\alpha=\left(\dfrac{1}{2+\alpha},\ \alpha\right),\\ \langle -k\rangle_\alpha=\left[-\dfrac{1}{k-1+\alpha},\ -\dfrac{1}{k+\alpha}\right), & \\ \quad\langle k\rangle_\alpha=\left(\dfrac{1}{k+\alpha},\ \dfrac{1}{k-1+\alpha}\right], & \text{for}\quad k\geqq 3\ ,\end{cases}$$

that is, P_α is the partition generated by cylinder sets of length 1. We extend P_α to $\tilde{P}_\alpha$ of M_α as follows:

$$\tilde{P}_\alpha=\{\Delta_{\alpha,k};\ k=\pm 2,\ \pm 3,\ \pm 4,\ \cdots\} \tag{11}$$

where

$$\Delta_{\alpha,k}=\{(x,\ y)\in M_\alpha;\ x\in\langle k\rangle_\alpha\}\ .$$

Case (ii). $((\sqrt{5}-1)/2<\alpha\leqq 1)$. For each $\alpha\in((\sqrt{5}-1)/2,\ 1]$, we define

$$R_\alpha(x)=\begin{cases}\left[0,\ \dfrac{1}{2}\right) & \text{if}\quad x\in\left[\alpha-1,\ \dfrac{1-\alpha}{\alpha}\right]\\ [0,\ 1) & \text{if}\quad x\in\left(\dfrac{1-\alpha}{\alpha},\ \alpha\right)\end{cases}$$

here if $\alpha=1$, then $R_\alpha(x)=[0,\ 1)$ for all $x\in[0,\ 1)$. The domain M_α is defined in the same way as in case (i):

$$\begin{aligned} M_\alpha&=\bigcup_{x\in I_\alpha}(\{x\}\times R_\alpha(x))\\ &=\left(\left[\alpha-1,\ \frac{1-\alpha}{\alpha}\right]\times\left[0,\ \frac{1}{2}\right)\right)\\ &\quad\cup\left(\left(\frac{1-\alpha}{\alpha},\ \alpha\right)\times[0,\ 1)\right)\\ &\quad(\subset R^2)\ .\end{aligned} \tag{12}$$

The fundamental partition P_α of I_α with respect to f_α is defined by

$$P_\alpha=\{\langle k\rangle_\alpha\colon k=1,\ 2,\ \cdots,\ r-1,\ r,\ \pm(r+1),\ \pm(r+2),\ \cdots\}$$

where

$$r=r(\alpha)=a_{\alpha,2}(\alpha)$$

and

$$\begin{cases}\langle 1\rangle_\alpha=\left(\frac{1}{1+\alpha},\ \alpha\right), \quad \langle k\rangle_\alpha=\left(\frac{1}{k+\alpha},\ \frac{1}{k-1+\alpha}\right] \quad \text{for } k\geqq 2\,, \\ \langle -(r+1)\rangle_\alpha=\left[\alpha-1,\ -\frac{1}{r+1+\alpha}\right), \\ \langle -j\rangle_\alpha=\left[-\frac{1}{j-1+\alpha},\ -\frac{1}{j+\alpha}\right) \quad \text{for } j>r+1\,. \end{cases}$$

And we also consider $\tilde{P}_\alpha$ defined by

$$\tilde{P}_\alpha=\{\Delta_{\alpha,k};\ k=1,\ 2,\ \cdots,\ r-1,\ r,\ \pm(r+1),\ \pm(r+2),\ \cdots\} \tag{13}$$

where

$$\Delta_{\alpha,k}=\{(x,\ y)\in M_\alpha;\ x\in\langle k\rangle_\alpha\}\,.$$

REMARK. If $\alpha=1$, then $M_\alpha=[0,\ 1)\times[0,\ 1)$ and

$$\tilde{P}_\alpha=\left\{\Delta_{1,k};\ \Delta_{1,k}=\left[\frac{1}{k+1},\ \frac{1}{k}\right)\times[0,\ 1),\ k=1,\ 2,\ \cdots\right\}\,.$$

Now we define T_α on M_α, $(1/2\leqq\alpha\leqq 1)$, as follows:

$$T_\alpha(x,\ y)=\begin{cases}\left(f_\alpha(x),\ \frac{1}{k+y}\right) & \text{if } x\in\langle k\rangle_\alpha,\ k>0 \\ \left(f_\alpha(x),\ \frac{1}{-k-y}\right) & \text{if } x\in\langle k\rangle_\alpha,\ k<0 \\ (0,\ 0) & \text{if } x=0\end{cases} \tag{14}$$

for $(x,\ y)\in M_\alpha$. Furthermore let μ_α be the absolutely continuous probability measure with the density function $C_\alpha\cdot(1/(1+xy))^2$, where C_α is a normalizing constant. To show that T_α is a one-to-one and onto mapping on M_α (except for a set of Lebesgue measure zero), we need the following two lemmas.

LEMMA 4. *For any* $\alpha\in(1/2,\ (\sqrt{5}-1)/2)$, *we have*

(i) $a_{\alpha,1}(\alpha)=2$ *and* $a_{\alpha,1}(\alpha-1)=-2$

(ii) $a_{\alpha,2}(\alpha-1)\geqq 2$ *and* $a_{\alpha,2}(\alpha)=-(a_{\alpha,2}(\alpha-1)+1)$

(iii) $f_\alpha^2(\alpha-1)=f_\alpha^2(\alpha)$.

PROOF. If $1/2<\alpha<(\sqrt{5}-1)/2$, then $1+\alpha\leqq 1/\alpha\leqq 2+\alpha$ and $1+\alpha\leqq 1/(1-\alpha)<2+\alpha$. Thus (i) is true. Moreover, since $f_\alpha(\alpha)=(1-2\alpha)/\alpha<0$

and $f_\alpha(\alpha-1)=(2\alpha-1)/(1-\alpha)>0$, (ii) and (iii) are obtained by simple calculations.

LEMMA 5. *For any* $\alpha\in((\sqrt{5}-1)/2, 1)$, *we have*
(i) $a_{\alpha,1}(\alpha)=1$,
(ii) $a_{\alpha,2}(\alpha)\geqq 2$ *and* $a_{\alpha,1}(\alpha-1)=-(a_{\alpha,2}(\alpha)+1)$,
(iii) $f_\alpha^2(\alpha)=f_\alpha(\alpha-1)$.

PROOF. If $(\sqrt{5}-1)/2<\alpha<1$, then $\alpha<1/\alpha<1+\alpha$ and this means that $a_{\alpha,1}(\alpha)=1$. Moreover, (ii) and (iii) follow from the facts that $f_\alpha(\alpha)=(1-\alpha)/\alpha>0$ and $1+\alpha\leqq\alpha/(1-\alpha)$.

THEOREM 1. *For each* $\alpha\in[1/2, 1]$, *we have*
(i) T_α *is a one-to-one, onto, bi-measurable and non-singular mapping on* M_α *except for a set of Lebesgue measure zero.*
(ii) μ_α *is the invariant measure of* T_α *and*

$$C_\alpha=\begin{cases}\dfrac{1}{\log(\sqrt{5}+1)/2}, & \dfrac{1}{2}\leqq\alpha\leqq\dfrac{\sqrt{5}-1}{2}\\[2ex] \dfrac{1}{\log(1+\alpha)}, & \dfrac{\sqrt{5}-1}{2}<\alpha\leqq 1\,.\end{cases}$$

PROOF. First we assume $\alpha\in(1/2, (\sqrt{5}-1)/2)$. We put $r=r(\alpha)=a_{\alpha,2}(\alpha-1)$ and $z=z(\alpha)=f_\alpha^2(\alpha)$. Let us consider the partition $Q_{\alpha,x}$ of $R_\alpha(x)$ defined by

$$Q_{\alpha,x}=\begin{cases}\{\langle k\rangle_\alpha^-(x);\ k=\pm3, \pm4, \cdots\} & \text{if}\quad x\in\left[\alpha-1, \dfrac{1-2\alpha}{\alpha}\right]\\[2ex] \{\langle k\rangle_\alpha^-(x);\ k=2, \pm3, \pm4, \cdots\} & \text{if}\quad x\in\left(\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}\right)\\[2ex] \{\langle k\rangle_\alpha^-(x);\ k=\pm2, \pm3, \pm4, \cdots\} & \text{if}\quad x\in\left[\dfrac{2\alpha-1}{1-\alpha}, \alpha\right)\end{cases}$$

where

$$\begin{cases}\langle r+1\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1+1/2}, \dfrac{1}{r+1}\right), \quad \langle -(r+1)\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1}, \dfrac{1}{r+(\sqrt{5}-1)/2}\right)\\[2ex] \langle r\rangle_\alpha^-(x)=\left(\dfrac{1}{r+(\sqrt{5}-1)/2}, \dfrac{1}{r}\right), \quad \langle -r\rangle_\alpha^-(x)=\left(\dfrac{1}{r}, \dfrac{1}{r+(\sqrt{5}-3)/2}\right)\\[2ex] \hfill \text{if}\quad x\leqq z\,,\end{cases}$$

$$\begin{cases} \langle r+1\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1+1/2}, \dfrac{1}{r+1}\right), & \langle -(r+1)\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1}, \dfrac{1}{r+1/2}\right) \\ \langle r\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1/2}, \dfrac{1}{r}\right), & \langle -r\rangle_\alpha^-(x)=\left(\dfrac{1}{r}, \dfrac{1}{r+(\sqrt{5}-3)/2}\right) \end{cases}$$

if $x>z$,

and

$$\begin{cases} \langle k\rangle_\alpha^-(x)=\left(\dfrac{1}{k+1/2}, \dfrac{1}{k}\right), \quad \langle -k\rangle_\alpha^-(x)=\left(\dfrac{1}{k}, \dfrac{1}{k-1/2}\right) & \text{for } k>r+1 , \\ \langle k\rangle_\alpha^-(x)=\left(\dfrac{1}{k+(\sqrt{5}-1)/2}, \dfrac{1}{k}\right), \quad \langle -k\rangle_\alpha^-(x)=\left(\dfrac{1}{k}, \dfrac{1}{k+(\sqrt{5}-3)/2}\right) & \end{cases}$$

for $2\leqq k\leqq r$.

We extend Q_α on $R_\alpha(x)$ to $\hat{Q}_\alpha$ on M_α by

$$\hat{Q}_\alpha=\{\hat{\Delta}_{\alpha,k};\ k=\pm 2,\ \pm 3,\ \pm 4,\ \cdots\} \tag{15}$$

where

$$\hat{\Delta}_{\alpha,k}=\{(x, y)\in M_\alpha;\ y\in\langle k\rangle_\alpha^-(x)\} .$$

From Lemma 4, $f_\alpha^2(\alpha)=f_\alpha^2(\alpha-1)=f_\alpha((2\alpha-1)/(1-\alpha))=f_\alpha((1-2\alpha)/\alpha)=z$. Furthermore $x\leqq z$ or $x>z$ is equivalent to

$$\text{“}\frac{1}{r+x}\in\left[\frac{2\alpha-1}{1-\alpha}, \alpha\right) \quad\text{and}\quad -\frac{1}{(r+1)+x}\in\left(\alpha-1, \frac{1-2\alpha}{\alpha}\right]\text{”}$$

or

$$\text{“}\frac{1}{r+x}\in\left[\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}\right) \quad\text{and}\quad -\frac{1}{(r+1)+x}\in\left(\frac{1-2\alpha}{\alpha}, \frac{2\alpha-1}{1-\alpha}\right]\text{”}$$

respectively. Thus T_α maps the interior points of $\Delta_{\alpha,r}$ and $\Delta_{\alpha,-(r+1)}$ in one-to-one manner, onto the interior points of $\hat{\Delta}_{\alpha,r}$ and $\hat{\Delta}_{\alpha,-(r+1)}$, respectively, because of (13) and (15). For $k\neq r,\ -(r+1)$, it is easy to see that T_α maps the interior points of $\Delta_{\alpha,k}$ onto the interior points of $\hat{\Delta}_{\alpha,k}$. If we denote the boundary of $\Delta_{\alpha,k}$ by $\partial\Delta_{\alpha,k}$, then we see that

$$\tilde{m}\Big(\bigcup_k \partial\Delta_{\alpha,k}\Big)=0 ,$$

where $\tilde{m}$ is the Lebesgue measure on M_α. And now the assertion of (i) is clear for $\alpha\in(1/2, (\sqrt{5}-1)/2)$.

Next we assume $\alpha\in((\sqrt{5}-1)/2, 1)$. Similarly to the above discussions, we put $r=r(\alpha)=a_{\alpha,2}(\alpha)$ and $z=z(\alpha)=f_\alpha^2(\alpha)$ and consider the partition $Q_{\alpha,x}$ of $R_\alpha(x)$ defined by

$$Q_{\alpha,x}=\begin{cases}\{\langle k\rangle_\alpha^-(x);\ k=2,3,4,\cdots,(r-1),r,\pm(r+1),\pm(r+2),\cdots\} & \text{if } x\in\left[\alpha-1,\dfrac{1-\alpha}{\alpha}\right]\\ \{\langle k\rangle_\alpha^-(x);\ k=1,2,3,\cdots,(r-1),r,\pm(r+1),\pm(r+2),\cdots\} & \text{if } x\in\left(\dfrac{1-\alpha}{\alpha},\alpha\right)\end{cases}$$

where

$$\begin{cases}\langle k\rangle_\alpha^-(x)=\left(\dfrac{1}{k+1},\dfrac{1}{k}\right) & \text{if } r>k>0\\ \langle r+1\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1+1/2},\dfrac{1}{r+1}\right) & \\ \langle k\rangle_\alpha^-(x)=\left(\dfrac{1}{k+1/2},\dfrac{1}{k}\right),\quad \langle -k\rangle_\alpha^-(x)=\left(\dfrac{1}{k},\dfrac{1}{k-1/2}\right) & \text{if } r+1<k,\end{cases}$$

and

$$\begin{cases}\langle r\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1/2},\dfrac{1}{r}\right),\quad \langle -(r+1)\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1},\dfrac{1}{r+1/2}\right) & \text{if } x\geqq z,\\ \langle r\rangle_\alpha^-(x)=\left(\dfrac{1}{r+1},\dfrac{1}{r}\right),\quad \langle -(r+1)\rangle_\alpha^-(x)=\varnothing & \text{if } x<z.\end{cases}$$

We also extend Q_α on $R_\alpha(x)$ to M_α by (15). Then we see once more that T_α maps the interior points of $\Delta_{\alpha,k}$ onto the interior points of $\hat{\Delta}_{\alpha,k}$ for each k by using the fact that $x\geqq z$ or $x<z$ is equivalent to

$$\text{“}\frac{1}{r+x}\in\left[\alpha-1,\frac{1-\alpha}{\alpha}\right]\quad\text{and}\quad -\frac{1}{r+1+x}\geqq\alpha-1\text{”}$$

or

$$\text{“}\frac{1}{r+x}\in\left(\frac{1-\alpha}{\alpha},\alpha\right)\quad\text{and}\quad -\frac{1}{r+1+x}<\alpha-1\text{”},$$

respectively, which follows from Lemma 5. In the case of $\alpha=1/2$, $(\sqrt{5}-1)/2$, or 1, the construction of Q_α is even simpler and it is easy to show (i) for each case.

Now we show that μ_α is the invariant measure for T_α. Suppose that (x, y) is an interior point of $\Delta_{\alpha,k}$, then

$$\frac{dT_\alpha^{-1}\mu_\alpha}{d\mu_\alpha}(x,y)=\frac{dT_\alpha^{-1}\mu_\alpha}{dT_\alpha^{-1}\tilde{m}}(x,y)\cdot\frac{dT_\alpha^{-1}\tilde{m}}{d\tilde{m}}(x,y)\cdot\frac{d\tilde{m}}{d\mu_\alpha}(x,y)$$

$$=\frac{d\mu_\alpha}{d\tilde{m}}(T_\alpha(x, y))\cdot\frac{dT_\alpha^{-1}\tilde{m}}{d\tilde{m}}(x, y)\cdot\frac{d\tilde{m}}{d\mu_\alpha}(x, y)$$

where $\tilde{m}$ is the Lebesgue measure on M_α. If k is a positive integer, then $T_\alpha(x, y)=(|1/x|-k, 1/(k+y))$ and so

$$\begin{aligned}(16)\qquad \frac{dT_\alpha^{-1}\mu_\alpha}{d\mu_\alpha}(x, y)&=C_\alpha\cdot\left(\frac{1}{1+(1/x-k)(1/(k+y))}\right)^2\cdot\left[\frac{1}{x^2}\cdot\left(\frac{1}{k+y}\right)^2\right]\\ &\quad\times C_\alpha^{-1}\cdot(1+xy)^2\\ &=1\,.\end{aligned}$$

If k is a negative integer, it also follows from $T_\alpha(x, y)=(|1/x|+k, 1/(-k-y))$ that

$$(17)\qquad \frac{dT_\alpha^{-1}\mu_\alpha}{d\mu_\alpha}(x, y)=1\,.$$

From (16) and (17), it follows that μ_α is the invariant measure for T_α.

Finally let us calculate C_α:
if $1/2\leqq\alpha\leqq(\sqrt{5}-1)/2$, then

$$\begin{aligned}C_\alpha^{-1}&=\int_{I_\alpha}dx\int_{R_\alpha(x)}\left(\frac{1}{1+xy}\right)^2dy\\ &=\log\left(\beta+1+\frac{1-2\alpha}{\alpha}\right)-\log(\beta+\alpha)+\log\left(2+\frac{2\alpha-1}{1-\alpha}\right)\\ &\quad-\log\left(2+\frac{1-2\alpha}{\alpha}\right)+\log(\beta+\alpha)-\log\left(\beta+\frac{2\alpha-1}{1-\alpha}\right)\\ &=\log\frac{\alpha(\beta-1)+1}{\alpha(2-\beta)+\beta+1}=\log\beta\end{aligned}$$

where $\beta=(\sqrt{5}+1)/2$;
if $(\sqrt{5}-1)/2<\alpha\leqq1$, then

$$C_\alpha^{-1}=\log(1+\alpha)\,,$$

which can be shown in the same way. Thus the proof of Theorem 1 is complete.

From the proof of Theorem 1, it is easy to see that the partition $\tilde{P}_\alpha(=T_\alpha^{-1}\hat{Q}_\alpha)$ is the generator of T_α, that is, $\bigvee_{n=-\infty}^{\infty}T_\alpha^{-n}\tilde{P}_\alpha$ separates any pair of points (x, y) and (x', y') belonging to M_α, ($\vee$ denotes the join of partitions). Let us define an equivalence relation in M_α as follows:

$$(x, y)\sim(x', y')\qquad\text{if}\quad x=x'$$

and consider the quotient space $\tilde{M}_\alpha = M_\alpha/\sim$. The definition of P_α implies that

$$\tilde{M}_\alpha = M_\alpha \Big/ \bigvee_{n=1}^{\infty} T_\alpha^{-n} \tilde{P}_\alpha$$

and by (14), the factor transformation T_α of M_α induced by T_α is well-defined. There is a natural correspondence between (M_α, T_α) and (I_α, f_α); thus for any measurable subset A of I_α, the probability measure ν_α on I_α defined by

$$\nu_\alpha(A) = \mu_\alpha\Big(\bigcup_{x \in A} (\{x\} \times R_\alpha(x))\Big) \tag{18}$$

gives an invariant measure for f_α, i.e., $\nu_\alpha(f_\alpha^{-1}(A)) = \nu_\alpha(A)$. Hence $(M_\alpha, T_\alpha, \mu_\alpha)$ is the natural extension automorphism of $(I_\alpha, f_\alpha, \nu_\alpha)$ in the sense of Rohlin [10] and it is easy to show that ν_α is an absolutely continuous measure.

COROLLARY 1. *The absolutely continuous invariant measure for f_α has the density function $C_\alpha \cdot h_\alpha(x)$, where $h_\alpha(x)$ is given by*:

(i) $1/2 \leqq \alpha \leqq (\sqrt{5}-1)/2$

$$h_\alpha(x) = \begin{cases} \dfrac{1}{x+\beta+1}, & x \in \left[\alpha - 1, \dfrac{1-2\alpha}{\alpha}\right] \\ \dfrac{1}{x+2}, & x \in \left(\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}\right) \\ \dfrac{1}{x+\beta}, & x \in \left[\dfrac{2\alpha-1}{1-\alpha}, \alpha\right) \end{cases}$$

(ii) $(\sqrt{5}-1)/2 < \alpha \leqq 1$

$$h_\alpha(x) = \begin{cases} \dfrac{1}{x+2}, & x \in \left[\alpha - 1, \dfrac{1-\alpha}{\alpha}\right] \\ \dfrac{1}{x+1}, & x \in \left(\dfrac{1-\alpha}{\alpha}, \alpha\right) \end{cases}$$

PROOF. From (18), the density function of ν_α is given by

$$C_\alpha \cdot h_\alpha(x) = C_\alpha \cdot \int_{R_\alpha(x)} \left(\frac{1}{1+xy}\right)^2 dy .$$

From the above corollary, it follows that there exists an absolute constant δ_3 such that for any measurable subset A of I_α,

$$\delta_3^{-1} \cdot m(A) \leqq \nu_\alpha(A) \leqq \delta_3 \cdot m(A) . \tag{19}$$

REMARK. If we define the transformation $f^*_{\alpha,x}(y)$ of $R_\alpha(x)$ by

$$f^*_{\alpha,x}(y)=\begin{cases}\dfrac{1}{y}-k & \text{if } \ y\in\langle k\rangle^-_\alpha(x)\\[2mm] -\dfrac{1}{y}+k & \text{if } \ y\in\langle -k\rangle^-_\alpha(x)\end{cases}$$

where k is a positive integer, then

$$T_\alpha^{-1}(x, y)=\begin{cases}\left(\dfrac{1}{k+x},\ f^*_{\alpha,x}(y)\right) & \text{if } \ y\in\langle k\rangle^-_\alpha(x)\\[2mm] \left(\dfrac{1}{-k-x},\ f^*_{\alpha,x}(y)\right) & \text{if } \ y\in\langle -k\rangle^-_\alpha(x)\ .\end{cases}$$

We call $\{f^*_{\alpha,x}\}_{x\in I_\alpha}$ the backward system which is a generalization of the backward transformation discussed in Nakada, Ito and Tanaka [6], (see also Schweiger [11]). To deduce $R_\alpha(x)$, it is useful to note the following fact: let Ω_n be the set of admissible sequences $(\omega_1, \omega_2, \cdots, \omega_n)$ of length n for which

$$m\{z\in I_\alpha;\ a_{\alpha,1}(z)=\omega_1,\ a_{\alpha,2}(z)=\omega_2,\ \cdots,\ a_{\alpha,n}(z)=\omega_n,\ f^n_\alpha(z)=x\}>0$$

and put $\Omega=\bigcup_{n=1}^\infty \Omega_n$ then $\{p_n(\omega)/q_n(\omega);\ \omega\in\Omega\}$ is dense in $R_\alpha(x)$.

§3. Some limit properties of q_n.

Since ν_α and m are equivalent, we use "a.e." or "a.a." with no distinction.

LEMMA 6. *For a.a. $x\in I_\alpha$, there exists a subsequence of natural numbers $\{n_1, n_2, n_3, \cdots\}$ depending on x such that*

$$f^{n_i}_\alpha(\langle a_{\alpha,1}(x),\ a_{\alpha,2}(x),\ a_{\alpha,3}(x),\ \cdots,\ a_{\alpha,n_i}(x)\rangle_\alpha)=I_\alpha \tag{20}$$

for all $i\geqq 1$.

REMARK. This implies that the cylinder sets $\langle\omega_1, \cdots, \omega_n\rangle_\alpha$ with $f^n_\alpha(\langle\omega_1, \cdots, \omega_n\rangle_\alpha)=I_\alpha$ generate Borel sets.

PROOF. To have $f^n_\alpha(\langle\omega_1, \omega_2, \cdots, \omega_n\rangle_\alpha)=I_\alpha$ for an admissible sequence $(\omega_1, \omega_2, \cdots, \omega_n)$, it is sufficient that

$$\begin{cases}(\omega_k,\ \omega_{k+1},\ \cdots,\ \omega_n)\neq(a_{\alpha,1}(\alpha),\ a_{\alpha,2}(\alpha),\ \cdots,\ a_{\alpha,n-k+1}(\alpha))\\ (\omega_k,\ \omega_{k+1},\ \cdots,\ \omega_n)\neq(a_{\alpha,1}(\alpha-1),\ a_{\alpha,2}(\alpha-1),\ \cdots\ a_{\alpha,n-k+1}(\alpha-1))\end{cases}$$

for all k, $1\leqq k\leqq n$. Thus the number of $(\omega_1, \omega_2, \cdots, \omega_n)$ such that

$$(21)\qquad f_\alpha(\langle\omega_1\rangle_\alpha)\neq I_\alpha\ ,\qquad f_\alpha^2(\langle\omega_1, \omega_2\rangle_\alpha)\neq I_\alpha,\ \cdots\ ,\qquad f_\alpha^n(\langle\omega_1, \omega_2, \cdots, \omega_n\rangle_\alpha)\neq I_\alpha$$

is at most 2^n. Let A_n be the union of cylinder sets satisfying (21), then it follows from Lemma 2 and (9) that

$$m(A_n)\leqq\delta_2\cdot D^{-n}\cdot 2^n\ .$$

Hence for any $\varepsilon>0$, we have by (19)

$$\nu_\alpha(A_n)<\varepsilon$$

for sufficiently large n, and so

$$\nu_\alpha(A)=0\ ,$$

where A denotes the set of x for which

$$f_\alpha^n(\langle a_{\alpha,1}(x), \cdots, a_{\alpha,n}(x)\rangle_\alpha)\neq I_\alpha \qquad \text{for all}\quad n\geqq 1\ .$$

Consequently it follows that

$$\nu_\alpha\Big(\bigcup_{n=0}^{\infty} f_\alpha^{-n}A\Big)=0$$

and this implies the assertion of the lemma.

THEOREM 2. *For any* $\alpha\in[1/2, 1]$, $(I_\alpha, f_\alpha, \nu_\alpha)$ *is ergodic and exact.*

PROOF. Let $A\subset I_\alpha$ be an f_α-invariant measurable subset, then for any cylinder set $B=\langle\omega_1, \cdots, \omega_n\rangle_\alpha$ with $f_\alpha^n B=I_\alpha$ we have

$$\begin{aligned} m(A\cap B)&=\int_{B\cap A}dx=\int_A \frac{d}{dy}\Big(\frac{p_n(\omega)+p_{n-1}(\omega)\cdot y}{q_n(\omega)+q_{n-1}(\omega)\cdot y}\Big)dy\\ &=\int_A\Big(\frac{1}{q_n(\omega)+q_{n-1}(\omega)\cdot y}\Big)^2 dy\\ &\geqq m(A)\cdot\frac{1}{4\cdot q_n^2(\omega)}\\ &\geqq\frac{1}{8}m(A)\cdot m(B)\ . \end{aligned}$$

Thus for any measurable subset B, we have

$$(22)\qquad m(A\cap B)\geqq\frac{1}{8}\cdot m(A)\cdot m(B)\ ,$$

since such cylinder sets $\langle \omega_1, \cdots, \omega_n \rangle_\alpha$ generate Borel subsets. So we have

$$m(A)=0 \quad \text{or} \quad 1 ,$$

by putting $B=A^c$.

To show the exactness of f_α, we only need the existence of a constant δ_4 such that

$$\nu_\alpha(f_\alpha^n A) \leqq \delta_4 \cdot \nu_\alpha(A)/\nu_\alpha(B)$$

for any $B=\langle \omega_1, \cdots, \omega_n \rangle_\alpha$ with $f_\alpha^n B=I_\alpha$ and $A \subset B$, (see Rohlin [10]). It is easy to calculate that

$$\begin{aligned} m(A) &= \int_{f_\alpha^n A} \left(\frac{1}{q_n(\omega)+q_{n-1}(\omega)\cdot y} \right)^2 dy \\ &\geqq \frac{1}{4 \cdot q_n^2(\omega)} \cdot m(f_\alpha^n A) \geqq \frac{1}{8} \cdot m(f_\alpha^n A) \cdot m(B) \end{aligned} \tag{23}$$

and we have δ_4 by using (19).

Corollary 2. *$(M_\alpha, T_\alpha, \mu_\alpha)$ is a Kolmogorov automorphism for each $\alpha \in [1/2, 1]$.*

Proof. This corollary follows from the fact that T_α is the natural extension of f_α.

Lemma 7. *For any $\alpha \in [1/2, 1]$, we have*

$$-\int_{I_\alpha} \log |x| \cdot h_\alpha(x) dm = \frac{\pi^2}{12} .$$

Proof. If we put

$$F(\alpha)=\int_{\alpha-1}^{\alpha} \log |x| \cdot h_\alpha(x) dm ,$$

then $F(\alpha)$ is continuous on $[1/2, 1]$ and differentiable on two open intervals $(1/2, (\sqrt{5}-1)/2)$ and $((\sqrt{5}-1)/2, 1)$ by virtue of Corollary 1. If $1/2 < \alpha < (\sqrt{5}-1)/2$, then

$$\begin{aligned} F(\alpha) &= \int_{\alpha-1}^{(1-2\alpha)/\alpha} \log(-x) \cdot \frac{dx}{x+\beta+1} + \int_{(1-2\alpha)/\alpha}^{0} \log(-x) \cdot \frac{dx}{x+2} \\ &\quad + \int_0^{(2\alpha-1)/(1-\alpha)} \log x \cdot \frac{dx}{x+2} + \int_{(2\alpha-1)/(1-\alpha)}^{\alpha} \log x \frac{dx}{x+\beta} \\ &= \int_{(2\alpha-1)/\alpha}^{1-\alpha} \log x \cdot \frac{dx}{\beta+1-x} + \int_0^{(1-2\alpha)/\alpha} \log x \cdot \frac{dx}{2-x} \end{aligned}$$

$$+\int_0^{(2\alpha-1)/(1-\alpha)} \log x \cdot \frac{dx}{x+2} + \int_{(2\alpha-1)/(1-\alpha)}^{\alpha} \log x \cdot \frac{dx}{x+\beta}$$

and

$$\begin{aligned}\frac{dF}{d\alpha} &= -\frac{1}{\beta+\alpha}\cdot\log(1-\alpha) - \frac{1}{\alpha^2}\frac{1}{\beta-1+1/\alpha}\cdot\log\frac{2\alpha-1}{\alpha} \\ &\quad + \frac{1}{\alpha}\cdot\log\frac{2\alpha-1}{\alpha} + \frac{1}{1-\alpha}\cdot\log\frac{2\alpha-1}{1-\alpha} + \frac{1}{\alpha+\beta}\cdot\log\alpha \\ &\quad - \frac{1}{(1-\alpha)^2}\frac{1}{\beta+(2\alpha-1)/(1-\alpha)}\cdot\log\frac{2\alpha-1}{1-\alpha} \\ &= \left[-\frac{1}{\alpha}+\frac{1}{\alpha+\beta}-\frac{1}{\alpha\cdot(\alpha\beta-\alpha+1)}\right]\cdot\log\alpha \\ &\quad + \left[-\frac{1}{\beta+\alpha}-\frac{1}{1-\alpha}+\frac{1}{(1-\alpha)(\beta-1+2\alpha-\alpha\beta)}\right]\cdot\log(1-\alpha) \\ &\quad + \left[-\frac{1}{\alpha\cdot(\alpha\beta-\alpha+1)}+\frac{1}{\alpha}+\frac{1}{1-\alpha}-\frac{1}{(1-\alpha)(\beta-1+2\alpha-\alpha\beta)}\right] \\ &\quad \times\log(2\alpha-1) \\ &= 0\,.\end{aligned}$$

For $(\sqrt{5}-1)/2<\alpha<1$, it is also straight forward to show $dF/d\alpha=0$. Thus $F(\alpha)$ is a constant function of $[1/2, 1]$ and we get $F(\alpha)=-\pi^2/12$ since

$$\int_0^1 \log x\cdot\frac{dx}{1+x} = -\frac{\pi^2}{12}\,.$$

PROPOSITION 2. *For each* $\alpha\in[1/2, 1]$,

$$\lim_{n\to\infty}\frac{1}{n}\log q_n(x;\alpha) = C_\alpha\cdot\frac{\pi^2}{12} \qquad (\text{a.a. } x)\,. \tag{24}$$

PROOF. Since $\varepsilon(x)\cdot p_{j+1}(x;\alpha)=q_j(f_\alpha(x);\alpha)$, we have

$$\frac{\varepsilon_1(x)\cdot\varepsilon_2(x)\cdots\varepsilon_n(x)}{q_n(x;\alpha)} = \prod_{k=1}^{n}\frac{p_{n+1-k}(f_\alpha^{k-1}(x);\alpha)}{q_{n+1-k}(f_\alpha^{k-1}(x);\alpha)}$$

and

$$\frac{1}{q_n(x;\alpha)} = \varepsilon_1(x)\cdot\varepsilon_2(x)\cdots\varepsilon_n(x)\cdot\prod_{k=1}^{n}\left(\frac{\varepsilon_k(x)}{||a_{\alpha,k}(x)|}\Big| + \frac{\varepsilon_{k+1}(x)}{||a_{\alpha,k+1}(x)|}\Big| + \cdots + \frac{\varepsilon_n(x)}{||a_{\alpha,n}(x)|}\Big|\right)\,. \tag{25}$$

By Lemma 3

$$\left|\log|f_\alpha^{k-1}(x)|-\log\left|\frac{\varepsilon_k(x)}{|a_{\alpha,k}(x)|}+\cdots+\frac{\varepsilon_n(x)}{|a_{\alpha,n}(x)|}\right|\right|\leqq\delta_2\cdot\frac{1}{D^{n+1-k}}\,. \tag{26}$$

From (25) and (26) we have

$$\begin{aligned}&\sum_{k=1}^{n}\log|f_\alpha^{k-1}(x)|-\sum_{k=1}^{n}\delta_2\cdot\frac{1}{D^{n+1-k}}\\&\quad\leqq\log\frac{1}{q_n(x;\alpha)}\leqq\sum_{k=1}^{n}\log|f_\alpha^{k-1}(x)|+\sum_{k=1}^{n}\delta_2\cdot\frac{1}{D^{n+1-k}}\end{aligned}$$

and

$$\begin{aligned}&-\frac{1}{n}\sum_{k=1}^{n}\log|f_\alpha^{k-1}(x)|-\frac{1}{n}\sum_{k=1}^{n}\delta_2\cdot\frac{1}{D^{n+1-k}}\\&\quad\leqq\frac{1}{n}\log q_n(x;\alpha)\\&\quad\leqq-\frac{1}{n}\sum_{k=1}^{n}\log|f_\alpha^{k-1}(x)|+\frac{1}{n}\sum_{k=1}^{n}\delta_2\cdot\frac{1}{D^{n+1-k}}\,.\end{aligned} \tag{27}$$

Furthermore from the ergodicity of f_α, Lemma 7 and the ergodic theorem, we get

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(-\log|f_\alpha^{k-1}(x)|)&=-C_\alpha\cdot\int_{I_\alpha}\log|x|\cdot h_\alpha(x)dx\quad\text{(a.a. }x)\\&=C_\alpha\cdot\frac{\pi^2}{12}\,.\end{aligned}$$

and thus (27) implies (24).

PROPOSITION 3. *For each* $\alpha\in[1/2,1]$, *we have*

$$\lim_{n\to\infty}\frac{1}{n}\log\left|x-\frac{p_n(x;\alpha)}{q_n(x;\alpha)}\right|=-C_\alpha\cdot\frac{\pi^2}{6}\qquad\text{(a.a. }x)\,.$$

PROOF. This follows from (6) and Proposition 2.

THEOREM 3. *For each* $\alpha\in[1/2,1]$, we have

$$\begin{aligned}&\lim_{n=\infty}\frac{1}{n}\log\nu_\alpha(\langle a_{\alpha,1}(x),a_{\alpha,2}(x),\cdots,a_{\alpha,n}(x)\rangle_\alpha)\\&\quad=\lim_{n\to\infty}\frac{1}{n}\log m(\langle a_{\alpha,1}(x),a_{\alpha,2}(x),\cdots,a_{\alpha,n}(x)\rangle_\alpha)\end{aligned}$$

$$=-C_\alpha\cdot\frac{\pi^2}{6}\qquad (\text{a.a. } x)$$

Thus the entropy of (f_α, ν_α) (*or* (T_α, μ_α)) *is* $C_\alpha\cdot\pi^2/6$.

PROOF. From (9), (19), Lemma 6 and Proposition 2, there exists a sequence $\{n_i\}$ depending on x such that

$$\lim_{i\to\infty}\frac{1}{n_i}\log\nu_\alpha(\langle a_{\alpha,1}(x), a_{\alpha,2}(x), \cdots, a_{\alpha,n_i}(x)\rangle_\alpha)$$
$$=-C_\alpha\cdot\frac{\pi^2}{6}$$

for a.a. x. On the other hand

$$\lim_{n\to\infty}\frac{1}{n}\log\nu_\alpha(\langle a_{\alpha,1}(x), a_{\alpha,2}(x), \cdots, a_{\alpha,n}(x)\rangle_\alpha)$$

exists for a.a. x by the Shannon-McMillan-Breiman's Theorem.

§4. Asymptotic behavior of orbits.

In §3 we have dealt with the metrical properties of f_α for each α. Now we shall discuss the orbits of $\{f_\alpha\}$ for a fixed point x and show that "a.a." is independent of α.

LEMMA 7. *For any* $\alpha\in((\sqrt{5}-1)/2, 1]$ *let us consider* α' *such that* $\alpha>\alpha'>1/(1+\alpha)$ (or $\alpha'=1/(1+\alpha)$) *and fix* $x\in[\alpha', \alpha)$, (*or* $x\in(\alpha', \alpha)$ *respectively*), *then we have*

$$f_{\alpha'}(x-1)=f_\alpha^2(x)\qquad (\text{mod. } 1)\ .$$

PROOF. The assumptions imply

$$a_{\alpha,1}(x)=1\ .$$

So we have

$$f_\alpha(x)=\frac{1-x}{x}>0$$

and

$$\left|\frac{1}{x-1}\right|-\left|\frac{1}{f_\alpha(x)}\right|=\frac{1}{1-x}-\frac{x}{1-x}=1\ .$$

It follows from the definition of f_α that

$$f_{\alpha'}(x-1)=f_\alpha^2(x) \qquad (\text{mod. } 1)\ .$$

Let us consider an ergodic invariant probability measure λ of (I_α, f_α). We put

$$N_{\alpha,\lambda}=\left\{x\in I_\alpha;\ \lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(x))=\lambda((a,b))\right.$$
$$\left.\text{for any open interval } (a,b)\subset I_\alpha\right\}$$

where $\chi_{(a,b)}$ is the indicator function of (a, b), then it follows from the ergodic theorem and separability of I_α that

$$\lambda(N_{\alpha,\lambda})=1\ .$$

THEOREM 4. *For any ergodic invariant probability measure* λ_1 *of* (I_1, f_1) *and for any* $\alpha\in[1/2, 1)$, *there exists an ergodic invariant probability measure* λ_α *such that* $x\in N_{1,\lambda_1}$ *if and only if* $\hat{x}\in N_{\alpha,\lambda_\alpha}$ *where* $\hat{x}=x$ (mod. 1). *And the converse is also true.*

PROOF. We assume that λ_1 is non-atomic, otherwise there exists a unique periodic orbit in N_{1,λ_1} and the following discussion is practically clear in such a case. We fix $x\in N_{1,\lambda_1}$, consider $\hat{x}=x$ (mod. 1), $\hat{x}\in I_\alpha$ and define

$$(28)\qquad \begin{cases} i_1=\min\{i;\ f_1^i(x)\neq f_\alpha^i(\hat{x}),\ i\geqq 0\} \\ i_n=\min\{i;\ i>i_{n-1},\ f_1^{i+n-1}(x)\neq f_\alpha^i(\hat{x})\} \qquad \text{for} \quad n\geqq 2\ , \end{cases}$$

here it could happen that $i_n=\infty$ for some $n\geqq 1$, however the following proof is easy in such cases so we assume $i_n<\infty$ for all $n\geqq 1$. If $i_n\leqq k<i_{n+1}$, then we have

$$(29)\qquad f_1^{k+n}(x)=f_\alpha^k(\hat{x})$$

by Lemma 7.

(i) $(\sqrt{5}-1)/2\leqq\alpha<1$. Let us consider an open interval (a, b), $0<a<b<1/\alpha-1$, and $i_n<m\leqq i_{n+1}$, then we have by using (29)

$$\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(\hat{x}))=\frac{(m+n)}{m}\cdot\frac{(M_1-M_2)}{m+n}$$

$$\text{where}\begin{cases} M_1=\#\{i;\ f_1^i(x)\in(a,b),\ 0\leqq i<m+n\} \\ M_2=\#\left\{i;\ f_1^i(x)\in\left(\dfrac{1}{1+b}, \dfrac{1}{1+a}\right),\ 0\leqq i<m+n\right\} \end{cases}$$

and for a set A, $\#A$ denotes the number of elements belonging to A. If m tends to ∞, then $m/(m+n)$ and $(M_1-M_2)/(m+n)$ converge to $\lambda_1([0, \alpha))$ and $\lambda_1((a, b))-\lambda_1((1/(1+b), 1/(1+a)))$ respectively. Thus we get

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(\hat{x}))=\frac{1}{\lambda_1((0,\alpha))}\left[\lambda_1((a,b))-\lambda_1\left(\left(\frac{1}{1+b},\frac{1}{1+a}\right)\right)\right] \tag{30}$$

By the same argument we have

$$\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(\hat{x}))=\frac{\lambda_1((a,b))}{\lambda_1((0,\alpha))} \tag{31}$$

for $(a, b)\subset(1/\alpha-1, \alpha)$ or $(a, b)\subset(\alpha-1, 0)$. From (30) and (31) we can define

$$\lambda_\alpha((a,b))=\lim_{m\to\infty}\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(\hat{x}))$$

for any open interval $(a, b)\subset I_\alpha$ and thus λ_α is extendable to a measure of I_α. It is clear from the construction of λ_α that λ_α is independent of choice of x and is an ergodic invariant probability measure of (I_α, f_α). Of course $\hat{x}$ belongs to $N_{\alpha,\lambda_\alpha}$. Moreover for a fixed $\hat{x}\in N_{\alpha,\lambda_\alpha}$ the reverse of the above discussion shows $x\in N_{1,\lambda_1}$.

(ii) $1/2\leqq\alpha<(\sqrt{5}-1)/2$. We put

$$\begin{cases}\omega_{-1}=\dfrac{1-\alpha}{\alpha}\\ \omega_0=\alpha\\ \omega_i=\dfrac{1}{1+\omega_{i-1}}, & i\geqq 1,\end{cases}$$

then $\lim_{i\to\infty}\omega_i=(\sqrt{5}-1)/2$. Moreover since $\alpha<(\sqrt{5}-1)/2$,

$$f_1(\omega_{-1})=\frac{\alpha}{1-\alpha}-1=\frac{2\alpha-1}{1-\alpha}<\alpha .$$

Suppose $(a, b)\subset(0, f_1(\omega_{-1}))$, then for $i_n<m\leqq i_{n+1}$, we have

$$\frac{1}{m}\sum_{i=0}^{m-1}\chi_{(a,b)}(f_\alpha^i(\hat{x}))=\frac{m+n}{m}\cdot\frac{M_1-M_2}{m+n} \tag{32}$$

and $(M_1-M_2)/(m+n)$ converges to $\lambda_1((a, b))-\lambda_1((1/(1+b), 1/(1+a)))$ because $f_1^{k+2}(x)\in(a, b)$ and $f_1^{k+1}(x)\in(1/(1+b), 1/(1+a))$ imply $f_1^k(x)\notin[\alpha, 1)$. Furthermore $n/(m+n)$ converges to

(33) $\lambda_\alpha^* = \lambda_1([\omega_0, 1)) - \lambda_1((\omega_0, \omega_1]) + \lambda_1([\omega_2, \omega_1)) - \lambda_1((\omega_2, \omega_3]) + \cdots$.

Since $(\omega_0, \omega_1] \cap [\omega_2, \omega_1) \cap \cdots = \{(\sqrt{5}-1)/2\}$ and λ_1 is non-atomic, the existence of the limit in (33) is ensured. Thus $\lim_{m\to\infty}(m+n)/m = 1/(1-\lambda_\alpha^*)$ exists and so (32) converges as m tends to ∞.

If $(a, b) \subset (f_1(\omega_{-1}), \alpha)$, then we put

$$\begin{cases} a_1 = \dfrac{1}{1+a}, & b_1 = \dfrac{1}{1+b}, \\ a_n = \dfrac{1}{1+a_{n-1}}, & b_n = \dfrac{1}{1+b_{n-1}}, \quad n \geqq 2 \end{cases}$$

and have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \chi_{(a,b)}(f_\alpha^i(\hat{x})) = \frac{1}{1-\lambda_\alpha^*}(\lambda_1((a, b)) - \lambda_1((b_1, a_1)) + \lambda_1((a_2, b_2)) - \cdots)$$

in the same way. It is also possible to calculate

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} \chi_{(a,b)}(f_\alpha^i(\hat{x}))$$

for $(a, b) \subset (\alpha-1, \omega_{-1}-1)$ and $(a, b) \subset (\omega_{-1}-1, 0)$. Consequently we can construct λ_α by the same argument.

REMARK. If $\lambda_1 = \nu_1$, then $\lambda_\alpha = \nu_\alpha$ and λ_α^* of (33) equals

$$\frac{(\log 2 - \log((\sqrt{5}-1)/2))}{\log 2} .$$

COROLLARY 3. *For any* $x \in N_{1,\nu_1}$, *let* $\hat{x} = x$ (mod. 1), $\hat{x} \in I_\alpha$, $1/2 \leqq \alpha \leqq 1$, *then*

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} g(f_\alpha^i(\hat{x})) = \int_{I_\alpha} g\,d\nu_\alpha$$

for all bounded continuous functions g.

PROOF. It follows from theorem 4.

Since "log" is not bounded on I_α, it is not possible to apply corollary 3 to the results of §3. In the sequel we shall treat this problem.

We fix $\alpha' < \alpha$ and $x \in I_\alpha$, and define i_n in the same way as in the proof of Theorem 4.

$$\begin{cases} i_0=-1 \\ i_1=\min\{i;\ f_\alpha^i(x)\neq f_{\alpha'}^i(x'),\ i\geqq 0\} \\ i_n=\min\{i;\ i>i_{n-1},\ f_\alpha^{i+n-1}(x)\neq f_{\alpha'}^i(x')\}\ , \qquad n\geqq 2 \end{cases}$$

where $x'=x$ (mod. 1), $x'\in I_{\alpha'}$ and we also assume $i_n<\infty$ for all $n\geqq 1$. Through i_n depends on α, α' and x, we do not bother mentioning this dependence in the following discussions.

LEMMA 8. *We fix* $\alpha\in[1/2, 1]$ *and irrational number* $x\in(0, 1)$, *then*

(34) $$q_n(\hat{x};\alpha)=q_{n+j}(x;1) \qquad \text{for} \quad i_j\leqq n<i_{j+1}\ , \qquad j\geqq 0$$

where $\hat{x}=x$ (mod. 1), $\hat{x}\in I_\alpha$.

LEMMA 8′. *We fix* $\alpha\in((\sqrt{5}-1)/2, 1]$, $\alpha'\in(1/(1+\alpha), \alpha)$ *and irrational number* $x\in I_\alpha$, *then*

$$q_n(\hat{x};\alpha')=q_{n+j}(x;\alpha) \qquad \text{for} \quad i_j\leqq n<i_{j+1},\ j\geqq 0$$

where $\hat{x}=x$ (mod. 1), $\hat{x}\in I_{\alpha'}$.

PROOF. The proof of Lemma 8 is same as that of Lemma 8′, so we only prove Lemma 8′.

If y belongs to $[\alpha', \alpha)$ then $\alpha>\alpha'>1/(1+\alpha)$ implies $a_{\alpha,1}(y)=1$. If $-1\leqq n<i_1$, then it is easy to see that

$$q_n(x';\alpha')=q_n(x;\alpha)$$

Since $f_\alpha^{i_1}(x)\neq f_{\alpha'}^{i_1}(x')$, we have

$$\begin{cases} |a_{\alpha',i_1}(x')|-|a_{\alpha,i_1}(x)|=1 \\ f_{\alpha'}^{i_1-1}(x')=f_\alpha^{i_1-1}(x)\ , \qquad \varepsilon(f_\alpha^{i_1-1}(x))=\varepsilon(f_{\alpha'}^{i_1-1}(x')) \\ \qquad\qquad\qquad\qquad\qquad \text{in the case of } i_1\neq 0 \end{cases}$$

and

(35) $$\begin{cases} f_{\alpha'}^{i_1}(x')=f_\alpha^{i_1}(x)-1\ , \\ f_\alpha^{i_1}(x)\in[\alpha', \alpha)\ . \end{cases}$$

Thus if we put $|a_{\alpha',i_1}(x')|=k$ then we get by (35) that

$$\begin{cases} q_{i_1}(x';\alpha')=k\cdot q_{i_1-1}(x';\alpha')+\varepsilon(f_{\alpha'}^{i_1-1}(x'))\cdot q_{i_1-2}(x';\alpha') \\ q_{i_1}(x;\alpha)=(k-1)\cdot q_{i_1-1}(x;\alpha)+\varepsilon(f_\alpha^{i_1-1}(x))\cdot q_{i_1-2}(x;\alpha)\ . \end{cases}$$

Moreover it follows from (35) that

$$a_{\alpha,i_1+1}(x)=1$$

and so

$$q_{i_1+1}(x;\alpha)=k\cdot q_{i_1-1}(x;\alpha)+\varepsilon(f_\alpha^{i_1-1}(x))\cdot q_{i_1-2}(x;\alpha)\ .$$

Consequently we have

$$q_{i_1}(x';\alpha')=q_{i_1+1}(x;\alpha) \tag{36}$$

$$q_{i_1}(x';\alpha')-q_{i_1}(x;\alpha)=q_{i_1-1}(x;\alpha)=q_{i_1-1}(x';\alpha') \qquad (\text{if}\quad i_1\neq 0)\ . \tag{37}$$

On the other hand, in the case of $i_1=0$, it follows that

$$a_{\alpha,1}(x)=1$$

and we also get (36) and (37).

Next we assume $i_1+1<i_2$. In this case we have

$$a_{\alpha',i_1+1}(x')=-(a_{\alpha,i_1+2}(x)+1)<0$$

and thus

$$\begin{aligned} q_{i_1+1}(x';\alpha')&=(a_{\alpha,i_1+2}(x)+1)\cdot q_{i_1}(x';\alpha')-q_{i_1-1}(x';\alpha')\\ &=a_{\alpha,i_1+2}(x)\cdot q_{i_1+1}(x;\alpha)+q_{i_1}(x';\alpha')-q_{i_1-1}(x';\alpha')\\ &=a_{\alpha,i_1+1}(x)\cdot q_{i_1+1}(x;\alpha)+q_{i_1}(x;\alpha)\\ &=q_{i_1+2}(x;\alpha)\end{aligned} \tag{38}$$

by virtue of (36) and (37). For n, $i_1+2\leqq n<i_2$, it is clear that $a_{\alpha',n}(x')=a_{\alpha,n+1}(x)$ and $q_n(x';\alpha')=q_{n+1}(x;\alpha)$.

Now we assume $i_1+1=i_2$, then

$$a_{\alpha',i_1+1}(x')=-(a_{\alpha,i_1+2}(x)+2)<0 \quad\text{and}\quad a_{\alpha,i_1+3}(x)=1\ .$$

Hence from (36) and (37),

$$\begin{aligned} q_{i_2}(x',\alpha')&=q_{i_1+1}(x';\alpha')\\ &=(a_{\alpha,i_1+2}(x)+2)\cdot q_{i_1}(x';\alpha')-q_{i_1-1}(x';\alpha')\\ &=(a_{\alpha,i_1+2}(x)+1)\cdot q_{i_1+1}(x;\alpha)+q_{i_1}(x;\alpha)\\ &=q_{i_1+3}(x;\alpha)\\ &=q_{i_2+2}(x;\alpha)\end{aligned}$$

and

$$q_{i_2}(x';\alpha')-q_{i_2+1}(x;\alpha)=q_{i_2}(x;\alpha)=q_{i_2-1}(x';\alpha')\ .$$

It follows inductively that

$$\begin{cases} q_{i_n}(x'; \alpha') = q_{i_n+n}(x; \alpha) \\ q_{i_n}(x'; \alpha') - q_{i_n+n-1}(x; \alpha) = q_{i_n-1}(x'; \alpha') = q_{i_n+n-2}(x; \alpha) \end{cases}$$

and as above it is possible to complete the proof of the assertion of this lemma.

THEOREM 5. *There exists* $N_0 \subset N_{1,\nu_1}$ *such that* $m(N_0)=1$ *and for any* $x \in N_0$ *and any* $\alpha \in [1/2, 1]$

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(\hat{x}; \alpha) = C_\alpha \cdot \frac{\pi^2}{12}$$

where

$$\hat{x} = x (\text{mod. } 1)\,, \qquad \hat{x} \in I_\alpha\,.$$

PROOF. We put

$$N_0 = \left\{x;\ \lim_{n\to\infty} \frac{1}{n} \log q_n(x; 1) = \frac{1}{\log 2} \cdot \frac{\pi^2}{12}\right\} \cap N_{1,\nu_1}\,.$$

From Proposition 2, it is clear that $m(N_0)=1$. We fix $x \in N_0$ and consider $\hat{x}=x$ (mod. 1), $\hat{x} \in I_\alpha$. By Lemma 8

$$\frac{1}{n} \log q_n(\hat{x}; \alpha) = \frac{(n+j)}{n} \cdot \frac{1}{(n+j)} \log q_{n+j}(x; 1)$$

for $i_j \leqq n < i_{j+1}$. Suppose $(\sqrt{5}-1)/2 \leqq \alpha < 1$, then

$$\lim_{n\to\infty} \frac{n}{n+j} = \frac{1}{\log 2} \cdot \int_0^\alpha \frac{1}{1+x} dx = \frac{1}{\log 2} \cdot \log(1+\alpha)\,.$$

Hence

$$\lim \frac{1}{n} \log q_n(\hat{x}; \alpha) = \frac{1}{\log(1+\alpha)} \cdot \frac{\pi^2}{12}\,.$$

On the other hand if $1/2 \leqq \alpha < (\sqrt{5}-1)/2$, then $j/(n+j)$ converges to $\nu_\alpha^* = (\log 2 - \log((\sqrt{5}+1)/2))/\log 2$ as n tends to ∞ and we get

$$\lim_{n\to\infty} \frac{1}{n} \log q_n(\hat{x}; \alpha) = \frac{1}{\log((\sqrt{5}+1)/2)} \cdot \frac{\pi^2}{12}\,.$$

Finally we consider the length $K=K(\alpha; x)$ of rational number x with respect to f_α. From Lemma 7,

$$K(\alpha'; x') \leqq K(\alpha; x) \qquad \text{for } \frac{\sqrt{5}-1}{2} \leqq \alpha' < \alpha \leqq 1$$

where $x = x'$ (mod. 1), $x \in I_\alpha$, $x' \in I_{\alpha'}$ and x is rational. Now we will show $K(\alpha'; x') = K(\alpha; x)$ for $1/2 \leqq \alpha' < \alpha \leqq (\sqrt{5}-1)/2$.

LEMMA 9. *For $\alpha \in [1/2, (\sqrt{5}-1)/2)$, $\alpha' \in (\alpha, (1+\alpha)/(2+\alpha)]$ and $x \in [\alpha, \alpha')$, we have*

$$f_\alpha^2(x-1) = f_{\alpha'}^2(x) \qquad (\text{mod. } 1)\ .$$

PROOF. The condition $\alpha' \leqq (1+\alpha)/(2+\alpha)$ implies

$$a_{\alpha,1}(x-1) = -2 \quad \text{and} \quad a_{\alpha',1}(x) = 2\ ,$$

so

$$(39) \qquad \begin{cases} f_{\alpha'}(x) = \dfrac{1}{x} - 2 = \dfrac{1-2x}{x} < 0 \\ f_\alpha(x-1) = \dfrac{1}{1-x} - 2 = \dfrac{2x-1}{1-x} > 0\ . \end{cases}$$

It follows from (39) that $f_\alpha^2(x-1) = f_{\alpha'}^2(x)$ (mod. 1).

From Lemma 9 (and (39)), it is easy to see

$$(40) \qquad K(\alpha'; x') = K(\alpha; x)$$

for such α, α' and rational numbers x, x' with $x = x'$ (mod. 1), $x \in I_\alpha$, $x' \in I_{\alpha'}$. Moreover for any α and α' with $1/2 \leqq \alpha < \alpha' \leqq (\sqrt{5}-1)/2$, there exists a finite sequence $\alpha = \alpha_1 < \alpha_2 < \cdots < \alpha_n = \alpha'$ such that

$$\alpha_{i+1} \leqq \frac{1+\alpha_i}{2+\alpha_i}$$

since $\alpha < (1+\alpha)/(2+\alpha) < (\sqrt{5}-1)/2$. Thus (40) is true for any α and α' belonging $[1/2, (\sqrt{5}-1)/2)$. For a rational number $y \in I_\eta$, $\eta = (\sqrt{5}-1)/2$, we put

$$z = \max\{y, f_\eta(y), f_\eta^2(y), \cdots, f_\eta^{K(\eta,y)}(y)\}$$

and fix $\alpha > z$, then

$$f_\alpha^i(y) = f_\eta^i(y) \qquad \text{for } i = 1, 2, \cdots, K(\eta, y)\ .$$

Hence $K(\eta, y) = K(\alpha, y)$ and from above we have the following.

THEOREM 6. *For any rational number $x \in [0, 1)$,*

$$K(\alpha'; x') \leqq K(\alpha''; x'')$$

where $x' \in I_{\alpha'}$, $x'' \in I_{\alpha''}$, $x' = x'' = x$ (mod. 1) *and* $1/2 \leqq \alpha' < \alpha'' \leqq 1$, *in particular*

$$K(\alpha'; x') = K(\alpha''; x'')$$

for $1/2 \leqq \alpha' < \alpha'' \leqq (\sqrt{5} - 1)/2$.

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