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On an Average of $\omega(n)$ with Respect to Some Sets of Composite Integers

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Throughout this paper we shall use the following notations:
$\boldsymbol{N}$: the set of all positive integers,
$\boldsymbol{P}$: the set of all rational prime numbers,
$\boldsymbol{N}(x)=\{n\in\boldsymbol{N};\ n\leqq x\}$ (for x: real),
$S(x)$: a subset of $\boldsymbol{N}(x)$,
$\#(S(x))$: the cardinal number of $S(x)$,
$\omega(n)$: the number of distinct prime factors of n,
$\Omega(n)$: the total number of prime factors of n,
$\|n\|$: $\min_{p\in\boldsymbol{P}}(|n-p|)$, i.e., the distance from n to its nearest prime.
The letters p, q will always denote prime numbers. We shall write $\log_2 x=\log\log x$ and $\log_3 x=\log\log\log x$, and use $\pi(x)$, $\pi(x;k,l)$ and $\mathrm{Li}(x)$ in the usual sense.

§1. Statement of results.

Since the value of $\omega(n)$ or that of $\{\Omega(n)-\omega(n)\}$ fluctuates irregularly, we shall observe

$$V(S(x))=\frac{\sum_{n\in S(x)}\omega(n)}{\#(S(x))}\,,\qquad W(S(x))=\frac{\sum_{n\in S(x)}\{\Omega(n)-\omega(n)\}}{\#(S(x))}\,,$$

each of which can be regarded as an average of $\omega(n)$ or that of $\{\Omega(n)-\omega(n)\}$ for a given subset $S(x)$. For $S(x)=\boldsymbol{N}(x)$, the value of $V(\boldsymbol{N}(x))$ or that of $W(\boldsymbol{N}(x))$ is, so to speak, "standard" average of $\omega(n)$ or that of $\{\Omega(n)-\omega(n)\}$. As is well known ([1: THEOREM 430]):

$$V(\boldsymbol{N}(x))=\log_2 x+A+O\left(\frac{1}{\log x}\right)\,, \tag{1.1}$$

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$$W(N(x))=\sum_p \frac{1}{p(p-1)}+O(x^{-1/2}) , \tag{1.2}$$

where $A=\gamma+\sum_p \{\log(1-1/p)+1/p\}$, and γ is Euler's constant. On the other hand, a few results are known as to the value of $V(S(x))$ or that of $W(S(x))$ for specially chosen set $S(x)$. For example, H. Halberstam ([2]) proved that if $g(X)$ is an irreducible polynomial with integral coefficients and $S^*(x)=\{g(p);\, p\in \boldsymbol{P},\, g(p)\leqq x\}$, then

$$V(S^*(x))\sim \log_2 x .$$

However, we can not decide whether $V(S^*(x))$ is larger than $V(N(x))$ or not, because no estimate is obtained for error terms for this $S^*(x)$.

In this paper we shall consider a positive valued non-decreasing function $f(x)$ majorized by $C(\log x)^{1-\varepsilon}$ with a constant C and a positive $\varepsilon\leqq 1$, *and the subset* $M_f(x)$ of $N(x)$, *whose elements* n *are composite numbers satisfying* $1\leqq \|n\|\leqq f(x)$, i.e., $n\notin \boldsymbol{P}$ and $n\in[p-f(x), p+f(x)]$, where p is the nearest prime to n. For this $M_f(x)$, we shall prove

THEOREM 1. *For the set $M_f(x)$ defined above, we have*

$$V(M_f(x))=\log_2 x+\left\{A+\sum_p \frac{1}{p(p-1)}-\log 2+\alpha_f(x)\right\} \tag{1.3}$$
$$+O((\log x)^{-\varepsilon}(\log_2 x)(\log_3 x)) ,$$

where $\alpha_f(x)$ is a function satisfying

$$\frac{1}{2}\leqq \alpha_f(x)\leqq 1 ,$$

and the constant implied by O-symbol depends at most on ε and C.

We obtain from this theorem and (1.1),

$$V(M_f(x))-V(N(x))\geqq \sum_p \frac{1}{p(p-1)}-\log 2+\frac{1}{2}+o(1) .$$

On the other hand, numerical calculation gives

$$0.773141<\sum_{p\leqq 10^4}\frac{1}{p(p-1)}<0.773149$$

and consequently, for sufficiently large x,

$$V(M_f(x))-V(N(x))>0.5799 .$$

Concerning the function $W(S(x))$, we obtain

THEOREM 2. *We have*

$$W(M_f(x))=\sum_p \frac{1}{(p-1)^2}+O((\log x)^{-\varepsilon}(\log_3 x)) ,$$

where the constant implied by O-symbol depends only on ε *and C.*

This result shows that

$$W(M_f(x))-W(N(x))\sim\sum_p \frac{1}{p(p-1)^2}>0.6019 .$$

The following two theorems concern special cases of above theorems where $f(x)$ is constant.

THEOREM 3. *Let* $N_d(x)=\{n\in N;\ 1\leqq\|n\|\leqq d,\ n\leqq x\}$ $(d>1)$, *then*

$$V(N_d(x))=\log_2 x+\left\{A+\sum_p \frac{1}{p(p-1)}-\log 2+\beta_d(x)\right\} \\ +O((\log x)^{-1}(\log_2 x)) , \tag{1.4}$$

where $\beta_d(x)$ *is a function satisfying*

$$\frac{1}{2}\leqq\beta_d(x)\leqq 1 ,$$

and the constant implied by O-symbol depends only on d.

THEOREM 4. *For the same* $N_d(x)$,

$$W(N_d(x))=\sum_p \frac{1}{(p-1)^2}+O((\log x)^{-1}(\log_2 x)) , \tag{1.5}$$

where the constant implied by O-symbol depends only on d.

Thus we can say that, if we restrict the domain of average to those composite integers in d-neighborhoods of primes, the corresponding average of $\omega(n)$ and that of $\{\Omega(n)-\omega(n)\}$ will be *definitely larger* than the "standard" averages given in (1.1) and (1.2) respectively (see also [6]).

I am grateful to Professor M. Tanaka for his kind advices.

§2. Some lemmas.

For an integer i, we put $P_i(x)=\{n;\, n=p+i,\ n\leqq x$ *and* $p\in \boldsymbol{P}\}$, i.e., a sequence of shifted primes.

LEMMA 1. *Suppose* $|i|\leqq\log x$, *then*

$$\sum_{n\in P_i(x)}\omega(n)=\left\{\log_2 x+A+\sum_p\frac{1}{p(p-1)}-\log 2+\delta_i(x)\right\}\cdot\frac{x}{\log x}, \tag{2.1}$$

where the function $\delta_i(x)$ *satisfies*

$$\frac{1}{2}+O\left(\frac{\log_2 x}{\log x}\right)\leqq\delta_i(x)\leqq 1+O\left(\frac{\log_2 x}{\log x}\right), \tag{2.2}$$

and the constants implied by O-symbols are absolute.

PROOF.

$$\sum_{n\in P_i(X)}\omega(n)=\sum_{p\leqq x-i}\sum_{\substack{q\leqq x\\ q|p+i}}1=\sum_{q\leqq x-i}\pi(x-i;\,q,\,-i)+O(i^2)\,.$$

Here we write $y=x-i$ and we divide the right-hand sum:

$$\sum_{q\leqq y}\pi(y;\,q,\,-i)=\sum_{q\leqq\sqrt{y}}\pi(y;\,q,\,-i)+\sum_{\sqrt{y}<q\leqq y}\pi(y;\,q,\,-i)\,.$$

We shall now evaluate $S_1=\sum_{q\leqq\sqrt{y}}\pi(y;\,q,\,-i)$ and $S_2=\sum_{\sqrt{y}<q\leqq y}\pi(y;\,q,\,-i)$. Bombieri's theorem ([3]) shows that

$$\sum_{q\leqq\sqrt{y}\,l^{-B}}\pi(y;\,q,\,-i)=\left\{\sum_{q\leqq\sqrt{y}\,l^{-B}}\frac{1}{\varphi(q)}\right\}\mathrm{Li}\,(y)+O\left(\frac{y}{\log^2 y}\right),$$

where $l=\log y$ and B is some suitably chosen positive number, and we have by Brun-Titchmarsh's theorem ([4: THEOREM 3.8])

$$\sum_{\sqrt{y}\,l^{-B}<q\leqq\sqrt{y}}\pi(y;\,q,\,-i)=O\left(\frac{y\log_2 y}{\log^2 y}\right).$$

Since $\sum_{\sqrt{y}\,l^{-B}<q\leqq\sqrt{y}}(1/\varphi(q))=O(\log_2 y/\log y)$, we have

$$\begin{aligned}S_1&=\left\{\sum_{q\leqq\sqrt{y}}\frac{1}{\varphi(q)}\right\}\frac{y}{\log y}+O\left(\frac{y\log_2 y}{\log^2 y}\right)\\&=\left\{\log_2 y+A+\sum_p\frac{1}{p(p-1)}-\log 2\right\}\frac{y}{\log y}+s_i(x)\,,\end{aligned}$$

where

$$s_i(x)=O\left(\frac{y\log_2 y}{\log^2 y}\right).$$

As to S_2, Goldfeld's result ([5]) shows that

$$\pi(y) \geqq S_2 \geqq \frac{1}{2} \frac{y}{\log y} + O\left(\frac{y \log_2 y}{\log^2 y}\right) ,$$

where the constant implied by O-symbol is absolute. Thus we have (2.1), if we put

$$\delta_i(x) = \{S_2 + s_i(x)\}/x(\log x)^{-1} .$$

$\delta_i(x)$ satisfies (2.2) since $|i| \leqq \log x$. q.e.d.

Let F be a positive number, let b_i $(1 \leqq i \leqq g)$ be integers satisfying $1 \leqq b_1 < b_2 < \cdots < b_g \leqq 2F$, and put

$$D_F = \max_{1 \leqq b \leqq 2F} \left(\prod_{p|b} \frac{p}{p-1}\right) ,$$

$$\boldsymbol{P}_{b_1,\dots,b_g}(x) = \{p;\ p \leqq x,\ p+b_i \in \boldsymbol{P},\ 1 \leqq i \leqq g\} ,$$

$$\boldsymbol{P}_{b_1,\dots,b_g}(x;\ k,\ l) = \{p;\ p \in \boldsymbol{P}_{b_1,\dots,b_g}(x),\ p \equiv l (\mathrm{mod}\ k)\} ,$$

where k and l are relatively prime integers.

LEMMA 2. *Let a be an integer such that $|a| \leqq \log x$, then*

I) $$\sum_{q \leqq x} \#(\boldsymbol{P}_{b_1,\dots,b_g}(x;\ q,\ a)) = (8D_F)^g \cdot O\left(\frac{x \log_2 x}{\log^{g+1} x}\right) .$$

II) $$\sum_{\substack{q^m \leqq x \\ m \geqq 2}} \#(\boldsymbol{P}_{b_1,\dots,b_g}(x;\ q^m,\ a)) = (8D_F)^g \cdot O\left(\frac{x}{\log^{g+1} x}\right) .$$

PROOF. We make use of the following two estimates, both of which are deduced from [4: THEOREM 2.4]:

(2.3) $$\#(\boldsymbol{P}_{b_1,\dots,b_g}(x)) = D_F{}^g \cdot O\left(\frac{x}{\log^{g+1} x}\right) ,$$

(2.4) $$\#(\boldsymbol{P}_{b_1,\dots,b_g}(x;\ k,\ l)) = D_F{}^g \left(\frac{k}{\varphi(k)}\right)^{g+1} \cdot O\left(\frac{x/k}{\log^{g+1}\left(\frac{x}{k}\right)}\right) .$$

We get from (2.3), by partial summation, that

(2.5) $$\sum_{p \in \boldsymbol{P}_{b_1,\dots,b_g}(x)} (\log p) = D_F{}^g \cdot O\left(\frac{x}{\log^g x}\right) .$$

For von Mangoldt's function $\Lambda(n)$ we have

$$\sum_{m \leqq x} \sum_{p \in \boldsymbol{P}_{b_1,\dots,b_g}(x;m,a)} \Lambda(m) = \sum_{p \in \boldsymbol{P}_{b_1,\dots,b_g}(x)} \sum_{\substack{m \leqq x \\ m | p-a}} \Lambda(m)$$

$$
\begin{aligned}
&= \sum_{p \in P_{b_1,\dots,b_g}(x)} \log(p-a) + O\Big(\sum_{x \leq q \leq x-a} \log q\Big) \\
&= \sum_{p \in P_{b_1,\dots,b_g}(x)} \Big\{\log p + O\Big(\frac{a}{p}\Big)\Big\} + a \cdot O(\log x) \\
&= \sum_{p \in P_{b_1,\dots,b_g}(x)} \log p + O(\log^2 x) .
\end{aligned}
$$

Then from (2.5)

$$
\sum_{m \leq x} \sum_{p \in P_{b_1,\dots,b_g}(x;m,a)} \Lambda(m) = D_F{}^g \cdot O\Big(\frac{x}{\log^g x}\Big) ,
$$

and especially, we obtain

$$
\sum_{q \leq x} \sum_{p \in P_{b_1,\dots,b_g}(x;q,a)} (\log q) = D_F{}^g \cdot O\Big(\frac{x}{\log^g x}\Big) . \tag{2.6}
$$

Now

$$
\begin{aligned}
\sum_{q \leq x} \#(P_{b_1,\dots,b_g}(x;\, q,\, a)) &= \{\sum_{q \leq x^{3/4}} + \sum_{x^{3/4} < q \leq x}\} \#(P_{b_1,\dots,b_g}(x;\, q,\, a)) \\
&= T_1 + T_2 .
\end{aligned}
$$

Then from (2.4), we get

$$
\begin{aligned}
T_1 &= \sum_{q \leq x^{3/4}} \#(P_{b_1,\dots,b_g}(x;\, q,\, a)) \\
&= D_F{}^g \cdot O\Big(\sum_{q \leq x^{3/4}} \frac{q^g}{(q-1)^{g+1}}\Big) \cdot O\Big(\frac{x}{\log^{g+1} x^{1/4}}\Big) = (8D_F)^g \cdot O\Big(\frac{x \log_2 x}{\log^{g+1} x}\Big) .
\end{aligned}
$$

And, concerning T_2, we obtain from (2.6)

$$
\begin{aligned}
T_2 &= \sum_{x^{3/4} < q \leq x} \#(P_{b_1,\dots,b_g}(x;\, q,\, a)) \\
&\leq \frac{4}{3} \sum_{q \leq x} \sum_{p \in P_{b_1,\dots,b_g}(x;q,a)} \frac{\log q}{\log x} = D_F{}^g \cdot O\Big(\frac{x}{\log^{g+1} x}\Big) .
\end{aligned}
$$

This concludes the proof of Lemma 2-I).

On the other hand,

$$
\begin{aligned}
&\sum_{\substack{q^m \leq x \\ m \geq 2}} \#(P_{b_1,\dots,b_g}(x;\, q^m,\, a)) \\
&\quad = \Big\{\sum_{\substack{q^m \leq x^{3/4} \\ m \geq 2}} + \sum_{\substack{x^{3/4} < q^m \leq x \\ m \geq 2}}\Big\} \#(P_{b_1,\dots,b_g}(x;\, q^m,\, a)) \\
&\quad = \tilde{T}_1 + \tilde{T}_2 .
\end{aligned}
$$

Then from (2.4), we get

$$\begin{aligned}\tilde{T}_1 &= \sum_{\substack{q^m \leqq x^{3/4} \\ m \geqq 2}} \#(P_{b_1,\cdots,b_g}(x;\, q^m,\, a)) \\ &= D_F{}^g \cdot O\Big(\sum_{\substack{q^m \leqq x^{3/4} \\ m \geqq 2}} \frac{q^{g+1}}{q^m(q-1)^{g+1}}\Big) \cdot O\Big(\frac{x}{\log^{g+1} x^{1/4}}\Big) \\ &= (8D_F)^g \cdot O\Big(\frac{x}{\log^{g+1} x}\Big) ,\end{aligned}$$

because

$$\sum_{q^m, m \geqq 2} \frac{q^{g+1}}{q^m(q-1)^{g+1}} = 2^g \cdot O(1) .$$

And

$$\begin{aligned}\tilde{T}_2 &= \sum_{\substack{x^{3/4} < q^m \leqq x \\ m \geqq 2}} \#(P_{b_1,\cdots,b_g}(x;\, q^m,\, a)) \\ &\leqq \sum_{m=2}^{\log x} \pi(x^{1/2}) \cdot \frac{x}{x^{3/4}} = O(x^{3/4}) .\end{aligned}$$

Lemma 2–II) follows immediately from these formulas.

LEMMA 3.

$$D_F = O(\log_2 F) , \qquad as \quad F \longrightarrow \infty .$$

PROOF. If we put $P_z = \prod_{p \leqq z} p$, it is sufficient to prove $\prod_{p \leqq z} (p/(p-1)) = O(\log_2 P_z)$. Mertens's theorem shows that $\prod_{p \leqq z} (p/(p-1)) = O(\log z)$, and as is well known, $\log_2 P_z = \log \{\sum_{p \leqq z} \log p\} = O(\log z)$. This concludes the proof of Lemma 3.

§3. Proofs of the theorems.

Now we start proving our theorems. We put $F = [f(x)]$, then, from the assumption on $f(x)$, we have $F \leqq C(\log x)^{1-\varepsilon}$. We define

$$I_j(x) = \left\{ n;\ \begin{array}{l} n \textit{ is contained in at least } j\textit{-sequences} \\ \textit{among } P_{-F}(x), \cdots, P_{-1}(x), P_1(x), \cdots, P_F(x) \end{array} \right\} ,$$

$$Q(x) = I_1(x) \cap \boldsymbol{P} .$$

It is easily seen that

$$I_1(x) \supset I_2(x) \supset \cdots , \tag{3.1}$$

and therefore

$$\#(I_1(x)) \geqq \#(I_2(x)) \geqq \cdots . \tag{3.1'}$$

Proof of Theorem 1. In order to evaluate $V(M_f(x))$, we decompose its denominator and numerator, respectively, into three terms:

$$\#(M_f(x)) = \sum_{|i|=1}^{F} \#(P_i(x)) - \sum_{j=2}^{2F} \#(I_j(x)) - \#(Q(x)) , \tag{3.2}$$

$$\sum_{n \in M_f(x)} \omega(n) = \sum_{|i|=1}^{F} \sum_{n \in P_i(x)} \omega(n) - \sum_{j=2}^{2F} \sum_{n \in I_j(x)} \omega(n) - \#(Q(x)) . \tag{3.3}$$

For the first term of the right-hand side of (3.2), we have obviously

$$\sum_{|i|=1}^{F} \#(P_i(x)) = 2F\left\{\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)\right\} . \tag{3.4}$$

To estimate the first term of the right-hand side of (3.3), we can apply Lemma 1, since $F \leqq \log x$ for sufficiently large x:

$$\sum_{|i|=1}^{F} \sum_{n \in P_i(x)} \omega(n) = 2F\left\{\log_2 x + A + \sum_p \frac{1}{p(p-1)} - \log 2 + \alpha'_f(x)\right\}\frac{x}{\log x} , \tag{3.5}$$

where $\alpha'_f(x) = (1/2F) \sum_{|i|=1}^{F} \delta_i(x)$. And, from Lemma 1, $\alpha'_f(x)$ can be written in the form

$$\alpha'_f(x) = \alpha_f(x) + \gamma_f(x) , \qquad \frac{1}{2} \leqq \alpha_f(x) \leqq 1 , \qquad \gamma_f(x) = O\left(\frac{\log_2 x}{\log x}\right) , \tag{3.5'}$$

where the constant implied by O-symbol depends only on C.

For the remaining terms of (3.2) and (3.3), we shall prove that

$$\sum_{j=2}^{2F} \#(I_j(x)) = F \cdot O(x(\log x)^{-1-\varepsilon}(\log_3 x)) , \tag{3.6}$$

$$\#(Q(x)) = F \cdot O(x(\log x)^{-2}(\log_3 x)) , \tag{3.7}$$

$$\sum_{j=2}^{2F} \sum_{n \in I_j(x)} \omega(n) = F \cdot O(x(\log x)^{-1-\varepsilon}(\log_2 x)(\log_3 x)) , \tag{3.8}$$

where the constants implied by O-symbols depend on ε and C. Once these formulas obtained, we can deduce from (3.4), (3.6) and (3.7) that

$$\#(M_f(x)) = 2F\{1 + O((\log x)^{-\varepsilon}(\log_3 x))\}\frac{x}{\log x} , \tag{3.2'}$$

and, form (3.5), (3.5′), (3.7) and (3.8), that

$$\sum_{n \in M_f(x)} \omega(n) = 2F\left\{\log_2 x + A + \sum_p \frac{1}{p(p-1)} - \log 2\right. \tag{3.3'}$$

$$+\alpha_f(x)+\gamma_f(x)+O((\log x)^{-\varepsilon}(\log_2 x)(\log_3 x))\Big\}\frac{x}{\log x}$$

$$=2F\Big\{\log_2 x+A+\sum_p \frac{1}{p(p-1)}-\log 2+\alpha_f(x)$$

$$+O((\log x)^{-\varepsilon}(\log_2 x)(\log_3 x))\Big\}\frac{x}{\log x}\ .$$

Then our Theorem 1 will be immediate from (3.2′) and (3.3′) .

PROOF OF (3.6). Concerning $\#(I_j(x))$, we obtain from (2.3) the following estimate:

$$\#(I_j(x))=\sum_{\substack{-F\leqq a_j<\cdots<a_1\leqq F\\ a_1\cdots a_j\neq 0}}\{\#(P_{a_1-a_2,\cdots,a_1-a_j}(x))+O(1)\}$$

$$=F^jD_F^{j-1}\cdot O\Big(\frac{x}{\log^j x}\Big)\ .$$

Since $F=O((\log x)^{1-\varepsilon})$ and $D_F=O(\log_3 x)$ (Lemma 3),

$$\#(I_j(x))=F\cdot O(x(\log x)^{-1-(j-1)\varepsilon}(\log_3 x)^{j-1}) \tag{3.9}$$

Let R be a natural integer satisfying $R>1+(1/\varepsilon)$. Then, by the aid of the relation (3.1′) and (3.9), we have

$$\begin{aligned}\sum_{j=2}^{2F}\#(I_j(x))&\leqq\sum_{j=2}^{R-1}\#(I_j(x))+\sum_{j=R}^{2F}\#(I_j(x))\\
&\leqq R\cdot\#(I_2(x))+2F\cdot\#(I_R(x))\\
&=F\cdot O(x(\log x)^{-1-\varepsilon}(\log_3 x))+F^2\cdot O(x(\log x)^{-1-(R-1)\varepsilon}(\log_3 x)^{R-1})\\
&=F\{O(x(\log x)^{-1-\varepsilon}(\log_3 x))+O(x(\log x)^{-R\varepsilon}(\log_3 x)^{R-1})\}\\
&=F\cdot O(x(\log x)^{-1-\varepsilon}(\log_3 x))\ .\end{aligned}$$

This proves (3.6).

(3.7) is proved directly from (2.3).

PROOF OF (3.8). Concerning $\sum_{n\in I_j(x)}\omega(n)$, we obtain, from our Lemma 2–I), the following estimate:

$$\begin{aligned}\sum_{n\in I_j(x)}\omega(n)&=\sum_{\substack{-F\leqq a_j<\cdots<a_1\leqq F\\ a_1\cdots a_j\neq 0}}\{\sum_{q\leqq x}\#(P_{a_1-a_2,\cdots,a_1-a_j}(x;\,q,\,-a_1))+O(\log x)\}\\
&=F^j(8D_F)^{j-1}\cdot O(x(\log x)^{-j}(\log_2 x))\\
&=F\cdot O(x(\log x)^{-(j-1)\varepsilon-1}(\log_2 x)(8\log_3 x)^{j-1})\ .\end{aligned}$$

We get, for the same R as in the proof of (3.6), that

$$\sum_{j=2}^{2F}\sum_{n\in I_j(x)}\omega(n)\leqq R\sum_{n\in I_2(x)}\omega(n)+2F\sum_{n\in I_R(x)}\omega(n)$$
$$=F\cdot O(x(\log x)^{-1-\varepsilon}(\log_2 x)(8\log_3 x))$$
$$+F\cdot O(x(\log x)^{-R\varepsilon}(\log_2 x)(8\log_3 x)^{R-1})$$
$$=F\cdot O(x(\log x)^{-1-\varepsilon}(\log_2 x)(\log_3 x))\ .$$

Consequently, we obtain (3.8) and this accomplishes the proof of Theorem 1.

PROOF OF THEOREM 2. Since we have already obtained the estimate of $\sharp(M_f(x))$ in (3.2′), it is sufficient to show

$$\sum_{|i|=1}^{F}\sum_{n\in P_i(x)}\sum_{\substack{q^m|n\\ m\geqq 2}}1=2F\left\{\sum_p\frac{1}{(p-1)^2}+O\left(\frac{1}{\log x}\right)\right\}\frac{x}{\log x}\ , \tag{3.10}$$

$$\sum_{j=2}^{2F}\sum_{n\in I_j(x)}\sum_{\substack{q^m|n\\ m\geqq 2}}1=F\cdot O(x(\log x)^{-1-\varepsilon}(\log_3 x))\ . \tag{3.11}$$

In fact, once we obtain these two formulas, we can deduce from them that

$$\sum_{n\in M_f(x)}\{\Omega(n)-\omega(n)\}=2F\left\{\sum_p\frac{1}{(p-1)^2}+O((\log x)^{-\varepsilon}(\log_3 x))\right\}\frac{x}{\log x}\ .$$

Then this formula and (3.2′) give a proof of Theorem 2.

We can prove (3.10) in a similar way as in our proof of Lemma 1; for any i $(1\leqq|i|\leqq[f(x)])$, we put $y=x-i$, then

$$\sum_{n\in P_i(x)}\sum_{\substack{q^m|n\\ m\geqq 2}}1=\left\{\sum_{\substack{q^m\leqq\sqrt{y}\\ m\geqq 2}}+\sum_{\substack{\sqrt{y}<q^m\leqq y^{3/4}\\ m\geqq 2}}+\sum_{\substack{y^{3/4}<q^m\leqq y\\ m\geqq 2}}\right\}\pi(y;q^m,-i)\ ,$$

and we can prove, again by the aid of Bombieri's theorem and Brun-Titchmarsh's theorem, that

$$\sum_{\substack{q^m\leqq\sqrt{y}\\ m\geqq 2}}\pi(y;q^m,-i)=\left\{\sum_p\frac{1}{(p-1)^2}+O\left(\frac{1}{\log y}\right)\right\}\frac{y}{\log y}\ ,$$
$$\sum_{\substack{\sqrt{y}<q^m\leqq y^{3/4}\\ m\geqq 2}}\pi(y;q^m,-i)=O(y^{7/8})\ ,$$
$$\sum_{\substack{y^{3/4}<q^m\leqq y\\ m\geqq 2}}\pi(y;q^m,-i)=O(y^{3/4})\ .$$

Since $y=x+O(\log x)$, these results give (3.10).

Concerning the formula (3.11), making use of Lemma 2–II), we get

$$\sum_{n\in I_j(x)}\sum_{\substack{q^m|n\\ m\geqq 2}}1=\sum_{\substack{-F\leqq a_j<\cdots<a_1\leqq F\\ a_1\cdots\cdots a_j\neq 0}}\sum_{\substack{q^m\leqq x\\ m\geqq 2}}\sharp(P_{a_1-a_2,\cdots,a_1-a_j}(x;q^m,-a_1))$$

$$= F\cdot O(x(\log x)^{-1-(j-1)\varepsilon}(8\log_3 x)^{j-1})\ .$$

Then we obtain (3.11) similarly as in our proof of (3.8).

Theorems 3 and 4 can be proved similarly as Theorems 1 and 2 respectively. We remark here that, if we take $\varepsilon=1$ in our Theorem 1, we obtain as a corollary that,

$$V(N_d(x)) = \log_2 x + \Big(A + \sum_p \frac{1}{p(p-1)} - \log 2 + \beta_d(x)\Big\} + O\Big(\frac{\log_2 x \log_3 x}{\log x}\Big)\ .$$

Theorem 3 shows that we can improve the estimate of the error term in this formula into $O((\log_2 x)(\log x)^{-1})$. In fact, in the case of $N_d(x)$, $D_F = \max_{1\leqq b\leqq F}(\prod_{p|b}(p/(p-1)))$ turns out to be a constant, and consequently, $\log_3 x$ does not appear.

Our Theorem 1 gives only a range of values of $\alpha_f(x)$. A more precise evaluate of $\alpha_f(x)$ would be obtained, if an asymptotic formula of the following form could be proved:

$$\#(S_i(x)) \sim C_i\cdot\pi(x)\ ,$$

where $S_i(x)=\{p;\ p\leqq x,\ p+i$ *has a prime factor greater than* $\sqrt{x}\}$ and C_i is a constant depending only on i. In this connection, we have a conjecture that

$$\#(S_i(x)) \sim (\log 2)\cdot\pi(x)\ .$$

If this is true, (1.1), (1.2) and (1.3) will give the following interesting relation:

$$V(M_f(x)) - V(N(x)) \sim W(N(x))$$

as $x\to\infty$. Incidentally *I* notice that the following asymptotic formula is easy to prove:

$$\#(T_i(x)) \sim (\log 2)\cdot x\ ,$$

where $T_i(x)=\{n;\ n\leqq x,\ n+i$ *has a prime factor greater than* $\sqrt{x}\}$.

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