TOKYO J. MATH.
VOL. 5, NO. 1, 1982

Asymptotic Strong Convergence of Nonlinear Contraction Semigroups

Hiroko OKOCHI

Waseda University
(Communicated by J. Wada)

Introduction

Let $S=\{S(t): t\geqq 0\}$ be a (nonlinear) contraction semigroup on a closed convex subset C of a Hilbert space H. In this note we study the asymptotic strong convergence of the orbits $S(t)x$ $(x\in C)$ of S. In 1975 Bruck [5] discussed this problem for a nonlinear contraction semigroup S under the assumption that S is generated by the subdifferential $\partial\varphi$ of a proper lower semicontinuous convex functional φ, and that φ is even in the sense that $\varphi(x)=\varphi(-x)$ on its effective domain $\mathfrak{D}(\varphi)=\{x\in H: \varphi(x)<+\infty\}$. Since then a number of extended forms of Bruck's conditions for the asymptotic strong convergence have been obtained, for instance, in the works of [1], [4], [7], [8] and [10]. Here some other sufficient conditions on the generator A of S for the existence of strong limits of Cèsaro means $(1/t)\int_0^t S(\tau+h)xd\tau$ as well as those of orbits $S(t)x$ are investigated.

The present paper contains three results. The first result (Theorem 1) provides a sufficient condition for the strong convergence of the orbit of S generated by the subdifferential of a proper lower semicontinuous functional φ. This result extends the author's previous result in [10] and so involves the case in which φ is even. On the other hand, if S is generated by the subdifferential of φ which assumes a minimum in H and if there exists a real number $\lambda>\min\varphi$ such that the set $M_\mu=\{x\in D(\partial\varphi): \varphi(x)\leqq\lambda$ and $\|x\|\leqq\mu\}$ is relatively compact for each $\mu>0$, then it is proved that $S(t)x$ converges strongly to a minimum point of φ as $t\to\infty$. Our result involves this case as well. It turns out that Theorem 1 extends the above-mentioned two results which are of completely different types. The second result (Theorem 2) is concerned with the asymptotic strong convergence of the Cèsaro means of S as well as the orbits of S them-

Received March 25, 1981

selves. In this theorem an extended form of Gripenberg's condition [7] (that involves the oddness condition) is employed. Accordingly, our second result extends a result of Gripenberg in the case of Hilbert spaces. Moreover, in Assertion (2°) of Theorem 2, the strong convergence of the orbits of S is obtained under the so-called Tauberian condition, by applying a result due to Lorentz [9]. These results are stated in Section 1 along with some comments and the proofs of the theorems are given in section 2. Finally, in section 3, we discuss the linear perturbation problem for the proper lower semicontinuous *even* convex functionals. This problem was raised in [4] by H. Brezis and closely related to Theorem 1. Here we shall show (in Proposition 4) by presenting a counterexample that perturbations of linear functionals to *even* convex functionals are *not* necessarily possible as for as the asymptotic strong convergence is concerned.

§1. Main results.

Throughout this paper we assume that H is a real Hilbert space. Our first result is stated as follows:

THEOREM 1. *Let φ be a proper lower semicontinuous convex functional on H which assumes a minimum in H. Suppose that there exists a real number $\lambda > \min \varphi$, a Fréchet differentiable operator B in H and a continuous functional $\alpha: \mathfrak{R}(B) \times (0, \infty) \to (0, 1]$ satisfying the following conditions*:

(a) *The domain $\mathfrak{D}(B)$ of B contains the set $\{x \in \mathfrak{D}(\partial\varphi): \varphi(x) \leqq \lambda\}$ and the inequality*

$$\varphi(x) \geqq \varphi(Bx - \alpha_x(I-B)x) \tag{1}$$

holds for $x \in \mathfrak{D}(B)$ with $(I-B)x \neq 0$, where α_x denotes the value $\alpha(Bx, \|(B-I)x\|)$.

(b) *The set $\{Bx: x \in \mathfrak{D}(\partial\varphi), \varphi(x) \leqq \lambda$ and $\|x\| \leqq \mu\}$ is relatively compact for all $\mu > 0$.*

Then the solution $u(t) \equiv u(t, x)$ of the initial-value problem

$$\frac{d}{dt}u(t) \in -\partial\varphi(u(t)) \quad \text{a.e.} \quad t \in (0, \infty)\,, \qquad u(0) = x \tag{2}$$

converges strongly to a minimum point (which may depend upon x) of φ as $t \to \infty$ for every initial value $x \in cl(\mathfrak{D}(\varphi)) = cl\{x \in H: \varphi(x) < +\infty\}$.

REMARK 1. Given an affine subspace X in H, suppose that the inequality (1) holds with $B=\mathrm{Proj}_X$ and $\alpha_x \equiv 1$. Then $\varphi(x)=\varphi(Bx-(I-B)x)$ for all $x \in \mathfrak{D}(\varphi)$, which means that φ is symmetric with respect to the affine subspace X. In particular, if φ is even, i.e., $\varphi(x)=\varphi(-x)$, then our conditions (a) and (b) hold with $B=\mathrm{Proj}_{\{0\}}=0$ and $\alpha_x \equiv 1$. In the case where B is a constant mapping $B(x)\equiv x_0$, Theorem 1 is reduced to the author's previous result [10] and is essentially contained in the work of Gripenberg [7].

Let A be a maximal monotone operator in H and S the contraction semigroup generated by $-A$.

We permit ourselves the common abbreviations, "$x_n \to x$" and "$x_n \rightharpoonup x$" in refering respectively to the strong convergence of $\{x_n\}$ to x and the weak convergence of $\{x_n\}$ to x.

We now introduce a condition for the operator A which is an extended form of Gripenberg's condition treated in [7]:

(i) There exist an element $z_0 \in A^{-1}(0)$ and a linear projection P with the following properties: For every $\varepsilon \in (0, 1)$, there exist constants $c_i=c_i(\varepsilon)$, $i=1, 2, 3$, and

$$
\begin{aligned}
(3)\qquad & c_1\{(Py_1, x_1-z_0)+(Py_2, x_2-z_0)\}+c_2(P(y_1+y_2), x_1+x_2-2z_0) \\
& +c_3\{(y_1, x_1-z_0)+(y_2, x_2-z_0)\}+(y_1+y_2, x_1+x_2-2z_0)\geqq 0
\end{aligned}
$$

holds for all $[x_j, y_j] \in A$ with $\|x_j\| \leqq 1/\varepsilon$ and $\|(I-P)(x_j-z_0)\| \geqq \varepsilon$, $j=1, 2$.

Gripenberg treated the inequality (3) with $P=0$. The inequality (3) may be regarded as an extention of the condition that $A \subset H \times H$ is symmetric with respect to $X \times \{0\}$ (i.e., $[x, y] \in A$ iff $[\mathrm{Proj}_X x-(I-\mathrm{Proj}_X)x, -y] \in A$), where X is an affine subspace of H with $z_0 \in X$. If $P=0$ then $X=\{z_0\}$. In particular, if A is an *odd* mapping (i.e., $[x, y] \in A$ iff $[-x, -y] \in A$), then condition (i) holds with $P=0$ and $X=\{0\}$. In this sense condition (i) is an extended form of the oddness condition for A.

Employing the above-mentioned condition (i), our second result is stated as follows:

THEOREM 2. *Let $x_0 \in cl(\mathfrak{D}(A))$. (1°) Assume that condition* (i) *holds, and that*

(ii) *there exists a periodic function v with period ρ* (i.e., $v(t+\rho)=v(t)$ *for* $t \geqq 0$) *such that* $\lim_{t\to\infty}\|v(t)-z_0\|$ exists and $\lim_{t\to\infty}\|PS(t)x_0-v(t)\|=0$. *Then we have*

$$
s-\lim_{t\to\infty}\frac{1}{t}\int_0^t S(\tau+h)x_0 d\tau = z \in A^{-1}(0) \quad \textit{uniformly in} \quad h \in (0, \infty)\,.
$$

(2°) *Suppose that $A^{-1}(0)\neq\varnothing$, and that condition* (i) *holds for a* *compact linear projection P. Let*

(iii) $$s-\lim_{t\to\infty}\{S(t+h)x_0-S(t)x_0\}=0 \quad \textit{for all} \quad h>0\,.$$

Then $S(t)x_0$ converges strongly to some point of $A^{-1}(0)$ as $t\to\infty$.

COROLLARY 3. *Suppose that the graph of A is symmetric with respect to a closed affine subspace $X\times\{0\}$ of $H\times H$ in the sense that*

(4) $$[x, y]\in A \quad \textit{iff} \quad [\mathrm{Proj}_X\, x-(I-\mathrm{Proj}_X)x,\ -y]\in A\,.$$

Then, for each $x\in cl(\mathfrak{D}(A))$, we have the convergence

$$s-\lim_{t\to\infty}\frac{1}{t}\int_0^t S(\tau+h)x d\tau=z_x\in A^{-1}(0) \quad \textit{uniformly in} \quad h\in(0, \infty)\,.$$

REMARK 2. Some sufficient conditions for (ii) to hold are in order.

(A) If $PS(t)x_0$ converges strongly to a point y as $t\to\infty$, then condition (ii) is automatically satisfied with $v(t)\equiv y$. In particular, if $P=0$, then $PS(t)x_0\equiv 0$.

(B) Suppose that

(ii)′ P is a compact linear projection, $A^{-1}(0)\neq\emptyset$ and $S(t+h)x_0-S(t)x_0\rightharpoonup 0$ as $t\to\infty$ for every $h>0$.

Then it follows (see [6]) that $S(t)x_0$ converges weakly as $t\to\infty$, so that $PS(t)x_0$ converges strongly as $t\to\infty$.

(C) Assume:

(ii)″ A is demipositive (see [5]) and P is a compact linear projection. Then the weak convergence of $S(t)x_0$ as $t\to\infty$ is obtained as well.

(D) Suppose that

(ii)″ P is a compact linear projection, $A^{-1}(0)\neq\emptyset$ and

$$\lim_{t\to\infty}(\min\{\|y\|: y\in AS(t)x_0\})\leqq\lim_{t\to\infty}\left(\frac{1}{h}\|S(t+h)x_0-S(t)x_0\|\right)$$

for every $h>0$.

Then $S(t)x_0$ converges weakly as $t\to\infty$ (see [11; Theorem 10.5]).

REMARK 3. Assertion (2°) of Theorem 2 and Corollary 3 were obtained respectively in Gripenberg [6] and Baillon [1] in the case where $P=0$.

§2. Proof of theorems.

PROOF OF THEOREM 1. Let $x\in cl(\mathfrak{D}(\varphi))$ and let $u(t)$ be the associated solution of the equation (2). Let λ be a constant as mentioned in

Theorem 1. We recall that $u(t)\in\mathfrak{D}(\partial\varphi)$ for all $t>0$ and that $\varphi(u(t))$ converges decleasingly to the minimum value of φ as $t\to\infty$. Thus there exists a point $t_0>0$ such that $\varphi(u(t))\leqq\lambda$ for all $t\geqq t_0$. This means that $u(t)\in\mathfrak{D}(B)$ for all $t\geqq t_0$. Moreover, since φ has a minimum point z in H, it follows from the definition of $\partial\varphi$ that

$$\frac{d}{dt}\|u(t)-z\|^2=2(u'(t),\ u(t)-z)\leqq\{\varphi(z)-\varphi(u(t))\}\leqq 0 \quad \text{a.e.} \quad t\in(0,\ \infty)\ .$$

From this we obtain $\|u(t)\|\leqq\|u(t_0)\|+2\|z\|$ for $t\geqq t_0$, and so condition (b) implies that the set $\{Bu(t)\colon t\geqq t_0\}$ is relatively compact in H.

Assume that $\liminf_{t\to\infty}\|(I-B)u(t)\|=0$. Then there exists a squence $\{t_n\}$ in $(0,\ \infty)$ such that $t_n\to\infty$ and $u(t_n)-Bu(t_n)\to 0$ as $n\to\infty$. On the other hand, the set $\{Bu(t)\colon t\geqq t_0\}$ is relatively compact in H, so that there exists a subsequence $\{t_{n_j}\}$ of $\{t_n\}$ such that $\{Bu(t_{n_j})\}$ converges strongly. Therefore, the sequence $\{u(t_{n_j})\}$ converges strongly as $t_{n_j}\to\infty$. On the other hand, it is well-known (see [5]) that $u(t)$ converges weakly to a minimum point of φ as $t\to\infty$. Thus $u(t)$ converges strongly to a minimum point of φ as $t\to\infty$.

Next, assume that $\liminf_{t\to\infty}\|(I-B)u(t)\|\overline{\gtrless}\varepsilon>0$. Then there exists a point $T\geqq t_0$ such that $\|(I-B)u(t)\|\geqq\varepsilon/2$ for all $t\geqq T$, so that the sets $\{Bu(t)\colon t\geqq T\}$ and $\{\|(I-B)u(t)\|\colon t\geqq T\}$ are relatively compact in H and $(0,\ \infty)$, respectively. Thus, it follows from the continuity of the functional α that

$$\delta\equiv\min\{\alpha(Bu(t),\ \|(I-B)u(t)\|)\colon\quad t\geqq T\}>0$$

Moreover there exists a sequence $\{t_n\}$ such that $t_n\uparrow\infty$ and both $\{Bu(t_n)\}$ and $\{\|(I-B)u(t_n)\|\}$ converges as $n\to\infty$ in H and R, respectively.

Fix an arbitraly point $t_n>T$ and define a functional $g\colon[T,\ t_n]\to R$ by

$$g(t)=\frac{1+\delta}{2}\{\|Bu(t)\|^2-\|Bu(t_n)\|^2+\|(I-B)u(t)\|^2-\|(I-B)u(t_n)\|^2\}$$
$$-\frac{\delta}{2}\|u(t)-u(t_n)\|^2\ .$$

Then, using the definition of subdifferential, we have

$$g'(t)=(1+\delta)\Big\{(u'(t),\ u(t)-Bu(t))-\Big(\frac{d}{dt}Bu(t),\ u(t)\Big)\Big\}-\delta(u'(t),\ u(t)-u(t_n))$$
$$=(u'(t),\ u(t)-Bu(t_n)+\delta(I-B)u(t_n))+(1+\delta)\frac{d}{dt}(u(t),\ Bu(t_n)-Bu(t))$$
$$\leqq\varphi(Bu(t_n)-(I-B)u(t_n))-\varphi(u(t))+(1+\delta)\frac{d}{dt}(u(t),\ Bu(t_n)-Bu(t))\ .$$

On the other hand, one obtains

$$\varphi(Bu(t_n)-(I-B)u(t_n))\leqq\varphi(u(t_n)) \quad \text{for} \quad t_n\geqq t_1\,. \tag{5}$$

In fact, from the convexity of φ together with the inequality (1), it follows that for every $x\in\mathfrak{D}(\partial\varphi)$ with $(I-B)x\neq 0$ we have

$$\begin{aligned}\varphi(Bx)&=\varphi\Big(\frac{\alpha}{1+\alpha}x+\frac{1}{1+\alpha}((1+\alpha)Bx-\alpha x)\Big)\\&\leqq\frac{\alpha}{1+\alpha}\varphi(x)+\frac{1}{1+\alpha}\varphi((1+\alpha)Bx-\alpha x)\\&\leqq\frac{\alpha}{1+\alpha}\varphi(x)+\frac{1}{1+\alpha}\varphi(x)=\varphi(x)\,,\end{aligned}$$

where $\alpha=\alpha(Bx,\|(I-B)x\|)$. Hence, if $0\leqq\delta\leqq\alpha$, then

$$\begin{aligned}\varphi(Bx-\delta(I-B)x)&=\varphi\Big(\frac{\alpha-\delta}{\alpha}Bx+\frac{\delta}{\alpha}(Bx-\alpha(I-B)x)\Big)\\&\leqq\frac{\alpha-\delta}{\alpha}\varphi(Bx)+\frac{\delta}{\alpha}\varphi(Bx-\alpha(I-B)x)\\&\leqq\frac{\alpha-\delta}{\alpha}\varphi(x)+\frac{\delta}{\alpha}\varphi(x)=\varphi(x)\,.\end{aligned}$$

Now, by virtue of the inequality (5) and the fact that $\varphi(u(t))$ is monotone nonincleasing, we have

$$g'(t)\leqq(1+\delta)\frac{d}{dt}(u(t),\,Bu(t_n)-Bu(t)) \quad \text{a.e.} \quad t\in[t_1,\,t_n]\,.$$

Integrating both sides of this inequality over $[t_m,\,t_n]$ $(m<n)$, we obtain

$$\begin{aligned}&\frac{1+\delta}{2}\{-\|Bu(t_m)\|^2+\|Bu(t_n)\|^2-\|(I-B)u(t_m)\|^2+\|(I-B)u(t_n)\|^2\}\\&\qquad+\frac{\delta}{2}\|u(t_m)-u(t_n)\|^2\\&\leqq-(1+\delta)(u(t_m),\,Bu(t_n)-Bu(t_m))\,.\end{aligned}\tag{6}$$

Let $n,m\to\infty$ in (6). Then the convergence of the sequences $\{Bu(t_n)\}$ and $\{\|(I-B)u(t_n)\|\}$ and the boundedness of $\{u(t_n)\}$ together yield that $\|u(t_m)-u(t_n)\|\to 0$. Thus $\{u(t_n)\}$ is a Cauchy sequence in H. Consequently, recalling that $u(t)$ converges weakly to a minimum point of φ as $t\to\infty$, we conclude that $u(t)$ converges strongly to a minimum point of φ as $t\to\infty$.

PROOF OF THEOREM 2. Let $x_0 \in \mathfrak{D}(A)$ and put $u(t) \equiv S(t)x_0$. We may assume without loss of generality that $z_0 = 0 \in A^{-1}(0)$. Then we have

$$\frac{d}{dt}\|u(t)\|^2 = 2(u'(t)-0,\ u(t)-0) \leqq 0\ , \quad \text{and}$$

$$\frac{d}{dt}\|u(t+h)-u(t)\|^2 = 2(u'(t+h)-u'(t),\ u(t+h)-u(t)) \leqq 0$$

for all $h>0$ and a.e. $t \in (0, \infty)$. These relations imply that

$$(7) \qquad \|u(t)\| \downarrow d_1 \quad \text{and} \quad \|u(t+h)-u(t)\| \downarrow \delta_h \quad \text{as} \quad t \to \infty$$

for some nonnegative numbers d_1 and δ_h.

Condition (ii) implies that $\|v(t)\| \equiv \text{const.}$ and

$$\|Pu(t)\| \to \|v(0)\| \quad \text{as} \quad t \to \infty\ .$$

Hence, by (7), we obtain the convergence

$$\|(I-P)u(t)\| \to d_2 \equiv (d_1^2 - \|v(0)\|^2)^{1/2} \quad \text{as} \quad t \to \infty\ .$$

At this point there are two cases to check. Suppose that $d_2=0$. In this case condition (ii) yields that $u(t) \to v(t)$ as $t \to \infty$, and hence $(1/t)\int_0^t u(\tau+h)d\tau$ converges strongly to $(1/\rho)\int_0^\rho v(\tau)d\tau(=z)$, uniformly for $h \in (0, \infty)$ as $t \to \infty$. Next, assume that $d_2>0$. Then there exists a constant $T \geqq 0$ such that

$$\|(I-P)u(t)\| \geqq \frac{d_2}{2} \quad \text{for} \quad t \geqq T\ .$$

Put $\varepsilon = \min\{d_2/2,\ 1/\|u(0)\|\} > 0$. Since $[u(t), -u'(t)] \in A$ for a.e. t and

$$\|u(t)\| \leqq \|u(0)\| \leqq \frac{1}{\varepsilon} \qquad \|(I-P)u(t)\| \geqq \varepsilon \quad \text{for} \quad t \geqq T\ ,$$

we infer from (3) that

$$c_1\left\{\frac{d}{dt}\|Pu(t+h)\|^2 + \frac{d}{dt}\|Pu(t)\|^2\right\} + c_2\frac{d}{dt}\|Pu(t+h)+Pu(t)\|^2$$
$$+ c_3\left\{\frac{d}{dt}\|u(t+h)\|^2 + \frac{d}{dt}\|u(t)\|^2\right\} + c_2\frac{d}{dt}\|u(t+h)+u(t)\|^2 \leqq 0$$

for every $h \geqq 0$ and a.e. $t \in [T, \infty)$.

Integrate this inequality over $(j\rho, j\rho+k\rho)$ (where j and k are arbitrary positive integers with $j\rho \geqq T$) to obtain

$$c_1\{\|Pu(h+(1+k)\rho)\|^2-\|Pu(h+j\rho)\|^2+\|Pu((j+k)\rho)\|^2-\|Pu(j\rho)\|^2\}$$
$$+c_2\{\|Pu(h+(j+k)\rho)+Pu((j+k)\rho)\|^2-\|Pu(h+j\rho)+Pu(j\rho)\|^2\}$$
$$+c_3\{\|u(h+(j+k)\rho)\|^2-\|u(h+j\rho)\|^2+\|u((j+k)\rho)\|^2-\|u(j\rho)\|^2\}$$
$$-\|u(h+(j+k)\rho)-u((j+k)\rho)\|^2+\|u(h+j\rho)-u(j\rho)\|^2+2\|u(h+(j+k)\rho)\|^2$$
$$+2\|u((j+k)\rho)\|^2-2\|u(h+j\rho)\|^2-2\|u(j\rho)\|^2\leqq 0$$

Moreover, we see from condition (ii) that

$$Pu(h+n\rho)\to v(h) \quad \text{as} \quad n\to\infty \quad \text{for each} \quad h\geqq 0\,. \tag{10}$$

Let $k\to\infty$ in (9). Then, application of (7) and (10) yields

$$\begin{aligned}\delta_h^2 &\leqq \|u(h+j\rho)-u(j\rho)\|^2\\ &\leqq \delta_h^2-c_1\{2\|v(h)\|^2-\|Pu(h+j\rho)\|^2-\|Pu(j\rho)\|^2\}\\ &\quad -c_2\{\|v(h)+v(0)\|^2-\|Pu(h+j\rho)+Pu(j\rho)\|^2\}\\ &\quad -c_3\{2d_1^2-\|u(h+j\rho)\|^2-\|u(j\rho)\|^2\}-4d_1^2+2\|u(h+j\rho)\|^2+2\|u(j\rho)\|^2\,.\end{aligned}$$

Hence, (7) and (10) yield that $\lim_{j\to\infty}\|u(h+j\rho)-u(j\rho)\|=\delta_h$ uniformly in $h\geqq 0$, so that the convergence $\lim_{t\to\infty}\|u(h+t)-u(t)\|=\delta_h$ holds uniformly in $h\geqq 0$. Thus, by a result of Kobayasi and Miyadera [8] we have

$$s-\lim_{t\to\infty}\frac{1}{t}\int_0^t S(\tau+h)x_0\,d\tau=z\in A^{-1}(0) \quad \text{uniformly in} \quad h\geqq 0\,.$$

It is easy to see that this convergence also holds for $x_0\in cl(\mathfrak{D}(A))$. Thus, we proved Assertion (1°).

By virtue of (A) and (B) of Remark 2, Assertion (2°) is a direct consequence of Assertion (1°) and Lorentz [9].

PROOF OF COROLLARY 3. Since A is a maximal monotone operator and $cl(\mathfrak{D}(A))$ is a convex subset of H, condition (4) yields that $\{\mathrm{Proj}_X\, x: x\in cl(\mathfrak{D}(A))\}=X\cap cl(\mathfrak{D}(A))\subset A^{-1}(0)$. Hence, for $x\in cl(\mathfrak{D}(A))$, $\mathrm{Proj}_X\, S(t)x$ $(=\mathrm{Proj}_{X\cap cl(\mathfrak{D}(A))}S(t)x)$ converges strongly as $t\to\infty$. This is obtained in the same way as in the derivation of the convergence of $\mathrm{Proj}_{A^{-1}(0)}S(t)x$ (cf., [10; Lemma 9.2]).

Fix an arbitraly point $z_0\in X\cap cl(\mathfrak{D}(A))$ and define a linear projection P by $P=\mathrm{Proj}_{X-z_0}$. Then, by (4), we have

$$(y_1-(-y_2),\, x_1-(2Px_2-x_2+2z_0-2Pz_0))\geqq 0\,, \quad \text{and}$$
$$(y_2-(-y_1),\, x_2-(2Px_1-x_1+2z_0-2Pz_0))\geqq 0$$

for $[x_i, y_i]\in A$, $i=1, 2$. Here we have used the fact that $\mathrm{Proj}_X\, x=\mathrm{Proj}_{X-z_0}(x-z_0)+z_0=Px-Pz_0-z_0$. Summing up these inequalities, one obtains

$$(y_1+y_2,\ x_1+x_2-2z_0)-(y_1+y_2,\ Px_1+Px_2-2Pz_0)\geqq 0\ .$$

Thus, condition (i) holds with $c_1=c_3=0$ and $c_2=-1$. Moreover, since $PS(t)x$ $(=\mathrm{Proj}_X\, x+Pz_0+z_0)$ converges strongly as $t\to\infty$, it follows from (A) of Remark 2 that condition (ii) holds for each $x\in cl(\mathfrak{D}(A))$.

Therefore, the assertion follows from Assertion (1°) of Theorem 2.

REMARK 4. The relationship between Theorems 1 and 2 may be stated as follows: If the operator B in Theorem 1 is a compact linear projection, then Theorem 1 can be derived from Assertion (2°) of Theorem 2 (cf., [7]).

§3. Linear perturbation to even convex functionals.

Assume that ψ is a proper *l.s.c. even* convex functional and let $f\in H$ be such that $\min_{z\in H}\{\psi(z)-(f, z)\}$ is achieved. Let $u(t)$ be the solution of $du/dt+\partial\psi(u)\ni f$, $u(0)=x$. Brezis raised in [3] the problem as to whether $u(t)$ converges strongly as $t\to\infty$.

In virtue of Theorem 1, $u(t)$ converges strongly if $\varphi\equiv\psi-f$ satisfies conditions (a) and (b) with $B=\mathrm{Proj}_X$ and $X=\{rf\colon r\in \boldsymbol{R}\}$.

But, in general, $u(t)$ need not be strongly convergent, as stated below.

PROPOSITION 4. *There exists a proper l.s.c. convex functional ψ and $f\in H$ such that* (i) *ψ is even*; (ii) *$\psi-f$ assumes a minimum in H; but* (iii) *the solution $u(t)$ of the initial-value problem*

$$du/dt+\partial\psi(u)\ni f\ ,\qquad u(0)=x$$

does <u>not</u> converge strongly as $t\to\infty$ for some $x\in\mathfrak{D}(\psi)$.

The above-mentioned fact is asserted by the following example (which is motivated by that of Baillon):

EXAMPLE. We construct the aimed ψ and f in the space $H=l^2$. Let a_λ and b_λ be functionals on $\boldsymbol{R}^2$ defined by

$$a_\lambda(\xi,\eta)=-\frac{1}{2}\left(\frac{\pi}{2}\right)^{\lambda}\xi+\frac{\lambda}{2}\left(\frac{\pi}{2}\right)^{\lambda-1}\eta\ ,$$

$$b_\lambda(\xi,\eta)=\begin{cases}[\tan^{-1}(|\xi|/|\eta|)]^{\lambda}\cdot(\xi^2+\eta^2)^{1/2}+a_\lambda(|\xi|,|\eta|) & \text{if}\quad \xi\eta\geqq 0\\ a_\lambda(\xi,\eta) & \text{if}\quad \xi\leqq 0\ \text{ and }\ \eta\geqq 0\\ -a_\lambda(\xi,\eta) & \text{if}\quad \xi\geqq 0\ \text{ and }\ \eta\leqq 0\ .\end{cases}$$

It is easy to check that a_λ is linear, while b_λ is even. We then demonstrate that $c_\lambda \equiv b_\lambda - a_\lambda$ is a nonnegative convex functional on R^2 provided that $\lambda \geqq 1$. According to [1], c_λ is a convex functional on $R^+ \times R^+$ if $\lambda \geqq 1$. On the other hand, c_λ is differentiable, on $R^2 \setminus \{0\}$ and

$$\operatorname{grad} c_\lambda(\xi, 0) = 2a_\lambda \quad \text{for} \quad \xi > 0 .$$

Hence, from the defining inequality for the subdifferential, we have

$$c_\lambda(\xi, \eta) \geqq 2a_\lambda(\xi, \eta) , \qquad (\xi, \eta) \in R^+ \times R^+ .$$

But this inequality shows that $c_\lambda \geqq 0$ on R^2, and that c_λ is a convex functional on R^2.

Now put

$$\lambda_i = \frac{\pi^2}{8} \frac{b}{b-1} b^i \quad (b = 1/\log 2) , \qquad i = 1, 2, 3, \cdots , \tag{11}$$

and, for $\alpha = (\alpha_i)_{i \geq 1}$ $(\alpha_i > 0)$, define f_α and ψ_α by

$$f_\alpha(x) = \alpha_1 a_{\lambda_1}(x_1, x_2) + \cdots + \alpha_n a_{\lambda_n}(x_n, x_{n+1}) + \cdots$$
$$\psi_\alpha(x) = \alpha_1 b_{\lambda_1}(x_1, x_2) + \cdots + \alpha_n b_{\lambda_n}(x_n, x_{n+1}) + \cdots ,$$

where $x = (x_i)_{i \geq 1} \in l^2$. Then, $\min(\psi_\alpha - f_\alpha) = 0$ and ψ_α is even for every $\alpha = (\alpha_i)_{i \geq 1}$.

Next, let $T_1 \geqq 1, \cdots, T_n \geqq n, \cdots$ and $\beta_1, \beta_2, \cdots, \beta_n, \cdots > 0$ be as in Lemma 6 of [2] and put $\varphi_\alpha \equiv \psi_\alpha - f_\alpha$ and $\varepsilon = 1/4$ in Lemma 6. Then, as is mentioned in [2], it follows that for every $n \geqq 1$ one has

$$\| S_\alpha(T_i)x_1 - x_{i+1} \| \leqq \varepsilon + \cdots + \varepsilon^i \qquad i = 1, 2, \cdots, n \tag{12}$$

whenever $\alpha = (\alpha_i)_{i \geq 1}$ satisfies that $\alpha_i = \beta_i$, $i = 1, \cdots, n$, where S_α refers the semigroup generated by $\partial\varphi_\alpha$ and $x_i \in l^2$ $(i \geqq 1)$ are as follows:

$$x_1 = (1, 0, 0, \cdots)$$
$$x_2 = \left(0, \exp\left(-\frac{\pi^2}{8} \frac{1}{\lambda_1}\right), 0, 0, \cdots\right)$$
$$x_i = \left(0, 0, \cdots, 0, \exp\left[-\frac{\pi^2}{8}\left(\frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_{i-1}}\right)\right], 0, 0, \cdots\right)$$
$$\cdots\cdots$$

Since $a_{\lambda_i}(i \geqq 1)$ are linear functional on R^2, so is f_α with $\mathfrak{D}(f_\alpha) \subset l^2$. But a sequence $\alpha' = (\alpha'_i)_{i \geq 1}$ of positive numbers can be chosen so that $D(f_{\alpha'}) = l^2$ and $f_{\alpha'} \in l^2$. Put

$$\alpha_1=\min\{\beta_1, \alpha_1'\} \quad \text{and} \quad \alpha_i=\min\{\beta_i\alpha_{i-1}, \alpha_i'\} \quad \text{for} \quad i\geqq 2\,. \tag{13}$$

Then, since $0<\alpha_i\leqq\alpha_i'$ for $i\geqq 1$, we have $f_\alpha\in l^2$ again.

Finally, we show that the solution of the equation

$$du/dt+\partial\psi_\alpha(u)\ni f_\alpha\,, \qquad u(0)=(1, 0, 0, \cdots)$$

does not converge strongly as $t\to\infty$. According to the selection of T_i, β_i ($i\geqq 1$) as mentioned in the proof of Lemma 6 of [2], we infer from (12) and (13) that

$$\|u(T_n')-x_{n-1}\|\leqq\varepsilon+\cdots+\varepsilon^n \quad \text{for} \quad n\geqq 1\,, \tag{14}$$

where $T_n'=(\beta_1/\alpha_1)T_1+(\beta_2/\alpha_2)(T_2-T_1)+\cdots+(\beta_n/\alpha_n)(T_n-T_{n-1})$. Since $\varepsilon=1/4$ and $\|x_n\|=\exp[-(\pi^2/8)((1/\lambda_1)+\cdots+(1/\lambda_{n-1}))]$, the estimate (14) together with (11) yields

$$\liminf_{n\to\infty}\|u(T_n')\|\geqq\exp\left[-\frac{\pi^2}{8}\left(\frac{8}{\pi^2}\frac{b-1}{b}\frac{b}{b-1}\right)\right]-\frac{\varepsilon}{1-\varepsilon}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}\,.$$

Since $T_n'\geqq T_n\geqq n$, this means that $u(t)$ does not converge strongly to 0 as $t\to\infty$. On the other hand, $\varphi_\alpha(x)\equiv\psi_\alpha(x)-(f_\alpha, x)\geqq 0$, and $\varphi_\alpha(x)=0$ iff $x=0$. Thus we infer with the aid of the result of [5] that $w-\lim_{t\to\infty}u(t)=0$. Consequently, $u(t)$ does not converge strongly as $t\to\infty$.

ACKNOWLEDGEMENT. The author would like to thank Professors Y. Kōmura, S. Oharu and K. Kobayasi for their comments.

References

[1] J. B. BAILLON, Quelques propriétés de convergence asymptotique pour les contractions impaires, C. R. Acad. Sci. Paris Sér. A-B, **283** (1976), 587-590.

[2] J. B. BAILLON, Un example concernant le comportement asymptotique de la solution du problème $du/dt+\partial\varphi(u)\ni 0$, J. Functional Analysis, **28** (1978), 369-376.

[3] J. B. BAILLON et H. BREZIS, Une remarque sur le comportement asymptotique des semi groupes nonlinéaies, Houston J. Math., **2** (1976), 5-7.

[4] H. BREZIS, Asymptotic behavior of some evolution systems, *Nonlinear Evolution Equations* (ed. by M. Crandall), 141-154, Academic Press, New York, 1978.

[5] R. E. BRUCK, Asymptotic convergence of nonlinear contraction semigroups in Hilbert space, J. Functional Analysis, **18** (1975), 15-26.

[6] R. E. BRUCK, On the almost-convergence of iterates of non-expansive mapping in Hilbert space and structure of the ω-limit set, Israel J. Math., **29** (1978), 1-16.

[7] G. GRIPENBERG, On the asymptotic behavior of nonlinear contraction semigroups, Math. Scand., **44** (1979), 385-397.

[8] K. KOBAYASI and I. MIYADERA, On the strong convergence of the Cèsaro means of contractions in Banach spaces, Proc. Japan Acad., **56** (1980), 245-249.

[9] G. G. LORENTZ, A contribution to the theory of divergent sequences, Acta Math., **80**

(1948), 167–190.
[10] H. OKOCHI, A note on asymptotic strong convergence of nonlinear contraction semigroups, Proc. Japan Acad., **56** (1980), 83–84.
[11] A. PAZY, Semigroups of nonlinear contractions and their asymptotic behavior, Nonlinear Analysis and Mechanics; Heriot-Watt Symposium vol 3 (ed. by R. J. Knops), Pitman, San Fransisco, 1978, 36–134.

Present Address:
DEPARTMENT OF MATHEMATICS
SCHOOL OF SCIENCES AND ENGINEERINGS
WASEDA UNIVERSITY
NISHIOKUBO, SHINJUKU-KU, TOKYO 160