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On Closed Subalgebras Lying Between A and H^∞, II

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Introduction

Let D be the open unit disc in the complex plane C, $\bar{D}$ its closure and T its boundary, the unit circle. The basic algebras appearing in this paper are the algebra $C(T)$ of continuous functions on T, and the algebra L^∞ of essentially bounded, measurable functions with respect to the normalized Lebesgue measure $d\theta/2\pi$ on T. These are Banach algebras under the supremum and essential supremum norms, respectively. We denote by A and H^∞ the closed subalgebras of $C(T)$ and L^∞, respectively, whose Fourier coefficients with negative indices vanish. Let $M(B)$ be the maximal ideal space of a uniform algebra B, and $\partial(B)$ be the Silov-boundary. We recall that L^∞ is isometrically isomorphic to $C(M(L^\infty))$ and that $\partial(H^\infty)=M(L^\infty)$. Let us denote $M(L^\infty)$ by X.

The closed subalgebras of L^∞, called Douglas algebras, which contain H^∞ properly, are studied in connection with Toeplitz operators. There is a smallest such algebra, namely, the closed subalgebra of L^∞ generated by H^∞ and $C(T)$, which is denoted by $[H^\infty, C(T)]$. This algebra turns out to be equal to $H^\infty+C(T)$, the linear span of H^∞ and $C(T)$ ([14]). A. Chang [3] and D. E. Marshall [10] showed that every Douglas algebra B is characterized as an algebra $B=H^\infty+C_B$, where C_B is the C^*-algebra generated by the inner functions invertible in B. This fact inspired our interest in characterizing or classifying closed subalgebras of H^∞ containing A, which will be called hereafter analytic subalgebras in this paper. The algebras $H^\infty\cap C_B$, defined in [4] is only well-known class of analytic subalgebras, which are associated with the Douglas algebras $H^\infty+C_B$. The first thing for this purpose, we shall construct in § 1 a new class of analytic subalgebras and seek common properties for these algebras, hopefully, for all the analytic subalgebras. Second, we shall be concerned in § 2 and § 4 with two problems, which will be mentioned

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below. We denote the independent variable on T by z or by $e^{i\theta}$, according to convenience. For f in L^∞, the harmonic extention of f into D is defined by the Poisson's formula. If f is in H^∞, this coincides with the analytic extention of f to D. The restricted backward shift operator is the operator U^* on H^∞ defined by $U^*(f)(e^{i\theta})=e^{-i\theta}\cdot(f(e^{i\theta})-f(0))$, where $f(0)=(1/2\pi)\int_T f(e^{i\theta})d\theta$. Algebras A, H^∞ and $H^\infty\cap C_B$ are invariant under U^*. The key observation needed to solve the following problems is the U^*-invariance of an analytic subalgebra. Section 3 is devoted to the existence of analytic subalgebras which are not U^*-invariant.

D. Stegenga generalized in [17] the fact that $H^\infty+C(T)$ is closed, by replacing H^∞ by an w^*-closed, shift-invariant subspace M. In view of this, we shall attempt to generalize the fact by replacing H^∞ by an analytic subalgebra B;

PROBLEM P_1: Find necessary and sufficient conditions for an analytic subalgebra B to satisfy, in order that the linear span $B+C(T)$ is a closed subalgebra of L^∞, or in other words, that $B+C(T)$ coincides with the closed subalgebra $[B, C(T)]$.

Theorem 5 in § 2 is an answer to P_1.

J. Wermer [8] showed that if B is a closed subalgebra of $C(T)$ and contains A properly, then $B=C(T)$. This fact is often called Wermer's maximality theorem. K. Hoffman and I. M. Singer [8] showed that if B is a closed subalgebra of L^∞ containing H^∞ properly, then $C(T)\subset B$, namely, H^∞ is maximal in the algebra $H^\infty+C(T)$. D. Sarason [15] asks the possibility of getting a common generalization which contains the both maximality theorems as special cases. Let B be a closed subalgebra of L^∞ such that $A\subsetneqq B\not\subset H^\infty$. He gives a sufficient condition for B to contain $C(T)$ and also gives an example of such B satisfying $C(T)\not\subset B$. We will consider this question as a problem for analytic subalgebras;

PROBLEM P_2. Find sufficient conditions for an analytic subalgebra B to satisfy, in order that any closed superalgebra $\tilde{B}$ of B in L^∞ with $\tilde{B}\not\subset H^\infty$ contains $C(T)$.

Theorem 9 in § 4 is an answer to P_2.

A convenient reference book for the basic facts about closed subalgebras of L^∞ is K. Hoffman [8].

§ 1. Subalgebras $B_S(X')$.

At first we review the definition and some properties of the algebras

B_A and B_C obtained in [13]. By the separability of $\bar{D}$, there exists a countable, dense subset $\{x_i: i \in N\}$ of $\bar{D}$, where N is the set of natural numbers. Let α be a mapping from N into $\bar{D}$ defined by $\alpha(n)=x_n$. By the Čech-compactification βN of N, α can be extended to a unique continuous mapping from βN onto $\bar{D}$, which we also denote by α. Further α maps the growth $N^*=\beta N-N$ onto $\bar{D}$ by the compactness of $\beta N-N$. Fix an interpolating sequence $S=\{z_n: z_n \in D\}$ for H^∞, which converges to λ of T. Identify n in N with z_n in D. Then we obtain an embedding of βN into $M(H^\infty)$, by which N^* corresponds to a compact, totally disconnected subset Y of the fiber $M_\lambda=\{m \in M(H^\infty): m(z)=\lambda\}$. Thus given a mapping γ from Y onto $\bar{D}$, the mapping γ^* of $C(\bar{D})$ into $C(Y)$ is defined by $\gamma^*(f)=f \circ \gamma$. It is clear that γ^* is isometric.

We denote by ρ the restriction mapping of H^∞ into $C(Y)$, via the Gelfand transform $\hat{f}$ of f; $\rho(f)=\hat{f}|_Y$ for f in H^∞. As S is interpolating, ρ is surjective. Define B_A and B_C to be $\rho^{-1}\circ\gamma^*(A)$ and $\rho^{-1}\circ\gamma^*(C(\bar{D}))$, respectively. B_A and B_C are closed subalgebras satisfying the conditions $A\subsetneqq B_A\subsetneqq B_C\subsetneqq H^\infty$, $M(B_A)=M(B_C)$. The above is shown in [13].

LEMMA 1 (Newmann [8, p. 179]). *For m in $M(H^\infty)$, the following three conditions are equivalent;*

(a) *m is in the Silov-boundary,*

(b) *$|m(b)|=1$ for all Blaschke products b,*

(c) *$m(b)\neq 0$ for all Blaschke products b.*

PROPOSITION 1. *The Silov-boundaries of B_A and B_C are equivalent to the Silov-boundary of H^∞. B_A and B_C are not logmodular on X.*

PROOF. Define $y\sim y'$ for y, y' in Y by $\gamma(y)=\gamma(y')$. We can identify $M(B_C)$ with $(M(H^\infty)-Y)\cup(Y/\sim)$ and we have $Y/\sim\,\cong\bar{D}$. Lemma 1 says that the Silov-boundary X of H^∞ is disjoint with Y, since the interpolating Blaschke product $b(S)$ is zero on Y but its modulus is 1 on X, where $b(S)$ is associated with S and defined by

$$b(S)(z)=\prod \frac{|z_n|}{z_n}\,\frac{z_n-z}{1-\bar{z}_n z}\ .$$

Therefore X is embedded in $M(B_C)$. Hence X is a boundary for B_A and for B_C.

It remains to show that X is minimal one. For any open set U in X, there exists g in H^∞ such that

$$\max_{X-U} |\hat{g}(m)| \lneqq \|g\|\ .$$

Since the product $g\cdot b(S)$ is zero on Y in the case of $g\notin B_A$, $g\cdot b(S)$ is in B_A and it satisfies

$$\max_{X-U}|\hat{g}\hat{b}(S)|=\max_{X-U}|\hat{g}|\lneqq\|g\|=\|g\cdot b(S)\| .$$

Hence X is the Silov-boundary of B_A and B_C.

Further we shall show that B_A and B_C are not logmodular on X. In fact, it is sufficient to remark that H^∞ is logmodular on X and that the mapping from Y onto $\bar{D}$ is not injective. The last assertion is clear, because Y is disconnected and, in contrast, $\bar{D}$ is connected. Let y and y' be in Y with $y\sim y'$. Then for y (resp. y') there exists a unique representing measure μ (resp. μ') supported on X. Hence we have at least two different representing measures for $[y]$ in $Y/\sim$. This implied that neither of two algebras B_A and B_C is logmodular on X. The proof of Proposition 1 is complete.

Let X' be a separable, connected, compact, Hausdorff space. Then we define an analytic subalgebra $B_S(X')$ in the same manner as we did for B_A and for B_C; by fixing an arbitrary uniform algebra B on X', replacing $\bar{D}$ with X' and regarding γ^* as a mapping from $C(X')$ into $C(Y)$, we define

$$B_S(X')=\rho^{-1}\circ\gamma^*(B) .$$

When $X'=\bar{D}$, the analytic subalgebras B_A and B_C are nothing but $A_S(\bar{D})$ $(=\rho^{-1}\circ\gamma^*(A))$ and $C(\bar{D})_S(\bar{D})$ $(=\rho^{-1}\circ\gamma^*(C(\bar{D})))$, respectively. For notational convenience, we shall often use B_S instead of $B_S(X')$.

THEOREM 1. *The Silov-boundary of $B_S(X')$ is X. $B_S(X')$ is not logmodular.*

PROOF. The proof is exactly the same as the proof of Proposition 1 with $\bar{D}$ replaced by X' and with B_C (or B_A) replaced by $B_S(X')$.

By Theorem 1, the class of analytic subalgebras $B_S(X')$ is of different type from the class of $H^\infty\cap C_B$, because $H^\infty\cap C_B$ is logmodular on $\partial(H^\infty\cap C_B)$ $(=M(C_B))$ and $M(C_B)\neq M(C_{B'})$ if $C_B\neq C_{B'}$.

We can characterize $\partial(B_S(X'))$ in analogous way as for H^∞ in Lemma 1.

THEOREM 2. *For m in $M(B_S(X'))$, the following conditions are equivalent;*

(a) *m is in $\partial(B_S)$*

(b) *$|m(b)|=1$ for all Blaschke products in B_S*

(c) *$m(b)\neq 0$ for all Blaschke products in B_S.*

PROOF. By Lemma 1 and by Theorem 1, it is clear that (a) implies (b). That (b) implies (c) is trivial. Now suppose (c). So we may assume that $m(b(S))=1$ for the interpolating Blaschke product $b(S)$. Now define a complex homomorphism $\tilde{m}$ from H^∞ into C such that $\tilde{m}(f)=m(f\cdot b(S))$ for any f in H^∞. This mapping $\tilde{m}$ is well-defined because of $b(S)\cdot H^\infty\subset B_S$, and $\tilde{m}$ is in $M(H^\infty)$ (details are given in [13]). For any Blaschke product b' in H^∞, $\tilde{m}(b')$ is not zero, since $m(b'\cdot b(S))\neq 0$ by (c). Hence by Lemma 1, $\tilde{m}$ is in X. Since $X\cap Y=\emptyset$, $\tilde{m}$ is identified with m (see [13]). Thus (c) implies (a). This completes the proof.

We remark that the equivalence of (a) and (b) in Theorem 2 is also true for the case $H^\infty\cap C_B$ (see [4]).

The following theorem for the case A was proved by Fisher [7], for H^∞ by Marshall [9] and for a general $H^\infty\cap C_B$ by Chang and Marshall [4].

LEMMA 2 (Bernard, Garnett and Marshall [2]). *Let B be a uniform algebra on a compact Hausdorff space $\tilde{X}$, and let $U(B)$ be $\{u\in B; |u(x)|=1$ on $\tilde{X}\}$. If B satisfies the following conditions*

(1) *the elements of $U(B)$ separate the points of $\tilde{X}$,*

(2) *for f in $C(\tilde{X})$ with $\|f\|<1$, there is g in $C(\tilde{X})$ such that $|g|=1$ and $f-g$ is in B,*

then the unit ball of B is the norm closed convex-hull of $U(B)$.

THEOREM 3. *The closed unit ball of $B_S(X')$ is the norm closed convex-hull of the Blaschke products in $B_S(X')$.*

PROOF. First we remark that H^∞ satisfies (1) and (2) in Lemma 2 by considering H^∞ as a uniform algebra on X. By Theorem 1, $B_S(X')$ is considered as a uniform algebra on X, via the Gelfand transform. B_S satisfies the conditions (1) and (2). In fact, for m_1 and m_2 in X, we choose u in $U(H^\infty)$ such that $m_1(u)\neq m_2(u)$. If u is not in $U(B_S(X'))$, then we only consider the function $ub(S)$ in $B_S(X')$. Since $m_1(ub(S))\neq m_2(ub(S))$, $B_S(X')$ satisfies (1). Suppose that f be in L^∞ $(=C(X))$ with $\|f\|<1$. Then $f\bar{b}(S)$ is in L^∞ with $\|f\bar{b}(S)\|<1$, where $\bar{b}(S)$ is the complex conjugate of $b(S)$. From (2) for H^∞, there exists u in $U(L^\infty)$ such that $f\bar{b}(S)-u$ is in H^∞. Hence $f-ub(S)=gb(S)$ for some g in H^∞. That $gb(S)$ is in $B_S(X')$ implies (2). Therefore the unit ball of $B_S(X')$ is the norm closed convex-hull of $U(B_S(X'))$.

We remark that if u is an inner function of B_S, then by Frosteman's theorem [8, p. 176], u is the limit of the Blaschke products $u_\lambda=(u-\lambda)/(1-\bar{\lambda}u)$ for $\lambda\in D$, as $\lambda\to 0$. These Blaschke products are in B_S. Indeed, by the definition of $B_S(X')$, there exists F in the uniform algebra B on X' such that $\gamma^*(F)=\rho(u)$ and $\|F\|\leqq 1$, because γ^* is isometric. Since $1-\bar{\lambda}F\in B$ satisfies $\|1-(1-\bar{\lambda}F)\|<1$, the element $1-\bar{\lambda}\cdot F$ is invertible in B. Therefore $1/(1-\bar{\lambda}\cdot u)$ is in $B_S(X')$. Hence a Blaschke product u_λ is in B_S. This completes the proof.

THEOREM 4. *An analytic subalgebra $B_S(X')$ is U^*-invariant.*

PROOF. Let g be in B_S. Then there is a G in B such that $\gamma^*(G)=\rho(g)$ and $U^*(g)$ is in H^∞ because of the U^*-invariance of H^∞. As $Y=\beta N-N$ is a closed subset of the fiber M_λ, we have that

$$\rho(U^*(g))=\widehat{U^*(g)}|_Y=(\hat{g}|_Y-g(0))/\lambda=\gamma^*((G-g(0))/\lambda)\ .$$

This implies that $U^*(g)$ is in B_S.

§ 2. Answer to Problem P_1.

We introduce the algebras QC, defined by $QC=(H^\infty+C(T))\cap\overline{(H^\infty+C(T))}$, and QA, defined by $QA=H^\infty\cap QC$; here the bar denotes the complex conjugation.

First we quote a lemma from [5] which gives a necessary and sufficient condition for the linear span $E+F$ of two closed subspaces E and F of a general Banach space B to be closed. Let $\|\ \ \|$ be the norm in B, if E is any subspace of B, and let $f\in B$. Then the distance of f to E is given by

$$d(f, E)=\inf\{\|f-g\|,\ g\in E\}\ .$$

LEMMA 3 (Davie, Gamelin and Garnett [5]). *If E and F are arbitrary closed subspaces of a Banach space B, then the following assertions are equivalent*:

(1) *There is a constant $K>0$ such that*

$$d(f, E\cap F)=K\cdot d(f, E)\ ,\quad \forall f\in F\ .$$

(2) *$E+F$ is a closed subspace of B.*

The next lemma due to D. Sarason [14], together with Lemma 3, shows that the linear span $B+C(T)$ is a closed subspace of L^∞, when B is an analytic subalgebra.

LEMMA 4 (Sarason [14]). *For every f in $C(T)$,*

$$d(f, A)=d(f, H^\infty)\ .$$

We can now give an answer to Problem P_1, posed in the introduction. The next theorem is independently obtained by T. Nakazi [12].

THEOREM 5. *Let B be an analytic subalgebra. The linear span $B+C(T)$ is a closed subalgebra of L^∞ if and only if B is U^*-invariant.*

PROOF. If B is U^*-invariant, then $\bar{z}^n B$ is in $B+C(T)$ for any n in N. The subspace $\bigcup_{i=1}^\infty \bar{z}^n B$ is an algebra; its closure is also an algebra and contains $B+C(T)$. Since $B+C(T)$ is closed, it must equal the closure of $\bigcup_{i=1}^\infty \bar{z}^n B$, and therefore $B+C(T)$ is an algebra. Suppose $B+C(T)$ be an algebra. Then for $f \in B$ and $\bar{z} \in C(T)$, $\bar{z}(f-f(0))$ is written as $\bar{z}(f-f(0))=g+c$ for some function $g+c$ in $B+C(T)$. $f-f(0)-zg$ $(=zc)$ in is $B \cap C(T)$, namely, it is in A. From the U^*-invariance of A, we have $f-f(0)-zg=zk$ for some k in A. Hence $\bar{z}(f-f(0))=k-g$ is in B. This shows the U^*-invariance of B.

COROLLARY. *If B is a U^*-invariant analytic subalgebra, then B is maximal in the algebra $B+C(T)$.*

PROOF. Suppose $\tilde{B}$ is a closed subalgebra of $B+C(T)$ with $B \subsetneq \tilde{B}$. Then there is $h \in \tilde{B}-B$, which is written as $h=f+c$ for $f \in B$ and $c \in C(T)$. That $h-f$ is in $(\tilde{B} \cap C(T))-B$ implies $\tilde{B} \supset C(T)$ by the Wermer's maximality theorem. Therefore we conclude $\tilde{B}=B+C(T)$.

In [1], Adamiyan, Arov and Krein give an example of a function $v \in C(T)$ which has no nearest element in A, i.e., $d(v, A)<d(v, g)$, $\forall g \in A$. Their example can be used to show that $A \subsetneq QA$. However we prove this fact as a consequence of the above theorem.

LEMMA 5. *QA is U^*-invariant.*

PROOF. Let g be in QA. Then $g-g(0)$ can be written as $\bar{f}+\bar{c}$ for some $f \in H^\infty$ and $c \in C(T)$. Hence $\bar{z}(\bar{f}+\bar{c})$ is in $\overline{H^\infty+C(T)}$, namely, $U^*(g)=\bar{z}(g(z)-g(0))$ is in QA.

COROLLARY. *QA properly contains A.*

PROOF. We must remark that the mapping; $B \to B+C(T)$, from the family of analytic subalgebras B to the family of the linear span $B+C(T)$ is injective. Algebras A and QA are U^*-invariant. From Theorem 5 we have that $[A, \bar{z}]=C(T)$ and $[QA, \bar{z}]=QA+C(T)$ $(=QC)$. If $A=QA$, then $C(T)=QC$. This contradicts the fact $C(T) \subsetneq QC$.

§ 3. Non-U^*-invariant analytic subalgebras.

It would be interesting whether every analytic subalgebra is U^*-invariant. However Theorem 7 shows that this question has a negative answer. D. Stegenga reports in [16] the similar result to Theorem 7 but, to my knowledge, the details have not been published. We shall indicate the proof for the sake of completeness.

Let u be an inner function. The support of u, $\operatorname{supp} u$, is the set of points $\lambda \in T$, for which there is a sequence $\{z_n\}$ of points in D such that $z_n \to \lambda$ and $u(z_n) \to 0$. Clearly $\operatorname{supp} u$ is closed in T. A linear span $uH^\infty + A$ is an algebra lying between A and H^∞. Concerning the closedness of $uH^\infty + A$ and of $uH^\infty + C(T)$, we quote a corollary in [17].

LEMMA 6 (Stegenga [17]). *Let u be an inner function. A necessary and sufficient condition for the linear spans $uH^\infty + C(T)$ and $uH^\infty + A$ to be closed is that either* $\operatorname{supp} u$ *has Lebesgue measure zero or* $\operatorname{supp} u = T$.

We give here some properties of a class of analytic subalgebras of $(uH^\infty + A)$-type.

LEMMA 7. *Let u be an inner function. The algebra $uH^\infty + A$ coincides with H^∞ if and only if u is a finite Blaschke product, i.e.,* $\operatorname{supp} u = \emptyset$.

PROOF. The proof of sufficiency is due to Muto [11]. We now prove the necessity. Let us suppose that u is not a finite Blaschke product. Then $\operatorname{supp} u$ is nonempty set. By passing to a subsequence, we may assume that there is an interpolating sequence $S = \{z_n\}$ in D such that $z_n \to \lambda$ for some $\lambda \in \operatorname{supp} u$ and $u(z_n) \to 0$. Identify n in N with z_n in D. Then we obtain an embedding of βN into $M(H^\infty)$, by which $\beta N - N$ corresponds to a closed subset Y of M_λ. As S is an interpolating sequence, i.e.,

$$H^\infty|_S = l^\infty(S) = C(\beta N) ,$$

we have that $\widehat{H^\infty}|_Y = C(Y)$. On the other hand, $\widehat{uH^\infty + A}|_Y$ contains no fuuction which is not constant. It follows that $uH^\infty + A \subsetneqq H^\infty$, when $\operatorname{supp} u \neq \emptyset$.

THEOREM 6. *Suppose $uH^\infty + A$ is an analytic subalgebra with* $\operatorname{supp} u \neq \emptyset$. *Then $uH^\infty + A$ is neither QA nor $H^\infty \cap C_B$ for any Douglas algebra B.*

PROOF. R. A. Douglas remarks in [6] that an inner function in QA

is a finite Blaschke product. Hence we have $uH^\infty + A \neq QA$ from the above lemma.

If $uH^\infty + A = H^\infty \cap C_B$ for some Douglas algebra B, then for any g in H^∞ with $\|g\| < 1$, there exists a unimodular function v in $(g \cdot \bar{u} + H^\infty) \cap C_B$ (see Corollary 2.2 in [4]). Therefore v can be written as $v = g\bar{u} + h$ for some $h \in H^\infty$, i.e., $g = vu - hu$. From $uH^\infty \subset C_B$, g is in C_B, namely, H^∞ is contained in C_B. This contradicts the assumption that $uH^\infty + A \subsetneq H^\infty$. This completes the proof.

We now return to the characterization of U^*-invariant analytic subalgebras. To check the U^*-invariance of an algebra $uH^\infty + A$, the following lemma is useful.

LEMMA 8 (Muto [11]). *Let $uH^\infty + A$ be an analytic subalgebra. Then $uH^\infty + A$ is U^*-invariant if and only if $U^*(u)$ is in $uH^\infty + A$.*

THEOREM 7. *An algebra $uH^\infty + A$ is U^*-invariant analytic subalgebra if and only if* supp u *has Lebesgue measure zero.*

PROOF. From Lemma 6, $uH^\infty + A$ is analytic subalgebra if and only if the Lebesgue measure of supp u is 0 or 1. T. Muto [11] shows that if supp u has Lebesgue measure zero, then $U^*(u)$ is in $uH^* + A$. Hence, by Lemma 8, $uH^* + A$ is U^*-invariant. Now we shall show that if supp $u = T$, then $U^*(u)$ is not in $uH^* + A$. It will be done by contradiction. We remark by Privaloff's theorem [8, p. 58] that if supp u has positive measure, then $uH^\infty \cap A = \{0\}$. Suppose now that supp $u = T$ and $U^*(u) \in uH^\infty + A$. Then $U^*(u)$ can be written as

$$U^*(u) = e^{-i\theta} \cdot (u(e^{i\theta}) - u(0)) = (u \cdot h_1 + h_2)(e^{i\theta})$$

for some $h_1 \in H^\infty$ and some $h_2 \in A$. So the function

$$u(e^{i\theta})(1 - e^{i\theta} \cdot h_1(e^{i\theta})) = u(0) + e^{i\theta} \cdot h_2(e^{i\theta})$$

is in $A \cap u \cdot H^\infty$. Therefore we have that $1 - e^{i\theta} \cdot h_1(e^{i\theta}) = 0$ on T, namely $e^{i\theta}$ is invertible in H^∞. This is a contradiction. Hence for an inner function u with supp $u = T$, the analytic subalgebra $uH^\infty + A$ is not U^*-invariant. This completes the proof.

We remark that Theorem 7 can be stated as follows; an algebra $uH^\infty + A$ is an analytic subalgebra which is not U^*-invariant, if and only if supp $u = T$.

Theorem 7, together with Theorem 5, shows that the linear space $uH^\infty + C(T)$ is not an algebra when supp $u = T$, i.e., $uH^\infty + C(T) \subsetneq [uH^\infty +$

$A, \bar{z}]$. In the following corollary, we shall give a concrete form to the algebra $[uH^\infty + A, \bar{z}]$, and show that the maximality of $uH^\infty + A$ in $[uH^\infty + A, \bar{z}]$ fails, in contrast with the first corollary of Theorem 5.

For f and g in L^∞ and λ a point in T, we define a local distance;

$$\operatorname{dist}_\lambda(f, g) = \text{ess.} \limsup_{e^{i\theta} \to \lambda} |f(e^{i\theta}) - g(e^{i\theta})| .$$

If we extend f and g harmonically to D, then we also have

$$\operatorname{dist}_\lambda(f, g) = \limsup_{z \to \lambda, z \in D} |f(z) - g(z)| .$$

For f in L^∞ and for a closed subset B in L^∞, we define

$$\operatorname{dist}_\lambda(f, B) = \inf \{\operatorname{dist}_\lambda(f, h) : h \in B\} .$$

It is clear that $\operatorname{dist}_\lambda(f, g) \leqq \|f - g\|$, and that the function $\operatorname{dist}_\lambda(f, H^\infty)$ is upper semi-continuous with respect to λ. So this function attains a maximum on T. We quote a lemma from [15].

LEMMA 9 (Sarason [15]). *If f is in L^∞, then*

$$d(f, H^\infty + C(T)) = \max \{\operatorname{dist}_\lambda(f, H^\infty) : |\lambda| = 1\} .$$

COROLLARY. *Let u be an inner function with* $\operatorname{supp} u = T$. *Then the closed subalgebra $[uH^\infty + A, \bar{z}]$ is nothing but $u(H^\infty + C(T)) + C(T)$. Moreover there exists a closed subalgebra $u(H^\infty + C(T)) + A$ satisfying*

$$uH^\infty + A \subsetneqq u(H^\infty + C(T)) + A \subsetneqq u(H^\infty + C(T)) + C(T) .$$

PROOF. First we shall show that (a) implies (b);

(a) there is an $\varepsilon > 0$ such that $\operatorname{dist}_\lambda(\bar{u}, H^\infty + C(T)) \geqq \varepsilon$ for all λ on T,

(b) $u(H^\infty + C(T)) + C(T)$ is closed.

By the above lemma, we have that for any c' in $C(T)$,

$$\begin{aligned} \operatorname{dist}(\bar{u}c', H^\infty + C(T)) &= \max \{\operatorname{dist}_\lambda(\bar{u}c', H^\infty) : |\lambda| = 1\} \\ &\geqq \|c'\| . \end{aligned}$$

Thus we know that $u(H^\infty + C(T)) \cap C(T) = \{0\}$.

Let g be in $C(T)$. Suppose that $g(\lambda) \neq 0$, $\lambda \in T$ and that $h + c$ in $H^\infty + C(T)$. Then for z in T,

$$\begin{aligned} |g(\lambda)| \cdot |\bar{u}(z) - (h + c)(z)| &= |(g(\lambda) - g(z)) \cdot \bar{u}(z) - g(\lambda)(h + c)(z) + g(z)\bar{u}(z)| \\ &\leqq |g(\lambda) - g(z)| + \|g - g(\lambda)u(h + c)\| . \end{aligned}$$

Therefore

$$g(\lambda)\cdot \mathrm{dist}_\lambda(\bar{u}, H^\infty+C(T)) \leqq d(g, u(H^\infty+C(T))) .$$

Since

$$d(g, u(H^\infty+C(T))\cap C(T)) = \|g\|$$
$$\leqq (1/\varepsilon) d(g, u(H^\infty+C(T))) ,$$

the linear span $u(H^\infty+C(T))+C(T)$ is closed, from Lemma 3.

Now we must show that (a) is valid. In fact, for $h+c$ in $H+C(T)$, we have

$$\begin{aligned}\mathrm{dist}_\lambda(\bar{u}, h+c) &= \text{ess.} \limsup_{e^{i\theta}\to\lambda} |(\bar{u}-h-c)(e^{i\theta})| \\ &= \text{ess.} \limsup |1-u(e^{i\theta})\cdot(h+c)(e^{i\theta})| \\ &\geqq 1 .\end{aligned}$$

Once we know that $u(H^\infty+C(T))+C(T)$ is closed, it is easy to prove that this is an algebra generated by $\bar{z}$ and by $uH^\infty+A$. In fact, for any n in N, the subspace $z^{-n}(uH^\infty+A)$ is contained in $u(H^\infty+C(T))+C(T)$. The subspace $\bigcup_{i=n}^\infty z^{-n}(uH^\infty+A)$ is an algebra, and its closure is also algebra and contains $u(H^\infty+C(T))+C(T)$. Hence the closure of $\bigcup_{i=n}^\infty z^{-n}(uH^\infty+A)$ is equal to $u(H^\infty+C(T))+C(T)$.

(a) implies that the subspace $u(H^\infty+C(T))+A$ is closed. And it is easy to see that this is a desired algebra.

§ 4. Answer to Problem P_2.

We shall be concerned in the remainder of this paper with Problem P_2 posed in the introduction.

The following theorem shows that every analytic subalgebra $uH^\infty+A$ which is not U^*-invariant can always have a superalgebra of $uH^\infty+A$ in L^∞ which does not contain $C(T)$.

THEOREM 8. *Let u be an inner function with* $\operatorname{supp} u=T$. *And let E be a closed, nowhere dense subset of T, whose Lebesgue measure is positive, and χ_E be the characteristic function of E. Then the closed subalgebra $[uH^\infty+A, \chi_E]$ of L^∞ does not contain $C(T)$.*

PROOF. The functions $\chi_E\cdot k+g$ are dense in $[uH^\infty+A, \chi_E]$, where k and g are in $uH^\infty+A$. We have

$$\begin{aligned}\mathrm{dist}_\lambda(\bar{z}, \chi_E\cdot k+g) &= \mathrm{dist}_\lambda(\bar{z}, g) \quad \text{for } \lambda \text{ in } T-E , \\ &= \mathrm{dist}_\lambda(\bar{z}, f+uh) \quad \text{for } g=f+uh , \\ &= \limsup_{z\to\lambda, |z|<1} |1-zf-zuh| \geqq \limsup_{z\to\lambda, |z|<1} |1-zf| \\ &= |1-\lambda f(\lambda)| .\end{aligned}$$

Hence

$$\|\bar{z}-(\chi_E\cdot k+g)\|\geqq\sup_{\lambda\in T-E}|1-\lambda f(\lambda)|\ .$$

By using here the continuity of zf on T and the fact that zf is not invertible in A, we have

$$\sup_{\lambda\in T-E}|1-\lambda f(\lambda)|=\|1-zf\|\geqq 1\ .$$

This shows that $\bar{z}$ is not in $[uH^\infty+A, \chi_E]$, as is to be proved.

We can now obtain the main result of this paper, which is an answer to P_2.

THEOREM 9. *Let B be an analytic subalgebra which contains an ideal uH^∞ of H^∞, where u is an inner function with* $\operatorname{supp} u\subsetneqq T$. *Then every closed superalgebra $\tilde{B}$ of B in L^∞ satisfying $\tilde{B}\not\subset H^\infty$ contains $C(T)$.*

PROOF. From $uH^\infty\subset B$, we have $H^\infty\subset\bar{u}\tilde{B}$ and $\tilde{B}\subset\bar{u}\tilde{B}$. The closed subspace $\bar{u}\tilde{B}$ is not necessarily an algebra. However, from the property $H^\infty\cdot\tilde{B}\subset\bar{u}\tilde{B}$, we see that the algebra $[H^\infty, \tilde{B}]$ generated by H^∞ and $\tilde{B}$ is contained in $\bar{u}\tilde{B}$. Since $H^\infty+C(T)$ is the minimum superalgebra of H^∞ in L^∞, $H^\infty+C(T)$ is contained in $[H^\infty, \tilde{B}]$ and so in $\bar{u}\tilde{B}$. Therefore $C(T)$ is contained in $\bar{u}\tilde{B}$, namely, $uC(T)\subset\tilde{B}$. The condition $\operatorname{supp} u\subsetneqq T$ implies that $uC(T)\cap C(T)$ is not contained in A. Hence $\tilde{B}\cap C(T)$ contains A properly. By the Wermer's maximality theorem we conclude that $C(T)$ is contained in $\tilde{B}$.

The author is indebted to K. Izuchi for the proof of the above theorem. The original proof by the author has an obscure point and is more complicated.

Applying this theorem to certain U^*-analytic subalgebras, we have the following results.

COROLLARY. *Let $\tilde{B}$ be any superalgebra of $B_S(X')$ in L^∞ with $\tilde{B}\not\subset H^\infty$. Then $\tilde{B}$ contains the analytic subalgebra $B_S(X')+C(T)$.*

PROOF. $B_S(X')$ contains an ideal $b(S)\cdot H^\infty$ by the definition and is U^*-invariant by Theorem 4. Therefore we can apply Theorems 5 and 9 to $B_S(X')$ to obtain $B_S(X')+C(T)\subseteqq\tilde{B}$.

COROLLARY. *Let $uH^\infty+A$ be an analytic subalgebra. Then a necessary and sufficient condition for a superalgebra $\tilde{B}$ of $uH^\infty+A$ in L^∞ with $\tilde{B}\not\subset H^\infty$ to contain $C(T)$ is that* $\operatorname{supp} u$ *has Lebesgue measure zero.*

PROOF. We see by Lemma 6 that $\operatorname{supp} u$ has measure zero or $\operatorname{supp} u=T$. In the first case, we can apply Theorem 9 to $uH^{\infty}+A$. In the second case, Theorem 8 shows that there exists a superalgebra $\tilde{B}$ $(=[uH^{\infty}+A, \chi_E])$ of $uH^{\infty}+A$ such that $\tilde{B}\not\subset H^{\infty}$ and $C(T)\not\subset\tilde{B}$.

References

[1] V. M. ADAMIAN, D. Z. AROV and M. G. KREĬN, On infinite Hankel matrices and generalized problems of Carathéodory-Fejer and F. Riesz, Funkcional. Anal. i Priložen., **2** (1968) vyp. 1, 1–19.

[2] A. BERNARD, J. B. GARNETT and D. E. MARSHALL, Algebras generated by inner functions, J. Functional Analysis, **25** (1977), 275–289.

[3] S-Y. CHANG, A Characterization of Douglas subalgebras, Acta Math., **137** (1976), 81–89.

[4] S-Y. CHANG and D. E. MARSHALL, Some algebras of bounded analytic functions containing the disc algebra, Banach Spaces of Analytic Functions, Lecture Notes in Math., no. **604**, Springer, Berlin-Heidelberg-New York, 1977, 12–20.

[5] A. M. DAVIE, T. W. GAMELIN and J, GARNETT, Distance estimates and point-wise bounded density, Trans. Amer. Math. Soc., **175** (1973), 37–68.

[6] R. G. DOUGLAS, Banach Algebra Techniques in Operator Theory, Academic Press, New York, 1972.

[7] S. FISHER, The convex hull of the finite Blaschke products, Bull. Amer. Math. Soc., **74** (1968), 1128–1129.

[8] K. HOFFMAN, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, NJ, 1962.

[9] D. E. MARSHALL, Blaschke products generate H^{∞}, Bull. Amer. Math. Soc., **82** (1976), 494–496.

[10] D. E. MARSHALL, Subalgebras of L^{∞} containing H^{∞}, Acta Math., **137** (1976), 91–98.

[11] T. MUTO, Remarks on some algebras of H^{∞} containing A, preprint.

[12] T. NAKAZI, A note on maximal subalgebras on the circle, preprint.

[13] K. NISHIZAWA, On closed subalgebras between A^{∞} and H^{∞}, Tokyo J. Math., **3** (1980), 137–140.

[14] D. SARASON, Algebras of functions on the unit circle, Bull. Amer. Math. Soc., **79** (1973), 286–299.

[15] D. SARASON, On products of Toeplitz operators, Acta Sci. Math. (Szeged), **35** (1973), 7–12.

[16] D. STEGENGA, Sums of invariant subspaces, Notices Amer. Math. Soc., **22** (1975), A-736.

[17] D. STEGENGA, Sums of invariant subspaces, Pacific J. Math., **70** (1977), 567–584.

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