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Fourier Ultra-Hyperfunctions Valued in a Fréchet Space

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Introduction

When the theory of Sato-hyperfunctions appeared in 1958, J. Sebastiaõ e Silva attempted to construct a space of ultra-distributions which contains the space $\mathfrak{H}'$ of tempered ultra-distributions and the space H' of all distributions of exponential growth and is stable under the Fourier transformation. He defined the space which he named the space of ultra-distributions of exponential type and obtained some important results for one-dimensional case [10].

(On the other hand, he studied the space $\mathfrak{H}'$ of tempered ultra-distributions for the one-dimensional space. Hasumi [1] extended the space for the n-dimensional space and obtained some valuable results.)

The n-dimensional case was studied by Y. S. Park and M. Morimoto [11]. We defined the space $Q(\boldsymbol{C}^n)$ which was included and dense in the spaces $H(\boldsymbol{R}^n)$ and $\mathfrak{H}(\boldsymbol{C}^n)$ and stable under the Fourier transformation. The dual space $Q'(\boldsymbol{C}^n)$ of $Q(\boldsymbol{C}^n)$ includes the spaces $H'(\boldsymbol{R}^n)$ and $\mathfrak{H}'(\boldsymbol{C}^n)$. The elements of the dual space $Q'(\boldsymbol{C}^n)$ are called the Fourier ultra-hyperfunctions in the Euclidean n-space.

The extension of the theory of Fourier hyperfunctions in T. Kawai [5] to vector valued case was studied by Y. Ito [2], Y. Ito and S. Nagamachi [3], [4], and other mathematicians.

In this paper, we establish the theory of Fourier ultra-hyperfunctions valued in a Fréchet space.

Our results are roughly as follows. Let $Q_b(T^n(K); K')$ be the space of all continuous functions f on $\boldsymbol{R}^n+iK$ which are holomorphic in the interior of $\boldsymbol{R}^n+iK$ and satisfy the estimate:

$$\sup\{\exp(\langle x, \eta\rangle)|f(z)|;\ \eta\in K',\ z\in \boldsymbol{R}^n+iK\}<\infty\ ,$$

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where $K \subset \boldsymbol{R}_x^n$(resp. $K' \subset \boldsymbol{R}_\xi^n$) is a convex compact set with non-void interior $\mathring{K}$(resp. $\mathring{K}'$). We put

$$Q(T^n(K); K') = \lim_{\substack{K \subset\subset L \\ K' \subset\subset L'}} \operatorname{ind} Q_b(T^n(L); L'),$$

where $T^n(K) = \boldsymbol{R}^n + iK$ and $L \subset \boldsymbol{R}_x^n$ and $L' \subset \boldsymbol{R}_\xi^n$ are convex compact sets. For an open set U of $\boldsymbol{R}_x^n$ and an open set U' of $\boldsymbol{R}_\xi^n$, we put

$$Q(T^n(U); U') = \lim_{\substack{K \subset\subset U \\ K' \subset\subset U'}} \operatorname{proj} Q(T^n(K); K').$$

Then the spaces $Q(T^n(K); K')$ and $Q(T^n(U); U')$ are nuclear (Theorem 2.1). Therefore, the ε-topology coincides with the π-topology on the tensor product space $Q(T^m(U); U') \otimes Q(T^n(V); V')$. The space $Q(T^m(U); U') \otimes Q(T^n(V); V')$ is dense in the space $Q(T^{m+n}(U \times V); U' \times V')$ (Theorem 2.3). The induced topology on the tensor product space $Q(T^m(U); U') \otimes Q(T^n(V); V')$ from the original topology of the FS-space $Q(T^{m+n}(U \times V); U' \times V')$ coincides with the π-topology.

Consequently we have the canonical isomorphisms:

$$Q(T^m(U); U') \hat{\otimes} Q(T^n(V); V') \cong Q(T^{m+n}(U \times V); U' \times V') \quad \text{and}$$
$$Q(T^m(K); K') \hat{\otimes} Q(T^n(L); L') \cong Q(T^{m+n}(K \times L): K' \times L').$$

We have the similar canonical isomorphisms for the dual spaces $Q'(T^m(U); U')$, $Q'(T^m(K); K')$, $Q'(\boldsymbol{C}^m)$ and $Q'(\boldsymbol{R}^m)$.

For a Fréchet space E, we define $Q'(T^n(U), U'; E)$ to be the space of all continuous linear mappings from $Q(T^n(U); U')$ to E. Similarly we define $Q'(T^n(K), K'; E)$. $Q'(T^n(\boldsymbol{R}^n), \boldsymbol{R}^n; E)$ is denoted by $Q'(\boldsymbol{C}^n; E)$ and $Q'(T^n(0), (0); E)$ is denoted by $Q'(\boldsymbol{R}^n; E)$. An element of $Q'(\boldsymbol{C}^n; E)$ (resp. $Q'(\boldsymbol{R}^n; E)$) is called a Fourier ultra-hyperfunction (resp. Fourier hyperfunction) valued in the Fréchet space E. We prove the canonical isomorphisms:

$$Q'(T^n(U), U'; E) \cong Q'(T^n(U); U') \hat{\otimes} E, \quad \text{etc.}.$$

For $f \in Q(T^n(U); U')$ and $g \in Q(T^n(V); V')$, the convolution $f * g$ is defined by $f * g(x+iy) \equiv \int_{\boldsymbol{R}^n} f(u+iv) g((x+iy)-(u+iv)) du$, where $x, u \in \boldsymbol{R}^n$, $v \in U$ and $y \in V+U$. Then $f * g$ belongs to $Q(T^n(U+V); U' \cap V')$, which allows us to define the convolution between $f \in Q(T^n(U'); U)$ and $S \in Q'(T^n(V'), V; E)$.

We have the useful diagrams for the test function spaces and their E-valued spaces (Proposition 4.2 and Theorem 4.8). The Fourier transformation $\mathscr{F}$ establishes a linear topological isomorphism: $Q'(T^n(U), U'; E) \xrightarrow{\sim}$

$Q'(T^n(-U'), U; E)$ (resp. $Q'(T^n(K), K'; E) \tilde{\to} Q'(T^n(-K'), K; E)$).

Section 1 is concerned with definitions and some properties of the test function spaces; section 2 with some theorems for the test function spaces and their tensor products; section 3 with Fourier ultra-hyperfunctions valued in a Fréchet space; and section 4 with multiplication, convolution, and Fourier transformation.

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§1. Definitions and some properties of the test function spaces.

We recall some definitions. $\mathscr{D}(\boldsymbol{R}^n)$ is, by definition, the space of all C^∞-functions on $\boldsymbol{R}^n$ with compact carrier. $\mathscr{S}(\boldsymbol{R}^n)$ is, by definition, the space of all rapidly decreasing C^∞-functions on $\boldsymbol{R}^n$. For an open convex subset U' of $\boldsymbol{R}^n$, let $H(\boldsymbol{R}^n; U')$ be the space of all C^∞-functions $\varphi(x)$ on $\boldsymbol{R}^n$ such that for every element η of U', the function $\exp(\langle\eta, x\rangle)\varphi(x)$ belongs to the space $\mathscr{S}(\boldsymbol{R}^n)$. Then, $\mathscr{D}(\boldsymbol{R}^n) \subset H(\boldsymbol{R}^n) = H(\boldsymbol{R}^n; \boldsymbol{R}^n) \subset H(\boldsymbol{R}^n; U')$ and $\mathscr{D}(\boldsymbol{R}^n)$ is dense in $H(\boldsymbol{R}^n)$ and in $H(\boldsymbol{R}^n; U')$. Let Ω be an open set in $\boldsymbol{C}^n$ and let $\mathscr{O}(\Omega)$ be the space of all holomorphic functions defined on Ω. We endow $\mathscr{O}(\Omega)$ with the topology of uniform convergence on every compact subset of Ω. $\mathscr{O}(\Omega)$ is a Fréchet space.

We will use the notation $T^n(A) = \boldsymbol{R}^n + iA \subset \boldsymbol{R}^n + i\boldsymbol{R}^n = \boldsymbol{C}^n$ for a subset A of $\boldsymbol{R}^n$ in order to clarify the dimension.

Let $\mathfrak{H}(T^n(U'))$ be the space of all holomorphic functions $\psi(\zeta)$ on $T^n(U')$ such that for any $K' \subset\subset U'$ and any $m = 0, 1, \cdots$, $\|\psi\|_{K',m} < \infty$, where $\|\psi\|_{K',m} = \sup\{|\zeta^p\psi(\zeta)|, \zeta \in T^n(K'), 0 \leqq |p| \leqq m\}$. An element of the space $\mathfrak{H}(T^n(U'))$ is called a rapidly decreasing holomorphic function on $T^n(U')$. Then, $\mathfrak{H}(T^n(U'))$ is an FS-space. We denote $\mathfrak{H}(\boldsymbol{C}^n) = \mathfrak{H}(T^n(\boldsymbol{R}^n))$.

DEFINITION 1.1. Suppose a convex compact set $K \subset \boldsymbol{R}^n_x$ (resp. $K' \subset \boldsymbol{R}^n_\xi$) has a nonempty interior $\mathring{K}$ (resp. $\mathring{K}'$). Then we denote by $Q_b(T^n(K); K')$ the space of all continuous functions on $T^n(K)$ which are holomorphic in the interior $T^n(\mathring{K})$ of $T^n(K)$ and satisfy the estimate:

$$\sup\{\exp(\langle x, \eta\rangle)|f(z)|; \eta \in K', z \in T^n(K)\} < \infty \,. \tag{1}$$

$Q_b(T^n(K); K')$ is a Banach space with the norm

$$\|f\|_{K,K'} = \sup\{\exp(\langle x, \eta\rangle)|f(z)|; \eta \in K', z \in T^n(K)\} \,. \tag{2}$$

If $K_1 \subset K_2$ and $K_1' \subset K_2'$ are convex compact sets of $\boldsymbol{R}^n_x$ and $\boldsymbol{R}^n_\xi$, then we have the canonical injection induced by the restriction mapping:

$$Q_b(T^n(K_2); K_2') \hookrightarrow Q_b(T^n(K_1); K_1') . \tag{3}$$

We put

$$Q(T^n(K); K') = \lim_{\substack{K \subset\subset L \\ K' \subset\subset L'}} \operatorname{ind} Q_b(T^n(L); L') , \tag{4}$$

where $L \subset R_x^n$ and $L' \subset R_\xi^n$ are convex compact sets.

For an open convex set U of R_x^n and an open convex set U' of R_ξ^n, we put

$$Q(T^n(U); U') = \lim_{\substack{L \subset\subset U \\ L' \subset\subset U'}} \operatorname{proj} Q_b(T^n(L); L') . \tag{5}$$

The inductive limit and the projective limit are taken following the canonical mappings (3). We will denote $Q(C^n) = Q(T^n(R^n); R^n)$, which is the space of entire functions supra-exponentially decreasing on any horizontal bands. We will denote $Q(R^n) = Q(T^n(0); (0))$. The space $Q(R^n)$ of infra-exponential real analytic functions was denoted by $\underset{\sim}{\mathcal{O}}(D^n)$ in Kawai [5].

PROPOSITION 1.2. *The space $Q(T^n(K); K')$ (resp. $Q(T^n(U); U')$) endowed with the locally convex inductive (resp. projective) limit topology is a DFS-space (resp. an FS-space).*

In fact, the restriction mapping (3) is continuous and compact. The space $Q(T^n(K); K')$ (resp. $Q(T^n(U); U')$) is the inductive (resp. projective) limit of the increasing (resp. decreasing) sequence of Banach spaces by the injective restriction mappings (3). Hence $Q(T^n(K); K')$ (resp. $Q(T^n(U); U')$) is a DFS-space (resp. an FS-space). (See Komatsu [6].)

Q.E.D.

The following proposition asserts the relation between the spaces $Q(T^n(U); U')$ and the spaces $\mathfrak{H}(T^n(U))$.

PROPOSITION 1.3. *For open convex sets $U \subset R_x^n$ and $U' \subset R_\xi^n$, $Q(T^n(U); U')$ is the space of all holomorphic functions $\varphi(z)$ on $T^n(U)$ such that for every $\eta \in U'$, the function $\exp(\langle \eta, z \rangle)\varphi(z)$ belongs to the space $\mathfrak{H}(T^n(U))$.*

We omit the proof of Proposition 1.3. The following lemma is standard and we need it in the proof of the following proposition.

LEMMA 1.4. *Suppose X_j (resp. X'_j) and Y_j (resp. Y'_j) are locally convex vector spaces (resp. dual spaces). Assume that we have the following commutative diagram:*

$$\begin{array}{ccccccc} X_1 & \longleftarrow & X_2 & \longleftarrow & X_3 & \longleftarrow & \cdots \\ \nwarrow & \swarrow & \nwarrow & \swarrow & \nwarrow & \swarrow & \\ & Y_1 & \longleftarrow & Y_2 & \longleftarrow & Y_3 & \longleftarrow \cdots \end{array}$$

$$(resp.\ \begin{array}{ccccccc} X'_1 & \longrightarrow & X'_2 & \longrightarrow & X'_3 & \longrightarrow & \cdots \\ \searrow & \nearrow & \searrow & \nearrow & \searrow & \nearrow & \\ & Y'_1 & \longrightarrow & Y'_2 & \longrightarrow & Y'_3 & \longrightarrow \cdots \end{array}) .$$

Then we have $\lim \operatorname{proj} X_j = \lim \operatorname{proj} Y_j$ *(resp.* $\lim \operatorname{ind} X'_j = \lim \operatorname{ind} Y'_j$*).*

PROPOSITION 1.5.

(i) $Q(T^n(K); K') = \lim\limits_{\substack{K \subset\subset U \\ K' \subset\subset U'}} \operatorname{ind} Q(T^n(U); U')$.

(ii) $Q(T^n(U); U') = \lim\limits_{\substack{K \subset\subset U \\ K' \subset\subset U'}} \operatorname{proj} Q(T^n(K); K')$.

PROOF. (i) By formula (4) in Definition 1.1, we have

$$Q(T^n(K); K') = \lim_{\substack{K \subset\subset L \\ K' \subset\subset L'}} \operatorname{ind} Q_b(T^n(L); L') .$$

We choose a sequence $\{U_j\}$ of open convex sets such that $\cdots L_j \supset U_j \supset L_{j+1} \supset U_{j+1} \supset L_{j+2} \cdots \to K$. We have

$$\begin{array}{ccccc} \cdots \longrightarrow Q_b(T^n(L_j); L'_j) & \longrightarrow & Q_b(T^n(L_{j+1}); L'_{j+1}) \cdots & \longrightarrow & Q(T^n(K); K) \\ \searrow \qquad \nearrow & & \searrow \qquad \nearrow & & \\ \cdots \longrightarrow Q(T^n(U_j); U'_j) & \longrightarrow & Q(T^n(U_{j+1}); U'_{j+1}) & \longrightarrow & \cdots . \end{array}$$

Hence we get, by Lemma 1.4, $\lim \operatorname{proj} Q(T^n(U_j); U'_j) = \lim \operatorname{proj} Q_b(T^n(L_j); L'_j)$, from which results (i).

The proof of (ii) depends also on Lemma 1.4 and is similar to the above. Q.E.D.

§2. Some theorems for the test function spaces and their tensor products.

THEOREM 2.1. *The spaces $Q(T^n(U); U')$ and $Q(T^n(K); K')$ are nuclear, where U (resp. U') is an open convex set in $\boldsymbol{R}^n_x$ (resp. $\boldsymbol{R}^n_\xi$) and K (resp.*

K') is a convex compact set in R^n_x (resp. R^n_ξ).

PROOF. We employ the fact that the space $\mathscr{S}(R^n)$ of rapidly decreasing C^∞-functions in R^n, is nuclear. Let L be a convex compact set in R^n with non-void interior. We define

$$(6)\qquad \mathscr{S}(T^n(L)) \equiv \{f \in C^\infty(T^n(L));\ \text{For any } N, p, q,\ \sup\{|(1+|x|)^N(\partial/\partial x)^p(\partial/\partial y)^q f(x, y)|;\ x \in R^n,\ y \in L\} < \infty\}\ ,$$

where $f \in C^\infty(T^n(L))$ means that f is a C^∞-function on the interior $T^n(\mathring{L})$ and for any multi-indices p, q, the derived functions $(\partial/\partial x)^p(\partial/\partial y)^q f(x, y)$ are continuous on $T^n(L)$. If we define a topology on the space $\mathscr{S}(T^n(L))$ by the seminorms:

$$\|f\|_{N,p,q} = \sup\{|x^N(\partial/\partial x)^p(\partial/\partial y)^q f(x, y)|;\ x \in R^n,\ y \in L\}\ ,$$

then $\mathscr{S}(T^n(L))$ is a nuclear space.

Indeed, $\mathscr{S}(T^n(L))$ is the quotient space of $\mathscr{S}_{x,y}(C^n)$, where $C^n = R^n + iR^n$, by its closed subspace consisting of the functions vanishing on $T^n(L)$. This follows from the Whitney extension theorem applied on the sphere S^n, the compactification of R^n.

For any convex compact set L' and $\varepsilon > 0$, we choose a C^∞-function $\varphi_{L',\varepsilon}(x)$ satisfying the condition

$$\sup_{\eta \in L'} \langle x, \eta \rangle \leqq \varphi_{L',\varepsilon}(x) \leqq \sup_{\eta \in L'_\varepsilon} \langle x, \eta \rangle\ .$$

Then $\mathscr{S}(T^n(L)) \exp(-\varphi_{L',\varepsilon}(x))$ is also a nuclear space. Hence the space $\mathscr{S}(T^n(L)) \exp(-\varphi_{L',\varepsilon}(x)) \cap \mathscr{O}(T^n(\mathring{L}))$ is nuclear. As we have

$$(7)\qquad Q(T^n(K); K') = \operatorname*{lim\,ind}_{\substack{K \subset\subset L \\ K' \subset\subset L' \\ \varepsilon > 0}} \mathscr{S}(T^n(L)) \exp(-\varphi_{L',\varepsilon}(x)) \cap \mathscr{O}(T^n(\mathring{L}))\ ,$$

the space $Q(T^n(K); K')$ is nuclear.

The nuclearity of the space $Q(T^n(U); U')$ results from the nuclearity of the space $Q(T^n(K); K')$ and Proposition 1.5. Q.E.D.

COROLLARY 2.2. *On the tensor product space $Q(T^m(U); U') \otimes Q(T^n(V); V')$ the ε-topology coincides with the π-topology, where $U \subset R^m_x$, $U' \subset R^m_\xi$, $V \subset R^n_x$ and $V' \subset R^n_\xi$ are open convex sets.*

PROOF. See Theorem 50.1(f), p. 511 in F. Trèves [12].

THEOREM 2.3. *The tensor product space $Q(T^m(U); U') \otimes Q(T^n(V); V')$ is dense in the space $Q(T^{m+n}(U \times V); U' \times V')$.*

PROOF. Let $f(z)\in Q(T^{m+n}(U\times V); U'\times V')$. For $\varepsilon>0$, we denote $C(\varepsilon)=(\pi\varepsilon)^{-(m+n)/2}$. Let us define the function $F_\varepsilon(z)$:

$$(8)\qquad \begin{aligned} F_\varepsilon(z)&=f(z) \underset{z}{*} C(\varepsilon)\exp\left(-\frac{1}{\varepsilon}z^2\right)\\ &\equiv C(\varepsilon)\int_{\operatorname{Im} w=(u_0,v_0)} f(w)\exp\left(-\frac{1}{\varepsilon}(z-w)^2\right)dw\,, \end{aligned}$$

where $(u_0, v_0)\in U\times V$ is a fixed vector, $z=(z_1,\cdots,z_{m+n})$, $w=(w_1,\cdots,w_{m+n})$ and $(z-w)^2=\sum_{j=1}^{m+n}(z_j-w_j)^2$. Then $F_\varepsilon(z)\in Q(T^{m+n}(U\times V); U'\times V')$, and $F_\varepsilon(z)$ converges to $f(z)$ in the space $Q(T^{m+n}(U\times V); U'\times V')$ as $\varepsilon\to 0$. On the other hand, the Riemann sum

$$(9)\qquad \sum_{i=1}^{N} f(w_i)C(\varepsilon)\exp\left(-\frac{1}{\varepsilon}(z-w_i)^2\right)\Delta w_i$$

converges to $F_\varepsilon(z)$ as $N\to\infty$ in the space $Q(T^{m+n}(U\times V); U'\times V')$, where $z=(z_1,\cdots,z_{m+n})$, $w_i=(w_{i1},\cdots,w_{i\,m+n})$, $(z-w_i)^2=\sum_{j=1}^{m+n}(z_j-w_{ij})^2$ and $\Delta w_i=\prod_{j=1}^{m+n}\Delta w_{ij}$. As

$$\begin{aligned} &f(w_i)C(\varepsilon)\exp\left(-\frac{1}{\varepsilon}(z-w_i)^2\right)\Delta w_i\\ &\quad=f(w_i)C(\varepsilon)\exp\left(-\frac{1}{\varepsilon}\sum_{j=1}^{m}(z_j-w_{ij})^2\right)\prod_{j=1}^{m}\Delta w_{ij}\\ &\quad\quad\times\exp\left(-\frac{1}{\varepsilon}\sum_{j=m+1}^{m+n}(z_j-w_{ij})^2\right)\prod_{j=m+1}^{m+n}\Delta w_{ij}\,, \end{aligned}$$

the Riemann sum $\sum_{i=1}^{N} f(w_i)C(\varepsilon)\exp(-(1/\varepsilon)(z-w_i)^2)\Delta w_i$ belongs to $Q(T^m(U); U')\otimes Q(T^n(V); V')$. Q.E.D.

THEOREM 2.4. *Let $\mathscr{T}$ be the induced topology on the space $Q(T^m(U); U')\otimes Q(T^n(V); V')$ from the original topology $\mathscr{T}_*$ of the FS-space $Q(T^{m+n}(U\times V); U'\times V')$. Then the topology $\mathscr{T}$ coincides with the π-topology.*

PROOF. By Corollary 2.2, the π-topology coincides with the ε-topology on $Q(T^m(U); U')\otimes Q(T^n(V); V')$. If we equip $Q(T^m(U); U')\otimes Q(T^n(V); V')$ with the topology $\mathscr{T}$, the bilinear map

$$(10)\qquad Q(T^m(U); U')\times Q(T^n(V); V')\longrightarrow Q(T^m(U); U')\otimes Q(T^n(V); V')$$

defined by $(f, g)\to f\otimes g$ is continuous. Since the π-topology on the space $Q(T^m(U); U')\otimes Q(T^n(V); V')$ is the strongest locally convex topology for which the bilinear map (10) is continuous, the π-topology is finer than the topology $\mathscr{T}$. On the other hand, if $\{f_\nu\}$ is a sequence of elements of $Q(T^m(U); U')\otimes Q(T^n(V); V')$ converging to zero for the topology $\mathscr{T}$, then the sequence $\{f_\nu\}$ converges to zero for the ε-topology. Hence the topology $\mathscr{T}$ is finer than the ε-topology. Q.E.D.

Consequently we have the following canonical isomorphisms:

THEOREM 2.5. (i) $Q(T^m(U); U')\hat{\otimes} Q(T^n(V); V')\cong Q(T^{m+n}(U\times V); U'\times V')$, *where* $U\subset \boldsymbol{R}^m_x$, $U'\subset \boldsymbol{R}^m_\xi$, $V\subset \boldsymbol{R}^n_x$ and $V'\subset \boldsymbol{R}^n_\xi$ *are open convex sets.*

(ii) $Q(T^m(K); K')\hat{\otimes} Q(T^n(L); L')\cong Q(T^{m+n}(K\times L); K'\times L')$, *where* $K\subset \boldsymbol{R}^m_x$, $K'\subset \boldsymbol{R}^m_\xi$, $L\subset \boldsymbol{R}^n_x$ *and* $L'\subset \boldsymbol{R}^n_\xi$ *are convex compact sets.*

As special cases, we have the following canonical isomorphisms.

COROLLARY 2.6. (i) $Q(T^m(U))\hat{\otimes} Q(T^n(V))\cong Q(T^{m+n}(U\times V))$.

(ii) $Q(\boldsymbol{C}^m)\hat{\otimes} Q(\boldsymbol{C}^n)\cong Q(\boldsymbol{C}^{m+n})$.

(iii) $Q(T^m(K))\hat{\otimes} Q(T^n(L))\cong Q(T^{m+n}(K\times L))$.

(iv) $Q(\boldsymbol{R}^m)\hat{\otimes} Q(\boldsymbol{R}^n)\cong Q(\boldsymbol{R}^{m+n})$.

As for the spaces of type $\mathfrak{H}$, the following theorem is valid. We mention it without proof.

THEOREM 2.7. $\mathfrak{H}(T^m(U))\hat{\otimes}\mathfrak{H}(T^n(V))\cong\mathfrak{H}(T^{m+n}(U\times V))$.

COROLLARY 2.8. $\mathfrak{H}(\boldsymbol{C}^m)\hat{\otimes}\mathfrak{H}(\boldsymbol{C}^n)\cong\mathfrak{H}(\boldsymbol{C}^{m+n})$.

§3. Fourier ulra-hyperfunctions valued in a Fréchet space.

DEFINITION 3.1. For a Fréchet space E, we define $Q'(T^n(U), U'; E)$ to be the space of all continuous linear mappings from $Q(T^n(U); U')$ to E, namely

$$Q'(T^n(U), U'; E)=L(Q(T^n(U); U'); E)\ .$$

Similarly we put

$$Q'(T^n(K), K'; E)=L(Q(T^n(K); K'); E)\ .$$

We will denote $Q'(\boldsymbol{C}^n; E)=Q'(T^n(\boldsymbol{R}^n), \boldsymbol{R}^n; E)$ and $Q'(\boldsymbol{R}^n; E)=Q'(T^n(0), (0); E)$. An element of $Q'(\boldsymbol{C}^n; E)$ (resp. $Q'(\boldsymbol{R}^n; E)$) is called *a Fourier ultra-hyperfunction* (*resp. Fourier hyperfunction*) *valued in the Fréchet space* E.

DEFINITION 3.2. Suppose $S\in Q'(T^n(U), U'; E)$ and $K\subset\subset U$, $K'\subset\subset U'$.

The product set $K\times K'\subset \boldsymbol{R}^n\times \boldsymbol{R}^n$ is said to be a carrier of S, if S can be extended to a continuous linear mapping of $Q(T^n(K); K')$ to E.

PROPOSITION 3.3. *Let E be a Fréchet space.*

(i) $Q'(T^n(U), U'; E)\cong Q'(T^n(U); U')\hat{\otimes}E$,

where $U\subset \boldsymbol{R}^n_x$ and $U'\subset \boldsymbol{R}^n_\xi$ are convex open sets.

(ii) $Q'(T^n(K), K'; E)\cong Q'(T^n(K): K')\hat{\otimes}E$,

where $K\subset \boldsymbol{R}^n_x$ and $K'\subset \boldsymbol{R}^n_\xi$ are convex compact sets. In the above formulas, $Q'(T^n(U); U')$ and $Q'(T^n(K); K')$ are dual spaces of $Q(T^n(U); U')$ and $Q(T^n(K); K')$ respectively.

PROOF. See Proposition 50.50, p. 522 in F. Trèves [12].

COROLLARY 3.4. (i) $Q'(\boldsymbol{C}^n; E)\cong Q'(\boldsymbol{C}^n)\hat{\otimes}E$.
(ii) $Q'(\boldsymbol{R}^n; E)\cong Q'(\boldsymbol{R}^n)\hat{\otimes}E$.

PROPOSITION 3.5. *We have the following canonical isomorphisms for the dual spaces:*
(i) $Q'(T^m(U); U')\hat{\otimes}Q'(T^n(V), V')\cong L(Q(T^m(U); U'); Q'(T^n(V); V'))\cong Q'(T^{m+n}(U\times V); U'\times V')$.
(ii) $Q'(T^m(K); K')\hat{\otimes}Q'(T^n(L); L')\cong L(Q(T^m(K); K'); Q'(T^n(L); L'))\cong Q'(T^{m+n}(K\times L); K'\times L')$.

PROOF. (i) is obvious by Proposition 50.7, and formula (50.16), p. 524 in F. Trèves [12]. The proof of (ii) is clear by Theorem 2.5 (i) and Proposition 50.7 and formula (50.16) in F. Trèves [12].

COROLLARY 3.6. (i) $Q'(\boldsymbol{C}^m)\hat{\otimes}Q'(\boldsymbol{C}^n)\cong Q'(\boldsymbol{C}^{m+n})$.
(ii) $Q'(\boldsymbol{R}^m)\hat{\otimes}Q'(\boldsymbol{R}^n)\cong Q'(\boldsymbol{R}^{m+n})$.

PROPOSITION 3.7. *We have the following canonical isomorphisms:*
(i) $Q'(T^m(U), U'; E)\hat{\otimes}Q'(T^n(V), V'; F)\cong$
$$Q'(T^{m+n}(U\times V), U'\times V'; E\hat{\otimes}F),$$
(ii) $Q'(T^m(K), K'; E)\hat{\otimes}Q'(T^n(L), L'; F)\cong$
$$Q'(T^{m+n}(K\times L), K'\times L'; E\hat{\otimes}F),$$
where E and F are Fréchet spaces and the tensor products are topologized by the ε-topology or the π-topology in each statement.

PROOF. Tensor products of locally convex Hausdorff spaces are commutative and associative. From Propositions 3.3, 3.5 and the above mentioned properties, the statements are obvious. Q.E.D.

COROLLARY 3.8. *Let E and F be Fréchet spaces.*

(i) $Q'(C^m; E)\hat{\otimes} Q'(C^n; F)\cong Q'(C^{m+n}; E\hat{\otimes} F)$,

(ii) $Q'(R^m; E)\hat{\otimes} Q'(R^n; F)\cong Q'(R^{m+n}; E\hat{\otimes} F)$,

where the tensor products are topologized by the ε-topology or the π-topology in each statement.

We mention without proof the results concerning the spaces of type $\mathfrak{H}$.

PROPOSITION 3.9. $\mathfrak{H}'(C^n; E)\cong \mathfrak{H}'(C^n)\hat{\otimes} E$.

PROPOSITION 3.10. $\mathfrak{H}'(C^m)\hat{\otimes}\mathfrak{H}'(C^n)\cong \mathfrak{H}'(C^{m+n})$.

PROPOSITION 3.11. $\mathfrak{H}'(C^m; E)\hat{\otimes}\mathfrak{H}'(C^n; F)\cong \mathfrak{H}'(C^{m+n}; E\hat{\otimes} F)$.

§4. Multiplication, convolution and Fourier transformation.

For $\varphi\in Q(T^n(U); U')$, we define the Fourier transformation $\mathscr{F}\varphi$ of φ by

$$\mathscr{F}\varphi(\zeta)=\tilde{\varphi}(\zeta)=\int_{R^n}\varphi(x+iy)\exp(-i\langle x+iy, \zeta\rangle)dx .$$

Then, $\mathscr{F}$ gives a linear topological isomorphism:

$$Q(T^n(U); U')\xrightarrow{\sim} Q(T^n(U'); -U) .$$

For every $f\in Q(T^n(U); U')$ and $g\in Q(T^n(V); V')$, $fg\in Q(T^n(U\cap V); U'+V')$.

PROPOSITION 4.1. *For $f\in Q(T^n(U); U')$, $g\in Q(T^n(V); V')$, we define the convolution $f*g$ of f and g as follows:*

$$\begin{aligned} f*g(x+iy) &\equiv \int_{R^n} f(u+iv)g((x+iy)-(u+iv))du \\ &\equiv \int_{R^n} f(u_1+iv_1, \cdots, u_n+iv_n) \\ &\quad \times g((x_1-u_1)+i(y_1-v_1), \cdots, (x_n-u_n)+i(y_n-v_n))du_1\cdots du_n , \end{aligned}$$

where $x, u\in R^n$, $v\in U$ and $y\in V+U$. Then $f*g$ is defined independently of v and belongs to $Q(T^n(U+V); U'\cap V')$. The correspondence $(f, g)\mapsto f*g$ is continuous.

PROOF. This can be seen directly from the above definition.

Alternatively, we can employ the obvious identity $\mathscr{F}(f*g)=\tilde{f}\cdot\tilde{g}$. Since $\tilde{f}\in Q(T^n(U'); -U)$ and $\tilde{g}\in Q(T^n(V'); -V)$, $\tilde{f}\cdot\tilde{g}$ clearly belongs to the space $Q(T^n(U'\cap V'); -(U+V))$. Hence $f*g\in Q(T^n(U+V); U'\cap V')$. The continuity is obvious. Q.E.D.

PROPOSITION 4.2. *We have the following diagram:*

$$\begin{array}{ccc} Q(T^n(U); U')\times Q(T^n(V); V') & \xrightarrow{(\cdot)} & Q(T^n(U\cap V); U'+V') \\ \downarrow{\scriptstyle \mathscr{F}\times\mathscr{F}} & & \downarrow{\scriptstyle \mathscr{F}} \\ Q(T^n(U'); -U)\times Q(T^n(V'); -V) & \xrightarrow{(*)} & Q(T^n(U'+V'); -U\cap V)\,, \end{array}$$

where the horizontal arrows $(\cdot)$ *and* $(*)$ *are the multiplication and the convolution respectively. The vertical arrows are the Fourier transformations.*

DEFINITION 4.3. Let $U, V, W\subset \boldsymbol{R}_x^n$ and $U', V', W'\subset \boldsymbol{R}_\xi^n$ be convex open sets and let $K, L, M\subset \boldsymbol{R}_x^n$ and $K', L', M'\subset \boldsymbol{R}_\xi^n$ be convex compact sets. We assume that $U\cap W\supset V$, $K\cap M\supset L$ and that $U'+W'\supset V'$, $K'+M'\supset L'$. If $f\in Q(T^n(U); U')$ (resp. $f\in Q(T^n(K); K')$) and $S\in Q'(T^n(V), V'; E)$ (resp. $S\in Q'(T^n(L), L'; E)$), then we define $fS\in Q'(T^n(W), W'; E)$ (resp. $fS\in Q'(T^n(M), M'; E)$) by the formula

(11) $(fS)(g)=S(fg)$ for all $g\in Q(T^n(W); W')$ (resp. $g\in Q(T^n(M); M')$) .

The legitimacy of this definition follows from the fact that the correspondence $g\mapsto fg$ from $Q(T^n(W); W')$ to $Q(T^n(V); V')$ is continuous. Remark that when we can take $W=V$ and $W'=V'$ (which is equivalent to assuming that $\bar{V}'\ni 0$), then $Q'(T^n(V), V'; E)$ becomes a $Q(T^n(U); U')$-module. Similar assertion holds for the other type.

We have the following definition of the tensor products of Fourier ultra-hyperfunctions valued in a Fréchet space.

DEFINITION 4.4. Let $S\in Q'(T^m(U), U'; E)$ and $T\in Q'(T^n(V), V'; F)$. We define $S\otimes T$ by the formula

(12) $S\otimes T(f\otimes g)=S(f)\otimes T(g)$ for $f\in Q(T^m(U); U')$ and $g\in Q(T^n(V); V')$. Then $S\otimes T\in Q'(T^{m+n}(U\times V), U'\times V'; E\hat{\otimes}F)$.

DEFINITION 4.5. For $S\in Q'(T^n(U), U'; E)$ (resp. $Q'(T^n(K), K'; E)$) we define the Fourier transformation $\mathscr{F}S$ of S as follows:

(13) $$\langle\mathscr{F}S, f\rangle=\langle S, \mathscr{F}f\rangle=S(\mathscr{F}f)$$

where $f\in Q(T^n(-U');U)$ (resp. $Q(T^n(-K');K)$).

THEOREM 4.6. *The Fourier transformation $\mathscr{F}$ establishes a linear topological isomorphism*:

$$Q'(T^n(U),U';E)\xrightarrow{\sim}Q'(T^n(-U'),U;E)\ (resp.$$
$$Q'(T^n(K),K';E)\xrightarrow{\sim}Q'(T^n(-K'),K;E))\,.$$

PROOF. We can easily prove it.

DEFINITION 4.7. Let U, V, W, U', V', W', K, L, M, K', L' and M' be as in Definition 4.3. If $f\in Q(T^n(U');U)$ (resp. $f\in Q(T^n(K'):K)$) and $S\in Q'(T^n(V'),V;E)$ (resp. $S\in Q'(T^n(L'),L;E)$), then we define $f*S\in Q'(T^n(W'),W;E)$ (resp. $f*S\in Q'(T^n(M'),M;E)$) by the formula

(14) $(f*S)(g)=S(f*g)$ for all $g\in Q(T^n(W');W)$ (resp. $g\in Q(T^n(M'),M)$).

The legitimacy of this definition follows from Proposition 4.1. Summing up, we have obtained

THEOREM 4.8. *Let U, V, W, U', V' and W' be as in Definition 4.3. We have the following diagram*:

$$\begin{array}{ccc} Q(T^n(U);U')\times Q'(T^n(V),V';E) & \xrightarrow{(\cdot)} & Q'(T^n(W),W';E) \\ \downarrow{\scriptstyle \mathscr{F}\times\mathscr{F}} & & \downarrow{\scriptstyle \mathscr{F}} \\ Q(T^n(U'),-U)\times Q'(T^n(V'),-V;E) & \xrightarrow{(*)} & Q'(T^n(W'),-W;E)\,. \end{array}$$

There are many other sorts of coupling concerning the convolution. For example, when $U'\supset V'$, the result of the convolution becomes an analytic function. In order to present these assertions in a systematic way, it will be convenient to introduce also the space of exponentially increasing analytic functions. Here we content ourselves by giving one additional remark:

THEOREM 4.9. *Let f be a hyperfunction with compact support. Then $f*$ defines a continuous linear mapping of $Q'(T^n(K),K';E)$ into itself. Assume further that f is slowly decreasing, that is, $(\mathscr{F}f)(\zeta)$ satisfies*:

(15) *given any $\varepsilon>0$, there exists n_ε such that for any $\xi\in R^n$ satisfying $|\xi|>n_\varepsilon$, we can find $\eta\in C$ such that*

i) $|\xi-\eta|<\varepsilon$,

ii) $|(\mathscr{F}f)(\eta)|\geqq\exp(-\varepsilon|\xi|)$.

Then $f*$ is surjective.

In fact, the convolution can be defined via the Fourier transformation by way of the multiplication of the entire function $(\mathscr{F}f)(\zeta)$ which is of infra-exponential growth on every tube with bounded base. The surjectivity of $f*$ under the assumption (15) is proved by Kawai [5] for the space $Q'(T^n(0); 0)$, i.e., for the Fourier hyperfunctions. The method applies to our space $Q'(T^n(K); K')$ without modification. Then the result extends to $Q'(T^n(K), K'; E) \cong Q'(T^n(K); K')\hat{\otimes}E$ by virtue of Proposition 43.9 in [12], because the operator $f*$ on this space is obviously equal to the tensor product of $f*$ on $Q'(T^n(K); K')$ with 1_E.

In view of the well known Malgrange inequality, a differential operator of finite order with constant coefficients satisfies the above assumptions as a convolution operator, hence operates surjectively on $Q'(T^n(K), K'; E)$. As another example, for given $f \in Q'(T^1(K), K'; E)$ we can always find $u \in Q'(T^1(K), K'; E)$ satisfying $u(x+1)-u(x)=f(x)$.

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