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Uniqueness for the Characteristic Cauchy Problem and its Applications

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Introduction

In this paper we will consider the local uniqueness for Fuchsian partial differential operators (See [2]) with C^∞-coefficients and as its applications we shall give some examples in the case of partial differential operators with characteristic or non-characteristic initial surfaces.

The local uniqueness for a Fuchsian partial differential operator has been obtained by Baouendi and Goulaouic [2], when its operator has analytic coefficients with respect to space variables x. Recently Alinhac and Baouendi [1] studied this problem for some characteristic pseudo-differential operators on a compact manifold. For other many works for characteristic operators, we wish the reader to consult references of [1] and [2].

On the other hand in the case of a non-characteristic initial surface there is well-known Holmgren's theorem for a differential operator with analytic coefficients. Calderón [3] showed the local uniqueness result for non-characteristic partial differential operators with non-analytic coefficients, assuming that coefficients of a principal symbol are real-valued and its characteristic roots are simple from each other. When characteristic roots have multiplicity, many works are found in Hörmander [4], Mizohata [6], Matsumoto [5], Watanabe [11], Zeman [13] and others. In the above refered papers, all the authors assume that a imaginary part of each characteristic root never vanishes or vanishes identically. When this assumption is not satisfied, Kumano-go [12], Nirenberg [7] studied some partial differential operators and recently Strauss and Trèves [8] considered a first order partial differential operator.

The aim of this paper is to show that for some differential operators with C^∞-coefficients we can treat the local uniqueness for a characteristic Cauchy problem and non-characteristic one in the same frame. In our

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theorem we shall prove the local uniqueness for certain Fuchsian partial differential operators, using a Carleman type estimate similar to Alinhac and Baouendi [1]. (This problem is treated by Baouendi and Goulaouic [2] in the case of analytic coefficients.) Consequently we may obtain local uniqueness results for characteristic operators which are not hyperbolic (cf. [1], [9]) and those for non-characteristic operators with degenerating imaginary parts of its characteristic roots.

An outline of this paper is as follows. In §1 we will state our theorems and their direct applications. In §2 we shall give Carleman type estimates for some first order pseudo-differential operators. In §3, using consequence of §2, we shall obtain the similar estimates for pseudo-differential operators of order m. The proof of the theorem will be given in §4. Finally in §5 we shall give generalizations of our theorems, and shall note that the local uniqueness for certain other differential operators is proved in the same method.

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§1. Statements of results and their applications.

Let $(t, x)\in[0, T]\times \boldsymbol{R}^n$ and $(D_t, D_x)=(D_t, D_{x_1}, \cdots, D_{x_n})$, where $D_t=-i(\partial/\partial t)$, $D_{x_j}=-i(\partial/\partial x_j)$ $(j=1, 2, \cdots, n)$. For multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)\in N^n$, we put $D_x^\alpha=D_{x_1}^{\alpha_1}\cdots D_{x_n}^{\alpha_n}$, $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$ and $|\alpha|=\sum_{j=1}^n \alpha_j$.

We shall define the following set of functions.

DEFINITION. For any integer $k\geqq 1$, we denote by C_k^∞ the set of functions $a(t, x)$ satisfying the property that $a(t, x)=\sum_{i=1}^p t^{l_i}a_i(t, x)$ where $l_i=m_i/k$, m_i is a positive integer and $a_i(t, x)\in C^\infty$ $(i=1, 2, \cdots, p)$.

We consider a differential polynomial $\tilde{P}$ with respect to (λ, ξ)

$$\tilde{P}=\tilde{P}(t, x, \lambda, \xi)=\lambda^m+\sum_{\substack{0\leq j\leq m-1\\ |\alpha|+j\leq m}} a_{\alpha,j}(t, x)\xi^\alpha\lambda^j\,, \tag{1.1}$$

where coefficients $a_{\alpha,j}(t, x)$ belong to C_k^∞. Then for a differential polynomial $\tilde{P}$ and a positive rational number $l=k'/k$ ($k'\geqq 1$ is an integer) we define the differential operator as follows.

$$\tilde{P}(t, x, tD_t, t^lD_x)=t^mD_t^m+\sum_{\substack{0\leq j\leq m-1\\ |\alpha|+j\leq m}} a_{\alpha,j}(t, x)t^{|\alpha|l+j}D_x^\alpha D_t^j\,. \tag{1.2}$$

Let the homogeneous part of degree m of $\tilde{P}(t, x, \lambda, \xi)$ with respect to (λ, ξ) be

(1.3) $$\tilde{P}_m(t, x, \lambda, \xi)=\lambda^m+\sum_{\substack{0\le j\le m-1\\ |\alpha|+j=m}} a_{\alpha,j}(t, x)\xi^\alpha\lambda^j .$$

Hence we assume the following conditions on $\tilde{P}_m$.

(A-I) The coefficients $a_{\alpha,j}(t, x)$ of $\tilde{P}_m$ are real-valued and belong to C^∞. If $|\alpha|l\notin \boldsymbol{Z}$, $a_{\alpha,j}(t, x)\equiv 0$.

(A-II) The equation $\tilde{P}_m(t, x, \lambda, \xi)=0$ has simple roots in regard to λ.

Then we obtain the basic theorem as follows.

THEOREM 1. *Under the assumptions* (A-I) *and* (A-II), *for a characteristic partial differential operator* $P=\tilde{P}(t, x, tD_t, t^lD_x)$ *we obtain the local uniqueness in the following sense*:

If a function u *of class* C^∞ *near the origin satisfies*

$$\begin{cases} Pu=0 \\ D_t^j u|_{t=0}=0 \quad \textit{for any} \quad j\ge 0 , \end{cases}$$

then u *vanishes identically in some neighbourhood of the origin in* $\boldsymbol{R}\times\boldsymbol{R}^n$.

We shall apply Theorem 1 to Fuchsian partial differential operators (see [2]).

Let $P=P(t, x, D_t, D_x)$ be a linear partial differential operator of order m whose coefficients are smooth of the form

(1.4) $$\begin{aligned} P&=P(t, x, D_t, D_x)\\ &=t^sD_t^m+a_{m-1}(t, x)t^{s-1}D_t^{m-1}+\cdots+a_{m-s}(t, x)D_t^{m-s}\\ &\quad+\sum_{\substack{0\le j\le m-1\\ 0<|\beta|\le m-j}} t^{\alpha(j,\beta)}a_{j,\beta}(t, x)D_x^\beta D_t^j \end{aligned}$$

for $0\le s\le m$ and $\alpha(j, \beta)=\max(0, s+j-m+1)$. Then P is said to be of Fuchsian type with weight $m-s$ with respect to t.

A characteristic polynomial associated with (1.4) is

(1.5) $$\begin{aligned} \mathscr{C}(\lambda, x)&=\lambda(\lambda-1)\cdots(\lambda-m+1)+ia_{m-1}(0, x)\lambda(\lambda-1)\cdots(\lambda-m+2)\\ &\quad+\cdots+i^sa_{m-s}(0, x)\lambda(\lambda-1)\cdots(\lambda-m+s+1) . \end{aligned}$$

Its roots, called characteristic roots, are denoted by

$$\lambda_1(x), \lambda_2(x), \cdots, \lambda_s(x), \lambda_{s+1}(x)=0, \cdots, \lambda_m(x)=m-s-1 .$$

Then we have

THEOREM 2. *Let* P *be of Fuchsian type with weight* $m-s$ *with respect to* t. *We suppose that there exist a differential polynomial* $\tilde{P}(t, x, \lambda, \xi)$ *satisfying* (A-I), (A-II) *and a positive rational number* $l=k'/k$

($k' \geqq 1$ is an integer) such that

$$t^{m-s}P(t, x, D_t, D_x) = \tilde{P}(t, x, tD_t, t^l D_x) . \tag{1.6}$$

Let $h \in N$ be such that

$$\mathrm{Re}\, \lambda_j(0) < m-s+h \quad \text{for} \quad 1 \leqq j \leqq m .$$

Then any function u of class C^∞ near the origin satisfying

$$\begin{cases} Pu = 0 \\ D_t^j u|_{t=0} = 0 \quad \text{for} \quad 0 \leqq j \leqq m-s+h-1 \end{cases} \tag{1.7}$$

vanishes identically near the origin in $R \times R^n$.

REMARK 1. If C^∞-solution u of homogeneous equation $Pu=0$ does not satisfy $D_t^j u|_{t=0}=0$ for $0 \leqq j \leqq m-s+h-1$, then the conclusions of Theorem 2 are false in general.

REMARK 2. When $l=0$ we cannot expect the local uniqueness generally even in the case of analytic coefficients (see [2]). On the other hand recently Alinhac and Baouendi [1] consider the Cauchy problem for some pseudo-differential operators on a compact manifold. Then they show the uniqueness theorem for such operators in the case of $l=0$.

REMARK 3. If $s=0$, we can take $h=0$. Therefore for a non-characteristic operator the Cauchy problem (1.7) is usual setting one.

Here we shall show that Theorem 1 implies Theorem 2 immediately.

The following lemma follows easily from [2].

LEMMA 3. *P is of Fuchsian type with weight $m-s$ with respect to t. Let $h \in N$ be such that*

$$\mathrm{Re}\, \lambda_j(0) < m-s+h \quad \text{for} \quad 1 \leqq j \leqq m .$$

Then any function u of class C^∞ near the origin satisfying

$$\begin{cases} Pu = 0 \\ D_t^j u|_{t=0} = 0 \quad \text{for} \quad 0 \leqq j \leqq m-s+h-1 \end{cases}$$

is flat at $t=0$ i.e. $D_t^j u|_{t=0}=0$ for any $j \geqq 0$.

Consequently we can reduce the problem of Theorem 2 to

$$\begin{cases} t^{m-s}P(t, x, D_t, D_x)u = \tilde{P}(t, x, tD_t, t^l D_x)u = 0 \\ u \text{ is a function of class } C^\infty \text{ and flat at } t=0 . \end{cases}$$

Then from Theorem 1 we can conclude that u vanishes near the origin in $\boldsymbol{R}\times\boldsymbol{R}^n$.

From Theorem 2 we can obtain uniqueness results for characteristic or non-characteristic differential operators. Hence we shall give some examples.

EXAMPLES.

(1) Let P be the operator with smooth coefficients

$$P(t, x, D_t, D_x)=t(D_t^2\pm D_x^2)+a(t, x)D_t+b(t, x)D_x+c(t, x) .$$

Then P satisfies our condition with $m=2$, $s=1$, $l=1$.

(2) Let P be the following operator with smooth coefficients

$$P(t, x, D_t, D_x)=tD_t^2\pm D_x^2+a(t, x)D_t+b(t, x)D_x+c(t, x) .$$

Then P satisfies our condition with $m=2$, $s=1$, $l=1/2$.

(3) Let P be the operator

$$P(t, x, D_t, D_x)=D_t^m\pm tD_x^m+(\text{any lower order terms}) .$$

Then P satisfies our condition with $m=m$, $s=0$, $l=1+1/m$.

(4) Let P be the operator

$$P(t, x, D_t, D_x)=D_t^2\pm t^{2k+1}D_x^2+t^k a(t, x)D_x+b(t, x)D_t + c(t, x) ,$$

where $a(t, x)$, $b(t, x)$ and $c(t, x)\in C^\infty$. Then P satisfies our condition with $m=2$, $s=0$, $l=k+3/2$.

(5) Let P be the operator

$$P(t, x, D_t, D_x)=D_t^2\pm t^{2k}D_x^2+t^{k-1}a(t, x)D_x+b(t, x)D_t + c(t, x)$$

where $a(t, x)$, $b(t, x)$ and $c(t, x)\in C^\infty$. Then P satisfies our condition with $m=2$, $s=0$, $l=k+1$.

Next we give a generalization of (5) (cf. [9], [10]).

(6) Let P be the operator of order m with smooth coefficients

$$P=P_m+P_{m-1}+\cdots+P_0 .$$

Its principal symbol P_m has real-valued coefficients and can be factored smoothly in the form:

$$P_m(t, x, \lambda, \xi)=\prod_{j=1}^{m}(\lambda-t^k\lambda_j(t, x, \xi))$$

where k is a non-negative integer, $\lambda_j(t, x, \xi) \in \mathscr{B}([0, T], S^1)$ (See §2) and $\lambda_i \neq \lambda_j$ when $i \neq j$. P_{m-j} $(1 \leqq j \leqq m)$ satisfies

$$P_{m-j}(t, x, \lambda, \xi) = \sum_{i=0}^{m-j} \sum_{|\alpha|=i} b_{i,j,\alpha}(t, x) t^{[ik-j]_+} \xi^\alpha \lambda^{m-j-i}$$

where $[A]_+ = \max(A, 0)$ and $b_{i,j,\alpha}(t, x) \in C^\infty$. Then P satisfies our condition with $m=m$, $s=0$, $l=k+1$.

§2. Carleman type estimates for first order pseudo-differential operators.

First we introduce classes of symbols of pseudo-differential operators. $S^m(=S^m_{1,0})$ is the set of well-known symbols of pseudo-differential operators with respect to space variable x. We denote by $\mathscr{B}([0, T], S^m)$ the set of functions $a(t, x, \xi)$ satisfying the condition: $a(t, x, \xi) \in S^m$ for any fixed $t \in [0, T]$ and the map: $[0, T] \ni t \to a(t, x, \xi) \in S^m$ is smooth.

The purpose of this section is to show the following basic proposition.

PROPOSITION 2.1. *Let $\tilde{A}(t, x, \xi)$ and $\tilde{B}(t, x, \xi)$ be real valued symbols of pseudo-differential operators belonging to $\mathscr{B}([0, T], S^1)$. Suppose that $\tilde{B}(t, x, \xi) \neq 0$ for any $\xi \neq 0$ or that $\tilde{B}(t, x, \xi) \equiv 0$. For a rational number $l = k'/k$ and a real-valued function $a(x) \in C^\infty(R^n)$ we define the pseudo-differential operator P as follows.*

$$P = tD_t - t^l(A(t, x, D_x) + iB(t, x, D_x)),$$

where $A(t, x, \xi) = a(x)\tilde{A}(t, x, \xi)$ and $B(t, x, \xi) = a(x)\tilde{B}(t, x, \xi)$. Then for T and N^{-1} sufficiently small there exists a constant c independent of T and N such that

$$N \int_0^T t^{-2N} \| u \|^2 dt \leqq c \int_0^T t^{-2N} \| Pu \|^2 dt \tag{2.1}$$

for any $u \in C^\infty$ with supp $u \subset \Omega$, *where $\Omega = [0, T] \times \{x; |x| \leqq r\}$ and $\| \quad \|$ is L^2-norm with respect to x.*

PROOF. We prove this proposition only in the case of $\tilde{B}(t, x, \xi) \neq 0$ for any $\xi \neq 0$. We leave the proof of the other case to the reader. We may read the proof of this proposition, supposing $\tilde{B}(t, x, \xi) \equiv 0$.

Set

$$Pu = f$$

and $v(t, x) = t^{-N} u(t, x)$. Then

$$(P-Ni)v=t^{-N}f\,.$$

We shall consider the integral $I=\int_0^T t^{-2N}\|f\|^2dt$.

$$\begin{aligned}(2.2)\qquad I&=\int_0^T t^{-2N}\|f\|^2dt\\&=\int_0^T \|tD_tv-t^lAv\|^2dt+\int_0^T \|t^lBv\|^2dt\\&\quad+N^2\int_0^T \|v\|^2dt+2\,\mathrm{Re}\int_0^T (tD_tv,\ -Niv)dt\\&\quad+2\,\mathrm{Re}\int_0^T (-t^lAv,\ -it^lBv-Niv)dt\\&\quad+2\,\mathrm{Re}\int_0^T(-it^lBv,\ -Niv)dt+2\,\mathrm{Re}\int_0^T(tD_tv,\ -it^lBv)dt\\&=I_1+I_2+\cdots+I_7\,.\end{aligned}$$

We proceed to calculate each I_i $(i\geqq 4)$.

$$I_4=-N\int_0^T \|v\|^2dt$$

$$\begin{aligned}I_5&=2\,\mathrm{Re}\int_0^T (-t^lAv,\ -it^lBv)dt+2\,\mathrm{Re}\int_0^T (-t^lAv,\ -Niv)dt\\&=\int_0^T (\{(-it^lB)^*(-t^lA)+(-t^lA)^*(-it^lB)\}v,\ v)dt\\&\quad+\int_0^T (\{(-Ni)^*(-t^lA)+(-t^lA)^*(-Ni)\}v,\ v)dt\,,\end{aligned}$$

where $(-it^lB)^*$, $(-t^lA)^*$ and $(-Ni)^*$ are the formal adjoint of $-it^lB$, $-t^lA$ and $-Ni$, respectively.

Since the symbols $A(t, x, \xi)$ and $B(t, x, \xi)$ are real-valued functions, it follows from the product and the adjoint formulae of pseudo-differential operators that

$$\begin{aligned}I_5\geqq&-c_{5,1}T^l\int_0^T \|t^la(x)\Lambda v\|\,\|v\|\,dt-c_{5,2}T^{2l}\int_0^T \|v\|^2dt\\&-c_{5,3}T^lN\int_0^T \|v\|^2dt\,,\end{aligned}$$

where Λ is the pseudo-differential operator defined by

$$\Lambda u(x)=(2\pi)^{-n}\int_{R^n} e^{ix\cdot\xi}(1+|\xi|^2)^{1/2}\hat{u}(\xi)d\xi\,,$$

$\hat{u}(\xi)$ being the Fourier transform of $u(x)$ i.e.

$$\hat{u}(\xi)=\int_{R^n} e^{-ix\cdot\xi}u(x)dx .$$

$$I_6=2N\,\mathrm{Re}\int_0^T (t^lBv,\ v)dt$$

$$\begin{aligned} I_7&=2\ \mathrm{Re}\int_0^T (tD_tv,\ -it^lBv)dt\\ &=\int_0^T (\{(-it^lB)^*+(-it^lB)\}(tD_tv),\ v)dt\\ &\quad-\int_0^T\Big(\frac{\partial}{\partial t}\{t^{l+1}B\}v,\ v\Big)dt\\ &=\int_0^T (\{(-it^lB)^*+(-it^lB)\}(tD_tv),\ v)dt\\ &\quad-(l+1)\int_0^T (t^lBv,\ v)dt-\int_0^T (t^{l+1}B_tv,\ v)dt , \end{aligned}$$

where B_t is the pseudo-differential operator with symbol $(\partial/\partial t)B(t, x, \xi)$. Therefore

$$\begin{aligned} I_7\geqq &-c_{7,1}T^l\int_0^T \|tD_tv\|\,\|v\|\,dt-(l+1)\,\mathrm{Re}\int_0^T (t^lBv,\ v)dt\\ &-c_{7,2}T\int_0^T \|t^la(x)\Lambda v\|\,\|v\|\,dt-c_{7,3}T^{l+1}\int_0^T \|v\|^2dt . \end{aligned}$$

Noting that

$$\begin{aligned} \|tD_tv\|&\leqq\|tD_tv-t^lAv\|+\|t^lAv\|\\ &\leqq\|tD_tv-tAv\|+\mathrm{const}\,(\|t^la(x)\Lambda v\|+\|v\|) , \end{aligned}$$

we have

$$\begin{aligned} I_7\geqq &-c_{7,4}T^l\int_0^T \|tD_t-t^lAvAv\|^2dt-(l+1)\,\mathrm{Re}\int_0^T (t^lBv,\ v)dt\\ &-c_{7,5}(T+T^l)\int_0^T \|t^la(x)\Lambda v\|\,\|v\|\,dt\\ &-c_{7,6}(T^l+T^{l+1})\int_0^T \|v\|^2dt . \end{aligned}$$

From $I_4, \cdots, I_7$ we easily derive the inequality

$$\begin{aligned} \text{(2.3)}\qquad I\geqq &I_1+I_2+I_3-N\int_0^T \|v\|^2dt-c_1T^mN\int_0^T \|v\|^2dt\\ &-c_2T^m\int_0^T \|t^la(x)\Lambda v\|\,\|v\|\,dt-c_3(T^m+T^{2m})\int_0^T \|v\|^2dt\\ &+(2N-l-1)\,\mathrm{Re}\int_0^T (t^lBv,\ v)dt-c_4T^m\int_0^T \|tD_t-t^lAv\|^2dt , \end{aligned}$$

where

$$m=\begin{cases} l & \text{when} \quad 0<l<1 \\ 1 & \text{when} \quad l\geqq 1 . \end{cases}$$

Since $\tilde{B}(t, x, \xi)$ is elliptic i.e., $\tilde{B}(t, x, \xi)\neq 0$ for any $\xi\neq 0$, we obtain

$$\|a(x)\Lambda v\|\leqq c'(\|Bv\|+\|v\|)$$

for some constant c'. Then by choosing T so small that $c_4T^m\leqq 1/2$ it follows from (2.3) that

$$\begin{aligned} I\geqq 1/2I_1+I_2+I_3-N\int_0^T \|v\|^2dt-c_1'T^mN\int_0^T \|v\|^2dt \\ -c_2'T^m\int_0^T \|t^lBv\|\,\|v\|\,dt-c_3'(T^m+T^{2m})\int_0^T \|v\|^2dt \\ -(2N-l-1)\int_0^T \|t^lBv\|\,\|v\|\,dt . \end{aligned}$$

Put $\varepsilon=N-l/2-1/2+c_2'T^m/2$. Then

$$2\varepsilon\int_0^T \|t^lBv\|\,\|v\|\,dt\leqq \int_0^T \|t^lBv\|^2\,dt+\varepsilon^2\int_0^T \|v\|^2dt .$$

Therefore

$$I\geqq(N^2-N-c_1'T^mN-c_3'T^m-c_3'T^{2m}-\varepsilon^2)\int_0^T \|v\|^2dt .$$

Here we can choose T and N^{-1} sufficiently small in such a way that

$$N^2-N-c_1'T^mN-c_3'T^m-c_3'T^{2m}-\varepsilon^2\geqq l/2N .$$

Consequently we have the desired inequality

$$I\geqq l/2N\int_0^T \|v\|^2dt . \qquad \text{Q.E.D.}$$

§3. Pseudo-differential operators of order m.

In this section we shall consider Carleman type estimates for some pseudo-differential operators of order m.

To begin with, we define the pseudo-differential operators of order 1 as follows (cf. [10]). We put

$$\partial_j=tD_t-t^la(x)\lambda_j(t, x, D_x) \qquad (1\leqq j\leqq m) , \tag{3.1}$$

where $l=k'/k$, $a(x)\in C^\infty$ is a real-valued function, $\lambda_j(t, x, \xi)\in \mathscr{B}([0, T], S^1)$

and $\lambda_i \neq \lambda_j$ when $i \neq j$. Suppose that $\mathrm{Im}\,\lambda_j(t, x, \xi) \neq 0$ for any $\xi \neq 0$ or $\mathrm{Im}\,\lambda_j(t, x, \xi) \equiv 0$.

Next we define the mudules W_i $(0 \leqq i \leqq m-1)$ over the ring of pseudo-differential operators in x of order 0. Let $\Pi_m = \partial_1 \cdots \partial_m$. Let W_{m-1} be the module generated by the monomial operators $\Pi_m / \partial_i = \partial_1 \cdots \partial_{i-1} \partial_{i+1} \cdots \partial_m$ of order $m-1$ and let W_{m-2} be the module generated by the operators $\Pi_m / \partial_i \partial_j$ $(i \neq j)$ of order $m-2$ and so on.

Here we introduce a new symbol class of pseudo-differential operators acting on C^∞-function u with $\mathrm{supp}\, u \subset \Omega$.

DEFINITION. For positive integer k $\mathscr{B}_k([0, T], S^m)$ is the set of functions $a(t, x, \xi)$ which are represented in the form:

$$a(t, x, \xi) = \sum_{i=1}^{p} t^{l_i} a_i(t, x, \xi) ,$$

where $l_i = m_i / k$ (m_i is a positive integer.) and $a_i(t, x, \xi) \in \mathscr{B}([0, T], S^m)$.

Then we immediately obtain the following basic lemma.

LEMMA 3.1. *For any* $a_1(t, x, \xi) \in \mathscr{B}_k([0, T], S^{m_1})$ *and* $a_2(t, x, \xi) \in \mathscr{B}_k([0, T], S^{m_2})$,

(i) $tD_t a_1(t, x, \xi) \in \mathscr{B}_k([0, T], S^{m_1})$

(ii) $D_x^\beta D_\xi^\alpha a_1(t, x, \xi) \in \mathscr{B}_k([0, T], S^{m_1 - |\alpha|})$

(iii) $a_1(t, x, \xi) a_2(t, x, \xi) \in \mathscr{B}_k([0, T], S^{m_1 + m_2})$

(iv) $\mathscr{B}([0, T], S^{m_1}) \subset \mathscr{B}_k([0, T], S^{m_1})$.

Hence we have

LEMMA 3.2. *For any i, j there exist* $A_{i,j}(t, x, \xi), B_{i,j}(t, x, \xi), C_{i,j}(t, x, \xi) \in \mathscr{B}_k([0, T], S^0)$ *such that*

$$[\partial_i, \partial_j] = A_{i,j} \partial_i + B_{i,j} \partial_j + C_{i,j} , \tag{3.2}$$

where $[\partial_i, \partial_j] = \partial_i \partial_j - \partial_j \partial_i$ *is the commutator of pseudo-differential operators* ∂_i *and* ∂_j.

PROOF. Let $\sigma_0([\partial_i, \partial_j])$ be the principal symbol of $[\partial_i, \partial_j]$. Then, by the formula of product of pseudo-differential operators, we obtain

$$\begin{aligned} \sigma_0([\partial_i, \partial_j]) = \sum_{\alpha=0}^{n} \{ & D_{\xi_\alpha}(t\xi_0 - t^l a(x)\lambda_i) \partial_{x_\alpha}(t\xi_0 - t^l a(x)\lambda_j) \\ & - D_{\xi_\alpha}(t\xi_0 - t^l a(x)\lambda_j) \partial_{x_\alpha}(t\xi_0 - t^l a(x)\lambda_i) \} \\ = {} & t^l a(x) D_{i,j}(t, x, \xi) , \end{aligned}$$

where $D_{i,j}(t, x, \xi) \in \mathscr{B}_k([0, T], S^1)$ from Lemma 3.1. Here we used the notations $x_0=t$ and $\xi_0=\lambda$.

If we define functions $A_{i,j}(t, x, \xi)$ and $B_{i,j}(t, x, \xi)$ for $i\neq j$ by $A_{i,j}=D_{i,j}(t, x, \xi)/(\lambda_j-\lambda_i)$ and $B_{i,j}=D_{i,j}(t, x, \xi)/(\lambda_i-\lambda_j)$ respectively, then $A_{i,j}, B_{i,j}\in \mathscr{B}_k([0, T], S^0)$ and the inequality:

$$A_{i,j}(t, x, \xi)(t\xi_0-t^l a(x)\lambda_i)+B_{i,j}(t, x, \xi)(t\xi_0-t^l a(x)\lambda_j)$$
$$=t^l a(x)D_{i,j}(t, x, \xi)$$

holds. Then we obtain

$$[\partial_i, \partial_j]=A_{i,j}\partial_i+B_{i,j}\partial_i+C_{i,j}$$

for some $C_{i,j}\in \mathscr{B}_k([0, T], S^0)$. Q.E.D.

Lemma 3.3. *For any monomial $\omega_s^\alpha\in W_s$ $(0\leqq s\leqq m-1)$ there exist ∂_i and $\omega_{s+1}^\beta\in W_{s+1}$ such that*

$$\partial_i\omega_s^\alpha=\omega_{s+1}^\beta+\sum_{j=1}^{s+1}\sum_\gamma c_{\gamma,j}\omega_{s+1-j}^\gamma\ , \tag{3.3}$$

where $c_{\gamma,j}\in \mathscr{B}_k([0, T], S^0)$ and $\omega_{s+1-j}^\gamma\in W_{s+1-j}$.

Proof. For any $\omega_s^\alpha=\partial_{j_1}\cdots\partial_{j_s}$ $(j_1<j_2<\cdots<j_s)$ there exists some $j\notin \{j_1, \cdots, j_s\}$ with $1\leqq j\leqq m$. Since $[\partial_i, \partial_j]=A_{i,j}\partial_i+B_{i,j}\partial_j+C_{i,j}$ by Lemma 3.2 we have immediately (3.3). Q.E.D.

We define the functionals of $u\in C^\infty$ with $\operatorname{supp} u\subset\Omega$ $\Phi_0, \Phi_1, \cdots, \Phi_m$ as follows. For any $N>0$

$$\Phi_0=\int_0^T t^{-2N}\|u\|^2dt$$
$$\Phi_1=\int_0^T t^{-2N}(\|\partial_1 u\|^2+\cdots+\|\partial_m u\|^2)dt$$
$$\vdots$$
$$\Phi_j=\int_0^T t^{-2N}\sum_\alpha\|\omega_j^\alpha u\|^2dt$$
$$\vdots$$
$$\Phi_m=\int_0^T t^{-2N}\|\Pi_m u\|^2dt\ .$$

Then we get

Lemma 3.4. *For T and N^{-1} sufficient small there exist constant $c_0, \cdots, c_{m-1}$ independent of T and N such that*

$$
\begin{aligned}
&(3.4.0) && N^{m}\Phi_0\leqq c_0N^{m-1}(\Phi_1+\Phi_0)\\
&(3.4.1) && N^{m-1}\Phi_1\leqq c_1N^{m-2}(\Phi_2+\Phi_1+\Phi_0)\\
&\ \vdots &&\\
&(3.4.j) && N^{m-s}\Phi_s\leqq c_sN^{m-s-1}(\Phi_{s+1}+\Phi_s+\cdots+\Phi_0)\\
&\ \vdots &&\\
&(3.4.m-1) && N\Phi_{m-1}\leqq c_{m-1}(\Phi_m+\Phi_{m-1}+\cdots+\Phi_0)\,.
\end{aligned}
$$

PROOF. We shall show

$$N\Phi_s\leqq c_s(\Phi_{s+1}+\Phi_s+\cdots+\Phi_0)\qquad (0\leqq s\leqq m-1)\,. \tag{3.5}$$

From Lemma 3.3, for any monomial $\omega_s^\alpha\in W_s$ there exist ∂_i, $\omega_{s+1}^\beta\in W_{s+1}$ and $c_{\gamma,j}\in\mathscr{B}_k([0,T],S^0)$ such that

$$\partial_i\omega_s^\alpha=\omega_{s+1}^\beta+\sum_{j=1}^{s+1}\sum_\gamma c_{\gamma,j}\omega_{s+1-j}^\gamma\,.$$

Putting

$$
\begin{aligned}
&U=\omega_s^\alpha u\\
&G=\omega_{s+1}^\beta u+\sum_{j=1}^{s+1}\sum_\gamma c_{\gamma,j}\omega_{s+1-j}^\gamma u\,,
\end{aligned}
$$

we have the equation $\partial_iU=G$. Hence it follows from Proposition 2.1 that

$$
\begin{aligned}
N\int_0^T t^{-2N}\|U\|^2dt&\leqq c\int_0^T t^{-2N}\|G\|^2dt\\
&\leqq c'\left\{\sum_{j=1}^{s+1}\sum_\gamma\int_0^T t^{-2N}\|\omega_{s+1-j}^\gamma u\|^2dt+\int_0^T t^{-2N}\|\omega_{s+1}^\beta u\|^2dt\right\}\\
&\leqq c'(\Phi_{s+1}+\cdots+\Phi_0)\,.
\end{aligned}
$$

Therefore we have

$$N\int_0^T t^{-2N}\|\omega_s^\alpha u\|^2dt\leqq c'(\Phi_{s+1}+\cdots+\Phi_0)\,. \tag{3.6}$$

Since (3.6) holds for any monomial $\omega_s^\alpha\in W_s$ we obtain the inequality (3.5).
Q.E.D.

Now we shall consider a pseudo-differential operator P of order m, satisfying some conditions.

DEFINITION. P is a pseudo-differential operator $P=P_m+\cdots+P_0$ whose principal symbol $P_m=P_m(t,x,\lambda,\xi)$ can be factored in the form:

$$P_m(t, x, \lambda, \xi)=K(t, x) \prod_{j=1}^{m} (t\lambda-t^l a(x)\lambda_j(t, x, \xi)) ,$$

where $a(x) \in C^\infty$ is a real-valued function, $l=k'/k$, $K(t, x)\neq 0$ on $[0, T]\times \boldsymbol{R}^n$, $\lambda_j(t, x, \xi) \in \mathscr{B}([0, T], S^1)$ and $\lambda_i \neq \lambda_j$ when $i\neq j$. Furthermore we assume that $\operatorname{Im} \lambda_j(t, x, \xi)\neq 0$ for any $\xi\neq 0$ or $\operatorname{Im} \lambda_j(t, x, \xi)\equiv 0$.

If the lower order terms $P_{m-j}(1\leqq j\leqq m)$ can be represented as follows:

$$P_{m-j}(t, x, \lambda, \xi)=\sum_{i=0}^{m-j} a(x)^i t^{il} b_{i,j}(t, x, \xi)(t\lambda)^{m-j-i} ,$$

where $b_{i,j}(t, x, \xi) \in \mathscr{B}_k([0, T], S^i)$, then we say that a pseudo-differential operator P satisfies ($\sharp$)-condition on $[0, T]$.

We shall proceed to obtain Carleman type estimates for pseudo-differential operators satisfying ($\sharp$)-condition on $[0, T]$.

PROPOSITION 3.5. *Suppose that P is a pseudo-differential operator of order m satisfying ($\sharp$)-condition on* $[0, T]$. *Then for T and N^{-1} sufficiently small there exists a constant c independent of T and N such that*

$$N^m \int_0^T t^{-2N} \| u \|^2 dt \leqq c \int_0^T t^{-2N} \| Pu \|^2 dt \tag{3.7}$$

for any $u \in C^\infty$ with $\operatorname{supp} u \subset \Omega$.

REMARK. Since $K(t, x)\neq 0$ on $[0, T]$ we may prove Proposition 3.5 only in the case of $K(t, x)\equiv 1$ on $[0, T]$. Therefore we suppose $K(t, x)\equiv 1$ on $[0, T]$ in the following.

Before the proof of Proposition 3.5 we prepare the following lemma.

LEMMA 3.6 (cf. [10]). *Let P be a pseudo-differential operator satisfying ($\sharp$)-condition on* $[0, T]$. *Then there exist some $c_{\alpha,j} \in \mathscr{B}_k([0, T], S^0)$ and $\omega_{m-j}^\alpha \in W_{m-j}$ such that*

$$P-\Pi_m=\sum_{j=1}^{m} \sum_{\alpha} c_{\alpha,j}\omega_{m-j}^\alpha . \tag{3.8}$$

PROOF. The proof of this lemma is done by two steps.

(I) Let $\Pi_s=\partial_{i_1}\cdots\partial_{i_s}$ $(1\leqq i_1<\cdots<i_s\leqq m)$. Then $\sigma(\Pi_s)$, the symbol of Π_s, can be written in the form:

$$\sigma(\Pi_s)=\prod_{j=1}^{s} (t\lambda-t^l a(x)\lambda_{i_j})+R_{s-1}+\cdots+R_0 , \tag{3.9}$$

where $R_{s-j}(t, x, \lambda, \xi) = \sum_{\beta=0}^{s-j} b_{\beta,j}(t, x, \xi)a(x)^\beta t^{l\beta}(t\lambda)^{s-j-\beta}$ and $b_{\beta,j}(t, x, \xi) \in$

$\mathscr{B}_k([0, T], S^{\beta})$.

We carry out the proof by induction on s. When $s=1$ (3.9) is trivial. Assume that (3.9) is valid for s. Since $\Pi_{s+1}=\Pi_s\partial_{i_{s+1}}$, we have, by the product formula for two pseudo-differential operators,

$$\sigma(\Pi_{s+1})=\sigma(\Pi_s)(t\xi_0-a(x)t^l\lambda_{i_{s+1}}) \\ +\sum_{\alpha\neq 0} D_\xi^\alpha\sigma(\Pi_s)\partial_x^\alpha(t\xi_0-a(x)t^l\lambda_{i_{s+1}}) .$$

Therefore by the assumption of induction we can easily get (3.9) with $s+1$ (see [10]).

(II) From (3.9) with $s=m$, we obtain

$$\sigma(P-\Pi_m)=\sum_{j=1}^{m}\sum_{i=0}^{m-j}\tilde{c}_{i,j}(t, x, \xi)a(x)^i t^{il}(t\lambda)^{m-i-j} ,$$

where $\tilde{c}_{i,j}\in\mathscr{B}_k([0, T], S^i)$. Let the part of order $m-1$ of $\sigma(P-\Pi_m)$ be

$$\tilde{P}_{m-1}(t, x, t\lambda, \xi)=\sum_{i=0}^{m-1}\tilde{c}_{i,1}(t, x, \xi)a(x)^i t^{il}(t\lambda)^{m-i-1} . \tag{3.10}$$

We want to determine $A_j(t, x, \xi)\in\mathscr{B}_k([0, T], S^0)$ so that

$$\tilde{P}_{m-1}(t, x, t\lambda, \xi)=\sum_{j=1}^{m}A_j(t, x, \xi)\prod_{i\neq j}(t\lambda-a(x)t^l\lambda_i(t, x, \xi)) . \tag{3.11}$$

From (3.10) we have

$$\tilde{P}_{m-1}(t, x, a(x)t^l\lambda_j(t, x, \xi), \xi)=a(x)^{m-1}t^{l(m-1)}\tilde{K}_j(t, x, \xi) ,$$

where $\tilde{K}_j(t, x, \xi)\in\mathscr{B}_k([0, T], S^{m-1})$. Putting $t\lambda=a(x)t^l\lambda_j(t, x, \xi)$ into (3.11) gives

$$a(x)^{m-1}t^{l(m-1)}\tilde{K}_j(t, x, \xi)=a(x)^{m-1}t^{l(m-1)}\prod_{i\neq j}(\lambda_j-\lambda_i)A_j(t, x, \xi) .$$

Then we can find

$$A_j(t, x, \xi)=[\prod_{i\neq j}(\lambda_j-\lambda_i)]^{-1}\tilde{K}_j(t, x, \xi)$$

in $\mathscr{B}_k([0, T], S^0)$. Then (3.9) for $s=m-1$ yields

$$\sigma\Big(P-\Pi_m-\sum_{j=1}^{m}A_j\prod_{i\neq j}\partial_j\Big) \\ =\sum_{j=2}^{m}\sum_{i=0}^{m-j}\tilde{d}_{i,j}(t, x, \xi)a(x)^i t^{il}(t\lambda)^{m-i-j} \tag{3.12}$$

where $\tilde{d}_{i,j}(t, x, \xi)\in\mathscr{B}_k([0, T], S^i)$. Repeating these steps we arrive at the relation (3.8). Q.E.D.

PROOF OF PROPOSITION 3.5. It follows from Lemma 3.6 that

$$\int_0^T t^{-2N}\|Pu-\Pi_m u\|^2dt \leqq c\sum_{i=0}^{m-1}\Phi_i . \tag{3.13}$$

Note that

$$\|\Pi_m u\|^2 \leqq 2\{\|Pu-\Pi_m u\|^2+\|Pu\|^2\} . \tag{3.14}$$

Then, using (3.13) and (3.14), we have from (3.4.$m-1$)

$$\begin{aligned} N\Phi_{m-1} &\leqq c_{m-1}(\Phi_m+\cdots+\Phi_0) \\ &\leqq c'_{m-1}\Big(\Phi_{m-1}+\cdots+\Phi_0+\int_0^T t^{-2N}\|Pu\|^2dt\Big) . \end{aligned}$$

Here we choose N sufficiently large to get

$$N\Phi_{m-1} \leqq c''_{m-1}\Big(\Phi_{m-2}+\cdots+\Phi_0+\int_0^T t^{-2N}\|Pu\|^2dt\Big) \tag{3.15}$$

for some different constant c''_{m-1}.

Next from (3.4.$m-2$) and (3.15) we obtain

$$\begin{aligned} N^2\Phi_{m-2} &\leqq c_{m-2}N(\Phi_{m-1}+\cdots+\Phi_0) \\ &\leqq c'_{m-2}\Big\{N(\Phi_{m-2}+\cdots+\Phi_0)+\int_0^T t^{-2N}\|Pu\|^2dt\Big\} . \end{aligned} \tag{3.16}$$

Choosing N sufficient large again, we have

$$N^2\Phi_{m-2} \leqq c''_{m-2}\Big\{N(\Phi_{m-3}+\cdots+\Phi_0)+\int_0^T t^{-2N}\|Pu\|^2dt\Big\} .$$

Repeating this steps we have

$$N^j\Phi_{m-j} \leqq c''_{m-j}\Big\{N^{j-1}(\Phi_{m-j-1}+\cdots+\Phi_0)+\int_0^T t^{-2N}\|Pu\|^2dt\Big\} .$$

Therefore we conclude that

$$N^m\Phi_0 \leqq c''_0\int_0^T t^{-2N}\|Pu\|^2dt .$$

The proof of Proposition 3.5 is completed.

§4. Proof of Theorem 1.

In this section we shall complete the proof of Theorem 1, using the consequence of §3.

It is sufficient for the proof of Theorem 1 that for arbitrary $\tilde{T}>0$ and $\tilde{r}>0$ any function u of class C^∞ satisfying

$$(4.1)\qquad \begin{cases} Pu=0 \text{ in } [0, \tilde{T}]\times\{x: |x|\leqq\tilde{r}\} \\ D_t^j u|_{t=0}=0 \text{ for any } j\geqq 0 \end{cases}$$

vanishes identically in $[0, \tilde{T}_1]\times\{x: |x|\leqq\tilde{r}_1\}$ for some $\tilde{T}_1>0$ and $\tilde{r}_1>0$.

Hence we shall take a singular change of variables as follows (cf. [1]).

$$(4.2)\qquad (t, x)\longrightarrow(T, X),$$

where $t=(r^2-|X|^2)^k T$, $x=X$ and $r=\tilde{r}/2$.

Hence we have

LEMMA 4.1. *Suppose that $P=P(t, x, D_t, D_x)$ satisfies the assumptions of Theorem 1. By the above change of variables, P is transformed to a new differential operator $P^1=P^1(T, X, D_T, D_X)$. Then there exists $T'>0$ so that $P^1(T, X, D_T, D_X)$ satisfies the ($\sharp$)-condition on $[0, T']$ with $a(x)=(r^2-|X|^2)^{k'}$, where $k'=lk$.*

The proof of this lemma is given in the last part of this section.

Now we set

$$(4.3)\qquad \tilde{u}(T, X)=\begin{cases} u((r^2-|X|^2)^k T, X) & \text{when } |X|\leqq r \\ 0 & \text{when } |X|\geqq r. \end{cases}$$

Then $u(T, X)$ is a function of the class $C^\infty([0, T']\times\boldsymbol{R}^n)$ satisfying

$$P^1(T, X, D_T, D_X)\tilde{u}(T, X)=0 \quad \text{in} \quad [0, T']\times\boldsymbol{R}^n$$

and $\operatorname{supp}\tilde{u}(T, X)$ is bounded set in $[0, T']\times\boldsymbol{R}^n$.

Though we are afraid of confusion, we denote again T, X by t, x, respectively. Using new variables we shall rewrite the above. C^∞-function $\tilde{u}(t, x)$ satisfies

$$(4.4)\qquad P^1(t, x, D_t, D_x)\tilde{u}(t, x)=0 \quad \text{in} \quad [0, T']\times\boldsymbol{R}^n$$

and $\operatorname{supp}\tilde{u}(t, x)$ is bounded set in $[0, T']\times\boldsymbol{R}^n$.

Next we set a function $\phi(t)\in C^\infty(\boldsymbol{R}^1)$ such that

$$(4.5)\qquad \phi(t)=\begin{cases} 1 & \text{if } 0\leqq t\leqq T'/2 \\ 0 & \text{if } 2T'/3\leqq t\leqq T'. \end{cases}$$

Then $\phi(t)\tilde{u}(t, x)\in C^\infty([0, T']\times\boldsymbol{R}^n)$ with $\operatorname{supp}\phi(t)\tilde{u}(t, x)\subset[0, 2T'/3]\times\{x: |x|\leqq r\}$. Therefore Proposition 3.5 implies the following inequality for $\phi(t)\tilde{u}(t, x)$.

$$N^m\int_0^{T'} t^{-2N}\|\phi(t)\tilde{u}(t, x)\|^2dt\leqq c\int_0^{T'} t^{-2N}\|P^1(\phi(t)\tilde{u}(t, x))\|^2dt .$$

It follows from (4.4) and (4.5) that

$$N^m\int_0^{T'/2} t^{-2N}\|\tilde{u}(t, x)\|^2dt\leqq c\int_{T'/2}^{T'} t^{-2N}\|P^1(\phi(t)\tilde{u}(t, x)\|^2dt .$$

Then, from this inequality, we can easily see

$$N^m(T'/2)^{-2N}\int_0^{T'/2} \|\tilde{u}(t, x)\|^2dt$$
$$\leqq c(T'/2)^{-2N}\int_{T'/2}^{T'} \|P^1(\phi(t)\tilde{u}(t, x))\|^2dt .$$

Therefore we have

$$\int_0^{T'/2}\|\tilde{u}(t, x)\|^2dt\leqq cN^{-m}\int_{T'/2}^{T'}\|P^1(\phi(t)\tilde{u}(t, x))\|^2dt . \tag{4.6}$$

Letting $N\to\infty$ in (4.6) we see that this is impossible unless $\tilde{u}(t, x)\equiv 0$ for $0\leqq t\leqq T'/2$. Hence we conclude that $u(t, x)\equiv 0$ in $[0, \tilde{T}_1]\times\{x: |x|\leqq\tilde{r}_1\}$ for some $\tilde{T}_1>0$ and $\tilde{r}_1>0$. The proof of Theorem 1 is completed.

PROOF OF LEMMA 4.1. Making the singular change of variables (cf. [1]), we have

$$\begin{cases} t=(r^2-|X|^2)^kT \\ x=X \end{cases}$$

with $tD_t=TD_T$ and $D_x=D_X+2kX/(r^2-|X|^2)TD_T$.

Here we denote the dual variable of (T, X) by (Λ, Ξ). Then principal symbol of $P^1(T, X, D_T, D_x)$ is written as follows.

$$P^1_m(T, X, \Lambda, \Xi)$$
$$=\tilde{P}_m\Big((r^2-|X|^2)^kT, X, T\Lambda, (r^2-|X|^2)^{k'}T^l\Big(\Xi+k\frac{2X}{r^2-|X|^2}T\Lambda\Big)\Big) ,$$

where $\tilde{P}_m(t, x, \lambda, \xi)=\lambda^m+\sum_{\substack{0\leqq j\leqq m-1\\|\alpha|+j=m}} a_{\alpha,j}(t, x)\xi^\alpha\lambda^j$. Then from (A-I) we see that coefficients of principal symbol $P^1_m(T, X, \Lambda, \Xi)$ are real-valued C^∞-functions.

Next we shall check that characteristic roots satisfy the properties of the (#)-condition. We proceed to calculate $P^1_m(T, X, \Lambda, \Xi)$.

$$P^1_m(T, X, \Lambda, \Xi)$$
$$=(T\Lambda)^m+\sum_{\substack{0\leqq j\leqq m-1\\|\alpha|+j=m}} a_{\alpha,j}((r^2-|X|^2)^kT, X)\times$$

$$\times\{(r^2-|X|^2)^{k'}T^l\Xi+2kX(r^2-|X|^2)^{k'-1}T^l(T\Lambda)\}^{\alpha}(T\Lambda)^j$$
$$=P_m((r^2-|X|^2)^kT,\ X,\ T\Lambda,\ (r^2-|X|^2)^{k'}T^l\Xi)$$
$$+\sum_{\substack{0\le j\le m-1\\|\alpha|+j=m}}\sum_{\alpha<\beta}a_{\alpha,j}((r^2-|X|^2)^kT,\ X)_\alpha C_\beta\{(r^2-|X|^2)^{k'}T^l\Xi\}^\beta$$
$$\times\{2kX(r^2-|X|^2)^{k'-1}T^l(T\Lambda)\}^{\alpha-\beta}(T\Lambda)^j$$
$$=P_m((r^2-|X|^2)^kT,\ X,\ T\Lambda,\ (r^2-|X|^2)^{k'}T^l\Xi)$$
$$+\sum_{\substack{0\le j\le m-1\\|\alpha|+j=m}}\sum_{\alpha<\beta}b_{\alpha,\beta,j}(T,\ X)T^{l|\alpha-\beta|}\{(r^2-|X|^2)^{k'}T^l\Xi\}^\beta(T\Lambda)^{m-|\beta|}$$

where $b_{\alpha,\beta,j}(T, X)$ is a smooth function.

Hence it is sufficient to see that the equation

$$P_m((r^2-|X|^2)^kT,\ X,\ \tilde{\Lambda},\ \tilde{\Xi})+\sum_{\substack{0\le j\le m-1\\|\alpha|+j=m}}\sum_{\alpha<\beta}b_{\alpha,\beta,j}(T,\ X)T^{l|\alpha-\beta|}\tilde{\Xi}^\beta\tilde{\Lambda}^{m-|\beta|}=0 \tag{4.7}$$

with respect to $\tilde{\Lambda}$ has simple roots and the imaginary part of its root never vanishes for $\tilde{\Xi}\neq 0$ or identically vanishes.

Putting $T=0$ in (4.7) we have the equation

$$\tilde{P}_m(0,\ X,\ \tilde{\Lambda},\ \tilde{\Xi})=0\ . \tag{4.8}$$

From the assumptions of Theorem 1, the roots of (4.8) are simple. On the other hand, roots of (4.7) are continuous with respect to T. Hence for some $T'>0$ roots of (4.7) are simple for $0\leqq T\leqq T'$. Since coefficients of the left-hand side of (4.7) are real-valued functions, it follows from distinctness that the imaginary parts of its roots never vanish for any $\tilde{\Xi}\neq 0$ or identically vanish.

From elementary calculation we can make sure that the lower order terms of $P^1(T, X, \Lambda, \Xi)$ satisfy the (♯)-condition on $[0, T']$ with $a(x)=(r^2-|X|^2)^{k'}$. Hence we shall omit its proof. Q.E.D.

§5. Concluding remarks.

We shall give generalization of Theorem 1 and 2.

Let $\sigma(x)\in C^\infty$ be a real-valued function. Then for a differential polynomial of $\tilde{P}(t, x, \lambda, \xi)$ of (1.1), $\sigma(x)$ and $l=k'/k$ we define the differential operator as follows.

$$\tilde{P}(t,\ x,\ tD_t,\ t^l\sigma(x)D_x)=t^mD_t^m+\sum_{\substack{0\le j\le m-1\\|\alpha|+j\le m}}a_{\alpha,j}(t,\ x)t^{|\alpha|l+j}\sigma(x)^{|\alpha|}D_x^\alpha D_t^j\ . \tag{5.1}$$

Then we obtain

THEOREM 1′. *Under the assumptions* (A-I) *and* (A-II), *for a characteristic partial differential operator* $P=\tilde{P}(t, x, tD_t, t^l\sigma(x)D_x)$ *we obtain the local uniqueness in the following sense.*

Any function u of class C^∞ *near the origin satisfying*

$$\begin{cases} Pu=0 \\ D_t^j u|_{t=0}=0 \quad \text{for any} \quad j\geqq 0 \end{cases}$$

vanishes identically in some neighbourhood of the origin in $\boldsymbol{R}\times\boldsymbol{R}^n$.

Note that this operator is translation-invariant under the singular change of variables (4.3).

The proof of this generalized theorem is easily seen if we read the proof of Theorem 1 replacing $a(X)=(r^2-|X|^2)^{k'}$ by $a(X)=(r^2-|X|^2)^{k'}\sigma(X)$ in Lemma 4.1.

Consequently from Theorem 1′ and Lemma 3 in §1 we also have a generalization of Theorem 2.

THEOREM 2′. *Let P be of Fuchsian type with weight* $m-s$ *with respect to t. We suppose that there exist a real-valued function* $\sigma(x)\in C^\infty$, *a differential polynomial* $\tilde{P}(t, x, \lambda, \xi)$ *satisfying* (A-I), (A-II) *and a positive rational number* $l=k'/k$ ($k'\geqq 1$ *is an integer*) *such that*

$$t^{m-s}P(t, x, D_t, D_x)=\tilde{P}(t, x, tD_t, t^l\sigma(x)D_x)\,. \tag{5.2}$$

Then, any function u of class C^∞ *near the origin satisfying*

$$\begin{cases} Pu=0 \\ D_t^j u|_{t=0}=0 \quad \text{for} \quad 0\leqq j\leqq m-s+h-1 \end{cases}$$

vanishes identically near the origin in $\boldsymbol{R}\times\boldsymbol{R}^n$, *where h is the same as in Theorem* 2.

Next we shall note the following fact. In our theorems we assumed that coefficients of principal symbol are real-valued functions. But let us review the proof of Theorems. Hence it is sufficient for the proof of the local uniqueness that the transformed operator under the singular change of variables satisfies the (♯)-condition. Consequently we can show the local uniqueness for certain other operators whose principal symbol has non-real coefficients. For example

$$P=D_t\pm it^sK(t, x)D_x+c(t, x)$$

where $s\geqq 0$ is an integer, $K(t, x)$ is a real valued C^∞-function, $K(0, 0)\neq 0$

and $c(t, x) \in C^{\infty}$ (see [8]).

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